LIBOR Market Model Lattice Study

Tim Xiao

ABSTRACT

This article proposes a lattice approach for pricing interest rate derivatives based on LIBOR Market Model. We introduce several fast drift approximation methods. The results show that Frozen Drift performs very badly. Arithmetic Average of Forward Rates and Geometric Average of Forward Rates do a little better but errors go in different directions. The same conclusions can be drawn for Arithmetic Average of the Drift Terms and Geometric Average of the Drift Terms. Both Conditional Expectation of the Forward Rate and *Conditional Expectation of the Drift Term* are the best.

Key Words: LIBOR Market Model, lattice model, drift approximation, risk management, calibration.

The LIBOR Market Model (LMM) is an interest rate model based on evolving LIBOR market forward rates under a risk-neutral forward probability measure. In contrast to models that evolve the instantaneous short rates (e.g., Hull-White, Black-Karasinski models) or instantaneous forward rates (e.g., Heath-Jarrow-Morton (HJM) model), which are not directly observable in the market, the objects modeled using the LMM are market observable quantities.

In general, it is believed that Monte Carlo simulation is the only viable numerical method available for the LMM (see Piterbarg [2003]). The Monte Carlo simulation is computationally expensive, slowly converging, and notoriously difficult to use for calculating sensitivities and hedges. Another notable weakness is its inability to determine how far the solution is from optimality in any given problem.

We introduce a shifted forward measure that uses a variable substitution to shift the center of a forward rate distribution to zero. This ensures that the distribution is symmetric and can be represented by a relatively small number of discrete points. The shift transformation is the key to achieve high accuracy in relatively few discrete finite nodes. In addition, we present several fast and novel drift approximation approaches. Other concepts used in the model are probability distribution structure exploitation, numerical integration and the long jump technique (we only position nodes at times when decisions need to be made).

The rest of this paper is organized as follows: The LMM is discussed in Section I. In Section II, the lattice model is elaborated. The calibration is presented in Section III. The numerical implementation is detailed in Section IV, which will enhance the reader's understanding of the model and its practical implementation. The conclusions are provided in Section V.

I. LIBOR MARKET MODEL

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ be a filtered probability space satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes a filtration.

The simply compounded forward rate reset at t for forward period (T_{k-1}, T_k) is defined by

$$F_k(t) := F(t; T_{k-1}, T_k) = \frac{1}{\delta_k} \left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1 \right) \quad (1)$$

where $P(t, T)$ denotes the time t price of a zero-coupon bond maturing at time T and $\delta_k := \delta(T_{k-1}, T_k)$ is the accrual factor or day count fraction for period (T_{k-1}, T_k) .

Inverting this relationship (1), we can express a zero coupon bond price in terms of forward rates as:

$$P(t, T_k) = P(t, T_{n(t)}) \prod_{j=n(t)}^k \frac{1}{1 + \delta_j F_j(t)} \quad (2)$$

LIBOR Market Model Dynamics

Consider a zero coupon bond numeraire $P(\bullet, T_i)$ whose maturity coincides with the maturity of the forward rate. The measure Q^i associated with $P(\bullet, T_i)$ is called T_i forward measure. Terminal measure Q^N is a forward measure where the maturity of the bond numeraire $P(\bullet, T_N)$ matches the terminal date T_N .

$$\text{If } i < k, t \leq T_i, \quad dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=i+1}^k \frac{\delta_j \sigma_j(t) F_j(t)}{1 + \delta_j F_j(t)} dt + \sigma_k(t) F_k(t) dX_t \quad (3a)$$

$$\text{If } i = k, t \leq T_{k-1}, \quad dF_k(t) = \sigma_k(t) F_k(t) dX_t \quad (3b)$$

$$\text{If } i > k, t \leq T_{k-1}, \quad dF_k(t) = -\sigma_k(t) F_k(t) \sum_{j=k+1}^i \frac{\delta_j \sigma_j(t) F_j(t)}{1 + \delta_j F_j(t)} dt + \sigma_k(t) F_k(t) dX_t \quad (3c)$$

where X_t is a Brownian motion.

There is no requirement for what kind of instantaneous volatility structure should be chosen during the life of the caplet. All that is required is (see Hull-White [2000]):

$$(\hat{\sigma}_k)^2 := (\hat{\sigma}_k(T_{k-1}, K))^2 = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k^2(u) du \quad (4)$$

where $\hat{\sigma}_k$ denotes the market Black caplet volatility and K denotes the strike. Given this equation, it is obviously not possible to uniquely pin down the instantaneous volatility function. In fact, this specification allows an infinite number of choices. People often assume that a forward rate has a piecewise constant instantaneous volatility. Here we choose the forward rate $F_k(t)$ has constant instantaneous volatility regardless of t (see Brigo-Mercurio [2006]).

Shifted Forward Measure

The $F_k(t)$ is a Martingale or driftless under its own measure $\mathcal{Q}^k$. The solution to equation (3b) can be expressed as

$$F_k(t) = F_k(0) \exp\left(-\frac{1}{2} \int_0^t \sigma_k^2(s) ds + \int_0^t \sigma_k(s) dX_s\right) \quad (5)$$

where $F_k(0) = F(0; T_{k-1}, T_k)$ is the current (spot) forward rate. Under the volatility assumption described above, equation (5) can be further expressed as

$$F_k(t) = F_k(0) \exp\left(-\frac{\sigma_k^2}{2} t + \sigma_k X_t\right) \quad (6)$$

Alternatively, we can reach the same Martingale conclusion by directly deriving the expectation of the forward rate (6); that is

$$\begin{aligned} E_0(F_k(t)) &= F_k(0) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_k^2}{2} t + \sigma_k X_t\right) \exp\left(-\frac{X_t^2}{2t}\right) dX_t \\ &= F_k(0) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(X_t - t\sigma_k)^2}{2t}\right) dX_t = F_k(0) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{Y_t^2}{2t}\right) dY_t = F_k(0) \end{aligned} \quad (7)$$

where X_t, Y_t are both Brownian motions with a normal distribution $(0, t)$ at time t , $E_t(\bullet) := E(\bullet | \mathcal{F}_t)$ is the expectation conditional on the $\mathcal{F}_t$, and the variable substitution used for derivation is

$$Y_t = X_t - t\sigma_k \quad (8)$$

This variable substitution that ensures that the distribution is centered on zero and symmetry is *the key to achieve high accuracy* when we express the LMM in discrete finite form and use numerical integration to calculate the expectation. As a matter of fact, without this linear transformation, a lattice method in the LMM either does not exist or introduces too much error for longer maturities.

After applying this variable substitution (8), equation (6) can be expressed as

$$F_k(t) = F_k(0) \exp\left(-\frac{\sigma_k^2}{2}t + \sigma_k X_t\right) = F_k(0) \exp\left(\frac{\sigma_k^2}{2}t + \sigma_k Y_t\right) \quad (9)$$

Since the LMM models the complete forward curve directly, it is essential to bring everything under a common measure. The terminal measure is a good choice for this purpose, although this is by no means the only choice. The forward rate dynamic under terminal measure Q^N is given by

$$dF_k(t) = -\sigma_k F_k(t) \sum_{j=k+1}^N \frac{\delta_j \sigma_j F_j(t)}{1 + \delta_j F_j(t)} dt + \sigma_k F_k(t) dX_t \quad (10)$$

The solution to equation (10) can be expressed as

$$F_k(t) = F_k(0) \exp\left(\mu_k(t) - \int_0^t \frac{\sigma_k^2}{2} ds + \int_0^t \sigma_k dX_s\right) = F_k(0) \exp\left(\mu_k(t) - \frac{\sigma_k^2}{2}t + \sigma_k X_t\right) \quad (11a)$$

where the drift is given by

$$\mu_k(t) = -\int_0^t \sum_{j=k+1}^N \psi_j(s) \sigma_k \sigma_j ds = -\int_0^t \sum_{j=k+1}^N \frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \sigma_k \sigma_j ds \quad (11b)$$

where $\psi_j(s) = \delta_j F_j(s) / [1 + \delta_j F_j(s)]$ is the drift term.

Applying (8) to (11a), we have the forward rate dynamic under the shifted terminal measure as

$$F_k(t) = F_k(0) \exp \left(\mu_k(t) + \frac{\sigma_k^2}{2} t + \sigma_k Y_t \right) \quad (12)$$

Drift Approximation

Under terminal measure, the drifts of forward rate dynamics are state-dependent, which gives rise to sufficiently complicated non-lognormal distributions. This means that an explicit analytic solution to the forward rate stochastic differential equations cannot be obtained. Therefore, most work on the topic has focused on ways to approximate the drift, which is the fundamental trickiness in implementing the Market Model.

Frozen Drift (FD). Replace the random forward rates in the drift by their deterministic initial values, i.e.,

$$\mu_k(t) = - \int_0^t \sum_{j=k+1}^N \frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \sigma_k \sigma_j ds \approx - \sum_{j=k+1}^N \frac{\delta_j F_j(0)}{1 + \delta_j F_j(0)} \sigma_k \sigma_j t \quad (13)$$

Arithmetic Average of the Forward Rates (AAFR). Apply the midpoint rule (rectangle rule) to the random forward rates in the drift, i.e.,

$$\mu_k(t) \approx - \sum_{j=k+1}^N \frac{\delta_j \frac{1}{2} (F_j(0) + F_j(t))}{1 + \delta_j \frac{1}{2} (F_j(0) + F_j(t))} \sigma_k \sigma_j t \quad (14)$$

Arithmetic Average of the Drift Terms (AADT). Apply the midpoint rule to the random drift terms, i.e.,

$$\mu_k(t) \approx - \sum_{j=k+1}^N \frac{1}{2} \left(\frac{\delta_j F_j(0)}{1 + \delta_j F_j(0)} + \frac{\delta_j F_j(t)}{1 + \delta_j F_j(t)} \right) \sigma_j \sigma_k t \quad (15)$$

Geometric Average of the Forward Rates (GAFR). Replace the random forward rates in the drift by their geometric averages, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^N \frac{\delta_j \sqrt{F_j(0) \times F_j(t)}}{1 + \delta_j \sqrt{F_j(0) \times F_j(t)}} \sigma_j \sigma_k t \quad (16)$$

Geometric Average of the Drift Terms (GADT). Replace the random drift terms by their geometric averages, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^N \sqrt{\frac{\delta_j F_j(0)}{1 + \delta_j F_j(0)} \times \frac{\delta_j F_j(t)}{1 + \delta_j F_j(t)}} \sigma_j \sigma_k t \quad (17)$$

Conditional Expectation of the Forward Rate (CEFR). In addition to the two endpoints, we can further enhance our estimate based on the dynamics of the forward rates. The forward rate $F_j(s)$ follows the dynamic (9) (The drift term is ignored). We can derive the expectation of the forward rate conditional on the two endpoints and replace the random forward rate in the drift by the conditional expectation of the forward rate.

Proposition 1. Assume the forward rate $F_j(s)$ follows the dynamic (9), with the two known endpoints given by $F_j(0)$ and $F_j(t)$. Based on the conditional expectation of the forward rate $F_j(s)$, the drift of $F_k(t)$ can be expressed as

$$\mu_k(t) \approx -\sum_{j=k+1}^N \int_0^t \frac{\delta_j E_0[F_j(s) | F_j(0), F_j(t)]}{1 + \delta_j E_0[F_j(s) | F_j(0), F_j(t)]} \sigma_j \sigma_k ds \quad (18a)$$

where the conditional expectation of the forward rate is given by

$$E_0[F_j(s) | F_j(0), F_j(t)] = F_j(0) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{s}{t}} \exp \left(\frac{\sigma_j^2 s(t-s)}{2t} \right) \quad (18b)$$

Proof. See Appendix A.

Conditional Expectation of the Drift Term (CEDT). Similarly, we can calculate the conditional expectation of the drift term and replace the random drift term by the conditional expectation.

Proposition 2. Assume the forward rate $F_j(s)$ follows the dynamic (9), with the two known endpoints given by $F_j(0)$ and $F_j(t)$. Based on the conditional expectation of the drift term ψ_j , the drift of $F_k(t)$ can be expressed as

$$\mu_k(t) \approx -\sum_{j=k+1}^N \int_0^t E_0 \left(\frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \middle|_{F_j(0), F_j(t)} \right) \sigma_j \sigma_k ds \quad (19a)$$

where the conditional expectation of the drift term is given by

$$E_0 \left(\psi_j(s) |_{F_j(0), F_j(t)} \right) = E_0 \left(\frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \middle|_{F_j(0), F_j(t)} \right) = 1 - \frac{1 + \nu_{C_j}(s) / \mu_{C_j}^2(s)}{\mu_{C_j}(s)} \quad (19b)$$

$$\mu_{C_j}(s) = 1 + \delta_j F_j(0) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{s}{t}} \exp \left(\frac{\sigma_j^2 s(t-s)}{2t} \right) \quad (19c)$$

$$\nu_{C_j}(s) = \delta_j^2 F_j^2(0) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{2s}{t}} \left(\exp \left(\frac{\sigma_j^2 s(t-s)}{t} \right) - 1 \right) \exp \left(\frac{\sigma_j^2 s(t-s)}{t} \right) \quad (19d)$$

Proof. See Appendix A.

The accuracy and performance of these drift approximation methods are discussed in section IV.

II. THE LATTICE PROCEDURE IN THE LMM

The “lattice” is the generic term for any graph we build for the pricing of financial products. Each lattice is a layered graph that attempts to transform a continuous-time and continuous-space underlying process into a discrete-time and discrete-space process, where the nodes at each level represent the possible values of the underlying process in that period.

This article presents a grid lattice model for the LMM. To illustrate the lattice algorithm, we use a callable exotic as an example. Callable exotics are a class of interest rate derivatives that have Bermudan style provisions that allow for early

exercise into various underlying interest rate products. In general, a callable exotic can be decomposed into an underlying instrument and an embedded Bermudan option.

$$\tilde{I}(t) := \frac{I(t)}{P(t, T_N)} = \sum_{i=n(t)}^N E_t \left(\frac{Z_i}{P(T_i, T_N)} \right) = \sum_{i=n(t)}^N E_t \left(\frac{\delta_i (F_i(T_{i-1}) - C_i)}{P(T_i, T_N)} \right) \quad (20)$$

where the ratio $\tilde{I}(t)$ is usually called the reduced value of the underlying instrument or the reduced exercise value or the reduced intrinsic value.

First we need to create the lattice. The random process we are going to model in the lattice is the LMM (12). Unlike traditional trees, we only position nodes at the determination dates (the payment and exercise dates). At each determination date, the continuous-time stochastic equation (12) shall be discretized into a discrete-time scheme. Such discretized schemes basically convert the Brownian motion into discrete variables. There is no restriction on discretization schemes. At any determination date t , for instance, we discretize the Brownian motion to be equally spaced as a grid of nodes $y_{i,t}$, for $i = 1, \dots, S_t$. The number of nodes S_t and the space between nodes $\varphi_t = y_{i,t} - y_{i-1,t}$ at each determination date can vary depending on the length of time and the accuracy requirement. The nodes should cover a certain number of standard deviations of the Gaussian distribution to guarantee a certain level of accuracy. We have the discrete form of the forward rate as

$$F_k(t; y_{i,t}) = F_k(0) \exp \left(\mu_k(t, y_{i,t}) + \frac{\sigma_k^2}{2} t + \sigma_k y_{i,t} \right) \quad (21)$$

The zero-coupon bond (2) can be expressed in discrete form as

$$P(t, T_k; y_{i,t}) = P(t, T_{n(t)}; y_{i,t}) \prod_{j=n(t)}^k \frac{1}{1 + \delta_j F_j(t; y_{i,t})} \quad (22)$$

We now have expressions for the forward rate (21) and discount bond (22), conditional on being in the state $y_{i,t}$ at time t , and from these we can perform valuation for the underlying instrument.

At the maturity date, the value of the underlying instrument is equal to the payoff, i.e.,

$$I(T_N, y_{i,T_N}) = Z_N(y_{i,T_N}) \quad (23)$$

The underlying state process X_t in the LMM (11) is a Brownian motion. The transition probability density from state $(x_{i,t}, t)$ to state $(x_{j,T}, T)$ is given by

$$p(x_{i,t}, t; x_{j,T}, T) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(x_{j,T} - x_{i,t})^2}{2(T-t)}\right] \quad (24)$$

Applying the variable substitution (8), equation (24) can be expressed as

$$p(y_{i,t}, t; y_{j,T}, T) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(y_{j,T} - y_{i,t} + \sigma_T T - \sigma_t t)^2}{2(T-t)}\right] \quad (25)$$

Equation (20) can be further expressed as a conditional value on any state $(y_{i,t}, t)$ as:

$$\frac{I(t; y_{i,t})}{P(t, T_N; y_{i,t})} = \sum_{j=n(t)}^N \frac{1}{\sqrt{2\pi(T_j - t)}} \int \frac{Z_j(y_{T_j})}{P(T_j, T_N; y_{T_j})} \exp\left[-\frac{(y_{T_j} - y_{i,t} + \sigma_{T_j} T_j - \sigma_t t)^2}{2(T_j - t)}\right] dy_{T_j} \quad (26)$$

This is a convolution integral. Some fast integration algorithms, e.g., Cubic Spline Integration, Fast Fourier Transform (FFT), etc., can be used for optimization. We use the Trapezoidal Rule Integration in this paper for ease of illustration.

Incomplete information handling. Convolution is widely used in Electrical Engineering, particularly in signal processing. The important part is that the far left and far right parts of the output are based on incomplete information. Any models that try to compute the transition values using integration will be inaccurate if this problem is not solved, especially for longer maturities and multiple exercise dates. Our solution is to extend the input nodes by padding the far end values on each side and only take the original range of the output nodes.

$$\frac{V(t_M^{ex}, y_{i,M})}{P(t_M^{ex}, T_N; y_{i,M})} = \max\left(\frac{I(t_M^{ex}; y_{i,M})}{P(t_M^{ex}, T_N; y_{i,M})}, 0\right) \quad (27)$$

Then, we conduct the backward induction process that is performed by iteratively rolling back a series of long jumps from the final exercise date t_M^{ex} across notification dates and exercise opportunities until we reach the valuation date. Assume that in the previous rollback step t_j^{ex} , we calculated the reduced option value:

$V(t_j^{ex}, y_{i,j}) / P(t_j^{ex}, T_N; y_{i,j})$. Now, we go to t_{j-1}^{ex} . The reduced option value at t_{j-1}^{ex} is

$$\frac{V(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} = \max\left\{\frac{I(t_{j-1}^{ex}; y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})}, \frac{V^c(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})}\right\} \quad (28a)$$

where the reduced continuation value is given by

$$\frac{V^c(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} = \frac{1}{\sqrt{2\pi(t_j^{ex} - t_{j-1}^{ex})}} \int \frac{V(t_j^{ex}, y_j)}{P(t_j^{ex}, T_N; y_j)} \exp\left[-\frac{(y_j - y_{i,j-1} + \sigma_j t_j^{ex} - \sigma_{j-1} t_{j-1}^{ex})^2}{2(t_j^{ex} - t_{j-1}^{ex})}\right] dy_j \quad (28b)$$

We repeat the rollback procedure and eventually work our way through the first exercise date. Then the present value of the Bermudan option is found by a final integration given by

$$pv_{Bermudan}(0) = P(0, T_N) \frac{1}{\sqrt{2\pi t_1^{ex}}} \int \frac{V(t_1^{ex}, y_1)}{P(t_1^{ex}, T_N; y_1)} \exp\left[-\frac{(y_1 + \sigma_1 t_1^{ex})^2}{2t_1^{ex}}\right] dy_1 \quad (29)$$

The present value or the price of the callable exotic from the coupon payer's perspective is:

$$pv_{payer}(0) = pv_{Bermudan}(0) - pv_{underly_instrument}(0) \quad (30)$$

This framework can be used to price any interest rate products in the LMM setting and can be easily extended to the Swap Market Model (SMM).

III. Calibration

First, if we choose the LMM as the central model, we need to price interest rate derivatives that depend on either or both of cap and swaption markets. Second, we will undoubtedly use various swaptions to hedge a callable exotic. It is a reasonable expectation that the calibrated model we intend to use to price our exotic, will at least correctly price the market instruments that we intend to hedge with. Therefore, in an exotic derivative pricing situation, recovery of both cap and swaption markets might be desired.

We introduced a parameter θ and set $\sigma_i = \theta \hat{\sigma}_i$ where $\hat{\sigma}_i$ denotes the market Black caplet volatility. One can choose different θ for different σ_i . For simplicity we describe one θ situation here. By choosing $\theta=1$, we have perfectly calibrated the LMM to the caplet prices in the market. However, our goal is to select a θ to minimize the sum of the squared differences of the volatilities derived from the market and the volatilities implied by our model for both caps and swaptions combined.

$$\begin{aligned} (v_{\alpha,\beta}^{LMM})^2 &= \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{ij}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt \\ &= \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\hat{\sigma}_i\hat{\sigma}_j\theta^2}{S_{\alpha,\beta}(0)^2} = \theta^2 (v_{\sigma,\beta}^{Rebonato})^2 \end{aligned} \quad (31a)$$

where $\rho_{ij}=1$ under one-factor LMM. The swap rate $S_{\alpha,\beta}(0)$ is given by

$$S_{\alpha,\beta}(0) = \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(0) \quad (31b)$$

$$w_i(0) = \frac{\delta_i \prod_{j=\alpha+1}^{\beta} (1 + \delta_j F_j(0))^{-1}}{\sum_{k=\alpha+1}^{\beta} \delta_k \prod_{j=\alpha+1}^k (1 + \delta_j F_j(0))^{-1}} \quad (31c)$$

Assume the calibration containing M caplets and G swaptions. The error minimization is given by

$$\min \left| \sum_{i=1}^M (\theta \hat{\sigma}_i - \hat{\sigma}_i)^2 + \sum_{j=1}^G (\theta v_{\alpha+j,N}^{Rebonato} - \sigma_{\alpha+j,N}^{sw})^2 \right| \quad (32)$$

where $\sigma_{\alpha+j,N}^{sw}$ denotes the market Black swaption volatility. The optimization can be found at a stationary point where the first derivative is zero; that is,

$$\theta = \frac{\sum_{i=1}^M \hat{\sigma}_i + \sum_{j=1}^G \sigma_{\alpha+j,N}^{sw}}{\sum_{i=1}^M \hat{\sigma}_i + \sum_{j=1}^G \nu_{\alpha+j,N}^{\text{Re bonap}}} \quad (33)$$

In terms of forward volatilities, we use the time-homogeneity assumption of the volatility structure, where a forward volatility for an option is the same or close to the spot volatility of the option with the same time to expiry. The time-homogeneous volatility structure can avoid non-stationary behavior.

IV. CONCLUSION

In this paper, we proposed a lattice model in the LMM to price interest rate products. Conclusions can be drawn, supported by the previous sections. First, the model is quite stable. The fast convergence behavior requires fewer discretization nodes. Second, this model has almost equivalent accuracy to the current pricing models in the market. Third, the implementation of the model is relatively easy. The calibration is very simple and straightforward. Finally, the performance of the model is probably the best among all known approaches at the time of writing.

For ease of illustration, we present the lattice model based on the Trapezoidal Rule integration. A better but slightly more complicated solution is to spline the payoff functions. The cubic spline of the option payoffs can achieve higher accuracy, especially for Greeks calculations, and higher speed. Although cubic spline takes some time, the lattice will require much fewer nodes (23 ~ 28 nodes are good enough) and can perform a much faster integration. In general, the spline method can provide a speedup factor around 3 ~ 5 times.

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APPENDIX A:

Proof of Proposition 1. We rewrite (9) as

$$Y(t) = \frac{1}{\sigma_j} \left(\ln \left(\frac{F_j(t)}{F_j(0)} \right) - \frac{\sigma_j^2 t}{2} \right) \quad (\text{A1})$$

In the general Brownian Bridge case when the Wiener process $Y(t)$ has $Y(t_1) = a$ and $Y(t_2) = b$, the distribution of $Y(t)$ at time $t \in (t_1, t_2)$ is normal given by

$$Y(t) \sim N \left(\mu_Y(t) = a + \frac{(t-t_1)(b-a)}{(t_2-t_1)}, \quad \nu_Y(t) = \frac{(t-t_1)(t_2-t)}{(t_2-t_1)} \right) \quad (\text{A2})$$

In our case: $t_1 = 0$, $t_2 = t$, $a=0$, $b=Y(t)$, $s \in (0, t)$, thus (A2) can be expressed as

$$Y(s) \sim N\left(\mu_Y(s) = \frac{s}{t}Y(t), \quad \nu_Y(s) = \frac{s(t-s)}{t}\right) \quad (\text{A3})$$

Let $A_j(s) = \sigma_j Y(s) + \sigma_j^2 s/2$. According to the linear transformation rule, $A_j(s)$ is a normal given by

$$A_j(s) \sim N\left(\mu_{A_j}(s) = \sigma_j \mu_Y(s) + \frac{\sigma_j^2 s}{2} = \frac{s}{t} \ln\left(\frac{F_j(t)}{F_j(0)}\right), \quad \nu_{A_j}(s) = \sigma_j^2 \nu_Y(s) = \frac{\sigma_j^2 s(t-s)}{t}\right) \quad (\text{A4})$$

Let $B_j(s) = \exp(A_j(s))$. By definition, $B_j(s)$ is a lognormal given by

$B_j(s) \sim \text{LogN}(\mu_{A_j}(s), \nu_{A_j}(s))$. According to the characterizations of the lognormal

distribution, the mean and variance of $B_j(s)$ are

$$\mu_{B_j}(s) = E_0(B_j(s)) = \exp\left(\mu_{A_j} + \frac{\nu_{A_j}(s)}{2}\right) = \left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \quad (\text{A5a})$$

$$\nu_{B_j}(s) = [\exp(\nu_{A_j}(s)) - 1] \exp(2\mu_{A_j}(s) + \nu_{A_j}(s)) = \left(\exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) - 1\right) \left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{2s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) \quad (\text{A5b})$$

We have the conditional expectation of the forward rate $F_j(s)$ as

$$E(F_j(s) |_{F_j(0), F_j(t)}) = F_j(0) E_0(B_j(s)) = F_j(0) \left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \quad (\text{A6})$$

Proof of Proposition 2. Let $C_j(s) = 1 + \delta_j F_j(s) = 1 + \delta_j F_j(0) B_j(s)$ where $B_j(s)$ is defined above. According to the linear transformation rule, $C_j(s)$ is a lognormal given by $C_j(s) \sim \text{LogN}(\hat{\mu}(s), \hat{\nu}(s))$. The mean and variance of $C_j(s)$ are

$$\mu_{C_j}(s) = 1 + \delta_j F_j(0) \mu_{B_j}(s) = 1 + \delta_j F_j(0) \left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \quad (\text{A7a})$$

$$\nu_{C_j}(s) = \delta_j^2 F_j(0)^2 \nu_{B_j}(s) = \delta_j^2 F_j(0)^2 \left(\exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) - 1 \right) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{2s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) \quad (\text{A7b})$$

On the other hand, according to the characterizations of the lognormal distribution, the mean and variance of $C_j(s)$ are

$$\mu_{C_j}(s) = \exp\left(\hat{\mu}(s) + \frac{\hat{\nu}(s)}{2}\right) \quad (\text{A8a})$$

$$\nu_{C_j}(s) = [\exp(\hat{\nu}(s)) - 1] \exp(2\hat{\mu}(s) + \hat{\nu}(s)) \quad (\text{A8b})$$

Solving the equation (A8a) and (A8b), we get

$$\hat{\mu}(s) = \ln\left(\frac{\mu_{C_j}(s)}{\sqrt{1 + \nu_{C_j}(s)/\mu_{C_j}^2(s)}}\right) \quad (\text{A9a})$$

$$\hat{\nu}(s) = \ln\left(1 + \frac{\nu_{C_j}(s)}{\mu_{C_j}^2(s)}\right) \quad (\text{A9b})$$

We know the first negative moment of the lognormal is $E(C_j^{-1}(s)) = \exp(-\hat{\mu}(s) + \hat{\nu}(s)/2)$

and have the conditional expectation of the drift term as

$$\begin{aligned} E_0\left(\frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \middle|_{F_j(0), F_j(t)}\right) &= E_0\left(1 - \frac{1}{1 + \delta_j F_j(s)}\right) = 1 - E_0\left(\frac{1}{C_j(s)}\right) \\ &= 1 - \exp\left(-\hat{\mu}(s) + \frac{\hat{\nu}(s)}{2}\right) = 1 - \frac{1 + \nu_{C_j}(s)/\mu_{C_j}^2(s)}{\mu_{C_j}(s)} \end{aligned} \quad (\text{A10})$$

where $\mu_{C_j}(s)$, $\nu_{C_j}(s)$ are given by (A7a) and (A7b).

EXHIBIT 1. The Grid/Rectangular Lattice

This exhibit defines the state space for the underlying process Y_t over the first two discrete time periods. The starting state y_0 at valuation date 0 is the single root of the lattice. At each date t_i the underlying process Y_{t_i} is discretized into a number of vertical nodes/states indexed by j .

The value y_{j,t_i} denotes the underlying process in state j at date t_i . The node y_{1,t_1} , for instance, can evolve to any discrete state in t_2 with certain transition probabilities. For a Brownian motion, the transition probability can be easily determined by (25).

EXHIBIT 2: The Callable Bond and Associated Spot Market Data

The callable bond has a one-year maturity, a \$100 principal, a quarterly payment frequency, and a 4% annual coupon rate. Delta = (end date – start date)/365 (day count: ACT/365). The discount bond $P(0, T_i)$ matures at the end date T_i . The call dates are 6, 9, and 12 months.

Cash flow index	1	2	3	4
Start date (days)	0	92	181	273
End date (days)	92 (T_1)	181 (T_2)	273 (T_3)	365 (T_4)
Delta (years)	0.252055	0.243836	0.252055	0.252055
Payoff (\$)	1	1	1	101
Call Schedule (days)	-	181	273	365
Discount bond $P(0, T_i)$	0.999313	0.998557	0.997293	0.995667
Black Volatility $\hat{\sigma}_i$	-	0.337631	0.344218	0.350878

EXHIBIT 3: The LMM Lattice Structure of the Callable Bond.

The callable bond is defined in Exhibit 2. $\tilde{V}_{i,j} := \tilde{V}(T_j, y_i)$ denotes the reduced value of the callable bond at any node (i, j) . V_1 denotes the coupon at T_1 . $\bar{V}(0)$ is the value calculated by the final integration. $V(0)$ is the final callable bond value that is equal to $\bar{V}(0)$ plus the present value of V_1 . The grid space is $\varphi = 0.5$ and the number of nodes is $S = 7$. This lattice has 3 steps and 7 nodes.

EXHIBIT 4: The Convergence Results for the Callable Bond.

The callable bond is defined in Exhibit 2. $\theta=1$ and drift approximation is AADT. Each curve represents the convergence behavior for a given grid space (ϕ) as nodes are added. All calculations are converged to 100.7518.