Various variants of Hamiltonian problems

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Abstract

Hamiltonian cycle and Hamiltonian path problems are famous hard problems. The Hamiltonian cycle seems to have received more attention from the mathematics community. In this short note, we want to mitigate this bias a little bit. Keeping on track with the Prasad and Sharygin kinds of doing mathematics, we give several simple constructions to show the hardness of some variants of Hamiltonian path problems.

I. NOTATION AND NECESSARY CONCEPTS

Given a graph $G = (V, E)$, we have $V = V(G)$ the *vertex set* and $E = E(G)$ the *edge set*. Throughout this paper, without any exception, all the graphs are loopless, simple, undirected; no multi-edge is allowed. We denote the vertices in the vertex set $V(G)$ by $v_0, v_1, \dots, v_{n-1}$. If an edge $uv \in E(G)$, we say that u and v are adjacent. Given two vertices $u, v \in V$, the *distance* between them is the length of the shortest path connecting the two. The *diameter* of a graph is the maximum value of the distance between every pair of its vertices. The *neighborhood* of a vertex is the set of its adjacent vertices.

A graph $G(V, E)$ is called 3-connected if the removal of any pair of its vertices cannot increase the number of connected components. A vertex u is called a *cut vertex* if its removal increases the number of connected components. The *degree* of a vertex is the number of its adjacent vertices, this is the same as the cardinality of its neighborhood as defined above. A graph is *semi-Hamiltonian* if it has a

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Hamiltonian path. We define a *Hamiltonian path* of a graph as a simple path going through each vertex once. A graph is *Hamiltonian* if it has a *Hamiltonian cycle*. And, we define a *Hamiltonian cycle* of a graph as a simple cycle including each vertex once.

One more concept is a *strict semi-Hamiltonian* graph. A graph satisfying this property is one that is semi-Hamiltonian non-Hamiltonian graph.

II. DECREASING THE DIAMETER

We show that deciding semi-Hamiltonicity of a graph with diameter 2 is no easier than general graphs. We denote the set of semi-Hamiltonian graphs having diameter of 2 by SEMI-HAMILTONIAN OF DIAMETER 2.

CLAIM: $\text{SAT} \leq_p \text{SEMI-HAMILTONIAN OF DIAMETER 2}$

Proof: Given a graph G of arbitrary diameter, we are asked to test for its semi-Hamiltonicity. We add a new vertex v_{-1} to this graph and connect it to every vertex in $V(G)$. Then, yet another new vertex v_{-2} is added, only is just this time it connected to v_{-1} . Call the new graph G'

Now, G is semi-Hamiltonian if and only if G' also is. Indeed, if G' has a Hamiltonian path, that path needs to start at v_{-2} and then go to v_{-1} before consuming all of $V(G)$. So by removing these two vertices, we are left with a Hamiltonian path of G . Conversely, if G has a Hamiltonian path starting from v_i ending at v_j , by adding v_{-2} and v_{-1} (in this order) before v_i , we successfully construct a Hamiltonian path for G' . **Q.E.D**

Figure 1. Construction for Diameter 2

Obviously, the G' graph constructed in the previous proof cannot be Hamiltonian, because of the degree of v_{-2} being 1. We have that $\text{SAT} \leq_p \text{STRICT-SEMI-HAMILTONIAN OF DIAMETER 2}$ as an easy corollary.

III. ONE OR TWO ENDPOINTS SPECIFIED

In this section, we want to specify one or two vertices of G as the endpoint(s) of the Hamiltonian path. As we will see, specifying the endpoint(s) does not affect the hardness of finding a Hamiltonian path. Before stating the claim below, we give another definition. By **ONE ENDPOINT SPECIFIED-SEMI-HAMILTONIAN** and **TWO ENDPOINTS SPECIFIED-SEMI-HAMILTONIAN** (resp.), we denote the set of semi-Hamiltonian graph in which one, two (resp.) endpoint(s) of the Hamiltonian path is (are) specified in the input.

CLAIM: The following claims hold:

- $\text{SAT} \leq_p \text{ONE ENDPOINT SPECIFIED-SEMI-HAMILTONIAN}$
- $\text{SAT} \leq_p \text{TWO ENDPOINTS SPECIFIED-SEMI-HAMILTONIAN}$

Proof: Given a graph G without any marked or specified vertex, we are asked to test for its Hamiltonicity. To do this, we may want to pick an arbitrary vertex u in $V(G)$. "Split" it into two vertices u_1 and u_2 , these two vertices are not connected to each other, but both having the same neighborhood of their "former" u , while u itself is deleted forever. Having done that splitting, we move on to create two new vertices v_{-1} and v_n . Then, v_{-1} is connected to u_1 , v_n is connected to u_2 .

Look at the constructed graph G' , a Hamiltonian path of it necessarily has two endpoints v_{-1} and v_n , since these two have degree of 1. We claim that G is Hamiltonian if and only if G' is semi-Hamiltonian. Indeed, for one direction, if G' has a Hamiltonian path going from v_{-1} to v_n , then by deleting these two endpoints and reuniting u_1 and u_2 as u again, we are left with a Hamiltonian cycle of G . Conversely, if G is Hamiltonian, we break its Hamiltonian cycle at u and connect v_{-1} and v_n to form a Hamiltonian path for G' .

Intuitively, we can specify one of v_{-1} or v_n as endpoint in the input if we are supposed to find a Hamiltonian path with one endpoint specified. Or, we can instead specify both v_{-1} and v_n as endpoints if we are supposed to find a Hamiltonian path with two endpoints specified. The claim thus follows. **Q.E.D**

We close this section by an illustration for the above intuitive construction. Figures of this particular kind are often seen in solutions in the ending chapter of maths books written by Prasolov or Sharygin. After trying to solve their problems for hours, if one cannot come to a solution, then the devoted ending chapter may give some hints on the solutions.

Figure 2. Construction for specified endpoint(s)

IV. TWO SIMPLE PATHS TO COVER A GRAPH

A Hamiltonian path can be seen as one single simple path that covers all the vertex set of a graph. If we are allowed to use at most two simple paths to cover the vertex set of a given graph, we have confronted yet another *NP*-complete problems. So, by **AT MOST 2 SIMPLE PATH COVER**, we denote the set of graphs that can be covered by at most two simple paths. And, by **EXACTLY 2 SIMPLE PATH COVER**, we denote the set of graphs that can be covered by two simple paths, but cannot be covered by one simple path.

CLAIM: The following claims hold:

- $\text{SAT} \leq_p \text{AT MOST 2 SIMPLE PATH COVER}$
- $\text{SAT} \leq_p \text{EXACTLY 2 SIMPLE PATH COVER}$

Proof: For both of these hard problem, we reduce from the original Hamiltonian path problem. So we are given a graph and are asked to test for its Hamiltonicity. Pick an arbitrary vertex $u \in V(G)$. Connect u by a single edge to (one vertex of) a square. By a *square*, we particularly want to indicate a 4-vertex graph with 4 edges, just like a normal square drawn on a board. So, we have used one of the vertices of the square to connect with u , we will now take the "opposite" vertex of the square and connect it to a (fixed) Hamiltonian graph. Call the resulting graph G' .

We claim that G is Hamiltonian if and only if G' can be covered by exactly 2 simple paths. First, note that independent of the graph G given in the input,

our G' needs at least two simple cycles to be covered. That is due to the way we connect our "square" to G and the fixed Hamiltonian graph. To cover the fixed Hamiltonian graph, we need one simple path starting from one of the two remaining vertices of our "square" (the two that are not directly connected to G or the fixed Hamiltonian graph) going through the vertex connected to the fixed Hamiltonian graph. We cannot add any vertex before the selected "unused" vertex in the "square", because that will leave us no choice to cover both the other "unused" vertex of the "square" and G itself. So, in order to cover G , we similarly start from the remaining "unused" vertex of the "square" and go the "opposite" way to cover G . That is the unavoidable strategy to cover G' by at most two simple paths. Second, having that settled, we are ready to proceed our logical reasoning in the next paragraph.

For the easier direction, if G' can be covered by two simple paths as described above, we just take the part of the path that covers G . This must be a Hamiltonian path of G . Conversely, if G has a Hamiltonian path, we follow the same strategy described above to construct the two simple paths covering G' .

Figure 3. Construction for two-simple-paths cover

As stated above, G' needs at least two simple paths to be covered. This is crucial to conclude that the above reduction works for both intended problems in the claim. The claim does hold. **Q.E.D**

V. GETTING RID OF THE UNWANTED FEATURES

A reader with keen eyes can notice that all the reductions above work by using some kinds of unwanted features to force the solution of the variants of the Hamiltonian path problems. These unwanted features are vertex of degree 1 and cut vertex. In this ending section, we shall humbly remove these features leaving only constructed beautiful graphs. We will do this in the following order: removing vertices of degree 1, and then removing cut vertices.

i. Vertex of degree 1

For each vertex u of degree 1, we replace it by an edge u_1u_2 , where u_1 and u_2 are new vertices. Then, connect u_1 and u_2 to the only adjacent vertex v of u in G . Literally, we replace the edge uv by a triangle u_1u_2v . These replacements does not change the semi-Hamiltonicity of the graph.

ii. Cut vertex

Without loss of generality, we can assume that the removal of each cut vertex in G leaves exactly 2 connected components. For otherwise, the given graph is surely a NO instance. Having that guaranteed, for each cut vertex u of G , its neighborhood $N_G(u)$ can be partitioned into two subsets $N_1(u)$ and $N_2(u)$, where each of the two subsets are the adjacent vertices of u that belongs to the same connected components on the removal of u .

We shall now split u into two vertices u_1 and u_2 . Then connect u_i to each vertex in $N_G(u)$, for $i = 1, 2$. Both u_1 and u_2 then are connected to yet another new vertex u_3 . After doing this for all the cut vertices of G , the obtained graph has no cut vertex. And, the semi-Hamiltonicity of both graphs are the same.

REFERENCES

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