Jump-Diffusion for Pricing Convertible Bonds

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ABSTRACT

This paper presents a new framework that relies on the probability distribution of a default jump rather than the default jump itself. As such, the model can back out the market prices of convertible bonds. The empirical results show that the model prices fluctuate randomly around the market prices, indicating the model is quite accurate. Our empirical evidence does not support a systematic underpricing hypothesis. Moreover, market participants almost always calibrate their models to the observed market prices using implied convertible volatilities. Therefore, underpricing may not be the main driver of profitability in convertible arbitrage.

Key Words: jump diffusion, convertible bond, convertible underpricing, convertible arbitrage, asset pricing and credit risk modeling.

1. Introduction

There is a rich literature on the subject of convertible bonds. Tsiveriotis and Fernandes (1998) argue that in practice one is usually uncertain as to whether the bond will be converted, and thus propose dividing convertible bonds into two components: a bond part that is subject to credit risk and an equity part that is free of credit risk. A simple description of this model and an easy numerical example in the context of a binomial tree can be found in Hull (2003).

Grimwood and Hodges (2002) indicate that the Goldman Sachs model is incoherent because it assumes that bonds are susceptible to credit risk but equities are not. Ayache, et al. (2003) conclude that the Tsiveriotis-Fernandes model is inherently unsatisfactory due to its unrealistic assumption of stock prices being unaffected by bankruptcy. To correct this weakness, Davis and Lischka (1999), Andersen and Buffum (2004), Bloomberg (2009), and Carr and Linetsky (2006) etc., propose a jump-diffusion model to explore defaultable stock price dynamics. They all believe that under a risk-neutral measure the expected rate of return on a defaultable stock must be equal to the risk-free interest rate. The jump-diffusion model characterizes the default time/jump directly.

The jump-diffusion model was first introduced by Merton (1976) in the market risk context for modeling asset price behavior that incorporates small day-to-day diffusive movements together with larger randomly occurring jumps. Over the last decade, people attempt to propagate the model from the market risk domain to the credit risk arena.

Zhou (1997), Hilberink and Rogers (2002), Chen and Kou (2009), etc. introduce the jump-diffusion mechanism into the structural models, while Davis and Lischka (1999), Andersen and Buffum (2004), and Bloomberg (2009), etc. add a default jump to the stock price dynamics. We refer to the formers as the structural jump-diffusion models and the latters as the reduced-form jump-diffusion models.

Although both the structural jump-diffusion model and the reduced-form model contain jumps, these jumps have different meanings: A jump in the structural jump-diffusion model corresponds to a

sudden change in the asset value that may or may not cause the firm to default, whereas a jump in the reduced-form model represents the default event itself.

In this paper, we mainly discuss the reduced-form jump-diffusion models. At the heart of the jump-diffusion models lies the assumption that the total expected rate of return to the stockholders is equal to the risk-free interest rate under a risk-neutral measure.

Although we agree that under a risk-neutral measure the market price of risk and risk preferences are irrelevant to asset pricing (see Hull (2003)) and thereby the expectation of a risk-free¹ asset grows at the risk-free interest rate, we are not convinced that the expected rate of return on a defaultable asset must be also equal to the risk-free rate. We argue that unlike market risk, credit risk actually has a significant impact on asset prices. This is why regulators, such as International Accounting Standards Board (IASB), Basel Committee on Banking Supervision (BCBS), etc. require financial institutions to report a credit value adjustment (CVA) in addition to the risk-free mark-to-market (MTM) value to reflect credit risk (see Xiao (2013)). By definition, a CVA is the difference between the risk-free value and the risky value of an asset/portfolio subject to credit risk. CVA implies that the risk-free value should not be equal to the risky value in the presence of default risk. As a matter of fact, we will prove that the expected return of a defaultable asset under a risk-neutral measure actually grows at a risky rate rather than the risk-free rate. This conclusion is very important for risky valuation.

Because of their hybrid nature, convertible bonds attract different type of investors. Especially, convertible arbitrage hedge funds play a dominant role in primary issues of convertible debt. In fact, it is believed that hedge funds purchase 70% to 80% of the convertible debt offered in primary markets. A prevailing belief in the market is that convertible arbitrage is mainly due to convertible underpricing (i.e., the model prices are on average higher than the observed trading prices) (see Ammann, et al (2003), Choi, et al. (2009), Loncarski, et al. (2009), etc.). However, Agarwal, et al. (2007) and Batta, et al. (2007) argue that the excess returns from convertible arbitrage strategies are not mainly due to underpricing, but rather

¹ Here, *risk-free* means free of credit risk, but not necessarily of market risk

partly due to illiquid. Calamos (2011) believes that arbitrageurs in general take advantage of volatility. A higher volatility in the underlying equity translates into a higher value of the equity option and a lower conversion premium. Multiple views reveal the complexity of convertible arbitrage, involving taking positions in the convertible bond and the underlying asset that hedges certain risks but leaves managers exposed to other risks for which they reap a reward.

We model both equities and bonds as defaultable in a consistent way. When a firm goes bankrupt, the investors who take the least risk are paid first. Secured creditors have the best chances of seeing the value of their initial investments come back to them. Bondholders have a greater potential for recovering some their losses than stockholders who are last in line to be repaid and usually receive little, if anything. The default proceedings provide a justification for our modeling assumptions: Different classes of securities issued by the same company have the same default probability but different recovery rates. Given this model, we are able to back out the market prices.

Valuation under our risky model can be solved by common numerical methods, such as, Monte Carlo simulation, tree/lattice approaches, or partial differential equation (PDE) solutions. The PDE algorithm is elaborated in this paper, but of course the methodology can be easily extended to tree/lattice or Monte Carlo.

The most important input parameter to be determined is the volatility for valuation. A common approach in the market is to use the at-the-money (ATM) implied Black-Scholes volatility to price convertible bonds. However, most liquid stock options have relatively short maturates (rarely more than 8 years). As a result, some authors, such as Ammann, et al. (2003), Loncarski, et al. (2009), Zabolotnyuk, et al. (2010), have to make do with historical volatilities. Therefore, we segment the sample into two sets according to the time to maturity: a short-maturity class (0 ~ 8 years) and a long-maturity class (> 8 years). For the short-maturity class, we use the ATM implied Black-Scholes volatilities for valuation, whereas for the long-maturity class, we calculate the historical volatility as the annualized standard deviation of the daily log returns of the last 2 years and then price the convertible bond based on this real-world volatility.

The empirical results show that the model prices fluctuate randomly around the market prices, indicating the model is quite accurate. Our empirical evidence does not support a systematic underpricing hypothesis. A similar conclusion is reached by Ammann and Wilde (2008) who use a Monte-Carlo simulation approach. Moreover, market participants almost always calibrate their models to the observed market prices using implied convertible volatilities. Therefore, underpricing may not be the main driver of profitability in convertible arbitrage.

It is useful to examine the basics of the convertible arbitrage strategy. A typical convertible bond arbitrage employs delta-neutral hedging, in which an arbitrageur buys a convertible bond and sells the underlying equity at the current delta (see Choi, et al. (2009), Loncarski, et al. (2009), etc.). With delta neutral positions, the sign of Gamma is important. If Gamma is negative, the portfolio profits so long as the underlying equity remains stable. If Gamma is positive, the portfolio will profit from large movements in the stock price in either direction (see Somanath (2011)).

2 Model

Convertible bonds can be thought of as normal corporate bonds with embedded options, which enable the holder to exchange the bond asset for the issuer's stock. Despite their popularity and ubiquity, convertible bonds still pose difficult modeling challenges, given their hybrid nature of containing both debt and equity features. Further complications arise due to the frequent presence of complex contractual clauses, such as, put, hard call, soft call, and other path-dependent trigger provisions. Contracts of such complexity can only be solved by numerical methods, such as, Monte Carlo simulation, tree/lattice approaches, or PDE solutions.

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes a filtration.

The risk-free stock price process can be described as

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW(t) \quad (1)$$

where $S(t)$ denotes the stock price, $r(t)$ denotes the risk-free interest rate, σ denotes the volatility, $W(t)$ denotes a Wiener process.

The expectation of equation (1) is

$$E(dS(t)|\mathcal{F}_t) = r(t)S(t)dt \quad (2)$$

where $E\{\bullet|\mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$.

Equation (2) tells us that in a risk-neutral world, the expected return on a risk-free stock is the risk-free interest rate $r(t)$, i.e., the discounted stock price under the risk neutral measure is a martingale process.

Next, we turn to a defaultable stock. The defaultable stock process proposed by Davis and Lischka (1999), Andersen and Buffum (2004), and Bloomberg (2009), etc., is given by

$$dS(t) = (r(t) + h(t))S(t_-)dt + \hat{\sigma}S(t_-)dW(t) - S(t_-)dU(t) \quad (3)$$

where $U(t)$ is an independent Poisson process with $dU(t) = 1$ with probability $h(t)dt$ and 0 otherwise, $h(t)$ is the hazard rate or the default intensity, $S(t_-)$ is the stock price immediately before any jump at time t . The expectation of $dU(t)$ is $E(dU(t)|\mathcal{F}_t) = h(t)dt$.

The expectation of equation (3) is given by

$$E(dS(t)|\mathcal{F}_t) = (r(t) + h(t))S(t)dt - S(t)h(t)dt = r(t)S(t)dt \quad (4)$$

It is shown in equation (4) that the expected return of a defaultable stock under a jump-diffusion model also grows at the risk-free interest rate. Equation (3) is a simpler version of the Merton's Jump-diffusion model where the number of jumps is 1.

The jump-diffusion model was first proposed in the context of market risk, which naturally exhibits high skewness and leptokurtosis levels and captures the so-called implied volatility smile or skew effects. Ederington and Lee (1993) find that the markets tend to have overreaction and underreaction to the outside news. The jump part of the model can be interpreted as the market response to outside news. If there is not any outside news, the asset price changes according to a geometric Brownian motion. Since the market

price of risk and risk preferences are irrelevant to asset pricing within the market risk context, the expected rate of return to the stockholders is equal to the risk-free rate under a risk-neutral measure.

The world of credit modeling is divided into two main approaches: structural models and reduced-form (or intensity) models. The structural models regard default as an endogenous event, focusing on the capital structure of a firm. The reduced-form models do not explain the event of default endogenously, but instead characterize it exogenously as a jump process. In general, structural models are based on the information set available to the firm's management, such as the firm's assets and liabilities; while reduced-form models are based on the information set available to the market, such as the firm's bond prices or credit default swap (CDS) premia. Many practitioners in the credit trading arena have tended to gravitate toward the reduced-form models given their mathematical tractability. The reduced-form models can be made consistent with the risk-neutral probabilities of default backed out from corporate bond prices or CDS spreads/premia.

In the reduced-form models, the stopping (or default) time τ of a firm is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined as,

$$\tau = \inf \left\{ t : \int_0^t h(s, \Phi_s) ds \geq \Delta \right\} \quad (5)$$

where $h(t)$ or $h(t, \Phi_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Φ_t , and Δ is a unit exponential random variable independent of Φ_t .

It is well-known that the survival probability from time t to s in this framework is defined by

$$p(t, s) := P(\tau > s \mid \tau > t, Z) = \exp\left(-\int_t^s h(u) du\right) \quad (6)$$

The default probability for the period (t, s) in this framework is defined by

$$q(t, s) := P(\tau \leq s \mid \tau > t, Z) = 1 - p(t, s) = 1 - \exp\left(-\int_t^s h(u) du\right) \quad (7)$$

We consider a defaultable asset that pays nothing between dates t and T . Let $V(t)$ and $V(T)$ denote its values at t and T , respectively. Risky valuation can be generally classified into two categories: the *default time approach* (DTA) and the *default probability (intensity) approach* (DPA).

The DTA involves the default time explicitly. If there has been no default before time T (i.e., $\tau > T$), the value of the asset at T is $V(T)$. If a default happens before T (i.e., $t < \tau \leq T$), a recovery payoff is made at the default time τ as a fraction of the market value² given by $\phi V(\tau)$ where ϕ is the default recovery rate and $V(\tau)$ is the market value at default. Under a risk-neutral measure, the value of this defaultable asset is the discounted expectation of all the payoffs and is given by

$$V(t) = E\left[\left(D(t, T)V(T)1_{\tau > T} + D(t, \tau)\phi V(\tau)1_{\tau \leq T}\right) \middle| \mathcal{F}_t\right] \quad (8)$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise, and $D(t, \tau)$ denotes the stochastic risk-free discount factor at t for the maturity τ given by

$$D(t, \tau) = \exp\left[-\int_t^\tau r(u)du\right] \quad (9)$$

Although the DTA is very intuitive, it has the disadvantage that it explicitly involves the default time/jump. We are very unlikely to have complete information about a firm's default point, which is often inaccessible. Usually, valuation under the DTA is performed via Monte Carlo simulation.

The DPA relies on the probability distribution of the default time rather than the default time itself. We divide the time period (t, T) into n very small time intervals (Δt) and assume that a default may occur only at the end of each very small period. In our derivation, we use the approximation $\exp(y) \approx 1 + y$ for very small y . The survival and default probabilities for the period $(t, t + \Delta t)$ are given by

$$\hat{p}(t) := p(t, t + \Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \quad (10)$$

$$\hat{q}(t) := q(t, t + \Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \quad (11)$$

² Here we use the recovery of market value (RMV) assumption.

The binomial default rule considers only two possible states: default or survival. For the one-period $(t, t + \Delta t)$ economy, at time $t + \Delta t$ the asset either defaults with the default probability $q(t, t + \Delta t)$ or survives with the survival probability $p(t, t + \Delta t)$. The survival payoff is equal to the market value $V(t + \Delta t)$ and the default payoff is a fraction of the market value: $\varphi(t + \Delta t)V(t + \Delta t)$. Under a risk-neutral measure, the value of the asset at t is the expectation of all the payoffs discounted at the risk-free rate and is given by

$$V(t) = E\left\{\exp(-r(t)\Delta t) [\hat{p}(t) + \varphi(t)\hat{q}(t)] V(t + \Delta t) \middle| \mathcal{F}_t\right\} \approx E\left\{\exp(-y(t)\Delta t) V(t + \Delta t) \middle| \mathcal{F}_t\right\} \quad (12)$$

where $y(t) = r(t) + h(t)(1 - \varphi(t)) = r(t) + c(t)$ denotes the risky rate and $c(t) = h(t)(1 - \varphi(t))$ is called the (short) credit spread.

Similarly, we have

$$V(t + \Delta t) = E\left\{\exp(-y(t + \Delta t)\Delta t) V(t + 2\Delta t) \middle| \mathcal{F}_{t+\Delta t}\right\} \quad (13)$$

Note that $\exp(-y(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t + \Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\left\{\exp(-y(t)\Delta t) V(t + \Delta t) \middle| \mathcal{F}_t\right\} \\ &= E\left\{\exp(-y(t)\Delta t) E\left[\exp(-y(t + \Delta t)\Delta t) V(t + 2\Delta t) \middle| \mathcal{F}_{t+\Delta t}\right] \middle| \mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-\sum_{i=0}^1 y(t + i\Delta t)\Delta t\right) V(t + 2\Delta t) \middle| \mathcal{F}_t\right\} \end{aligned} \quad (14)$$

By recursively deriving from t forward over T and taking the limit as Δt approaches zero, the risky value of the asset can be expressed as

$$V(t) = E\left\{\exp\left[-\int_t^T y(u)du\right] V(T) \middle| \mathcal{F}_t\right\} \quad (15)$$

Using the DPA, we obtain a closed-form solution for pricing an asset subject to credit risk. Another good example of the DPA is the CDS model proposed by J.P. Morgan (1999).

The derivation of equation (15) takes into account all credit characteristics: possibility of a jump to default and recovery rate. It tells us that a defaultable asset under the risk-neutral measure grows at a *risky*

rate. The risky rate is equal to a risk-free interest rate plus a credit spread. If the asset is a bond, the equation is the same as Equation (10) in Duffie and Singleton (1999), which is the market model for pricing risky bonds. The market bond model says that the value of a risky bond is obtained by discounting the promised payoff using the risk-free interest rate plus the credit spread³.

In asset pricing theory, the fundamental no-arbitrage theorems do not require expected returns to be equal to the risk free rate, but only that prices are martingales after discounting under the numeraire. For risk-free valuation, people commonly use a risk-free bond as the numeraire, whereas for risky valuation, they should choose an associated risky numeraire to reflect credit risk. The expected return is that of the numeraire.

According to equation (15), we propose a risky model that embeds the probability of the default jump rather than the default jump itself into the price dynamics of an asset. The stochastic differential equation (SDE) of a defaultable stock is defined as

$$dS(t) = (r(t) + h(t)(1 - \varphi_s(t)))S(t)dt + \sigma S(t)dW(t) = y(t)S(t)dt + \sigma S(t)dW(t) \quad (16)$$

where φ_s is the recovery rate of the stock and $y(t) = r(t) + h(t)(1 - \varphi_s(t))$ is the risky rate.

For most practical problems, zero recovery at default (or jump to zero) is unrealistic. For example, the stock of Lehman Brothers fell 94.3% on September 15, 2008 after the company filed for Chapter 11 bankruptcy. Similarly, the shares of General Motors (GM) plunged 32% on June 1, 2009 after the firm initiated Chapter 11 bankruptcy. A good framework should flexibly allow people to incorporate different recovery assumptions into risky valuation.

Equation (16) is the direct derivation of equation (15). The formula allows different assumptions concerning recovery on default. In particular, $\varphi_s = 0$ represents the situation where the stock price jumps to 0, and $\varphi_s = 1$ corresponds to the risk-free case. The expectation of equations (16) is

$$E(dS(t)|\mathcal{F}_t) = (r(t) + h(t)(1 - \varphi_s(t)))S(t)dt \quad (17)$$

³ There is a liquidity component in the bond spread. This paper, however, focuses on credit risk only.

Equation (17) says that the expected return of a stock subject to credit risk is equal to a risky rate rather than the risk-free rate. The risky rate reflects the compensation investors receive for bearing credit risk.

3. PDE Algorithm

The numerical solution of our risky model can be obtained by either PDE methods, tree approaches, or Monte Carlo simulation. In this paper, we introduce the PDE procedure, but of course the methodology can be easily extended to the tree/lattice or Monte Carlo algorithms.

The defaultable stock price process is given by

$$dS(t) = (r(t) - q(t) + h(t)(1 - \varphi_s(t)))S(t)dt + \sigma S(t)dW(t) = \mu(t)S(t)dt + \sigma S(t)dW(t) \quad (18)$$

where $q(t)$ is the dividend and $\mu(t) = r(t) - q(t) + h(t)(1 - \varphi_s(t))$.

Therefore, we arrange the valuation so that the value of the convertible at each node is divided into two components: a component of bond and a component of stock, i.e. $L(S, t) = G(S, t) + B(S, t)$ where $G(S, t)$ denotes the equity part of the convertible bond and $B(S, t)$ denotes the bond part of the convertible.

Suppose that $G(S, t)$ is some function of S and t . Applying Ito Lemma, we have

$$dG = \left(\mu S \frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt + \sigma S \frac{\partial G}{\partial S} dW \quad (19)$$

Since the Wiener process underlying S and G are the same, we can construct the following portfolio so that the Wiener process can be eliminated.

$$X = G - S \frac{\partial G}{\partial S} \quad (20)$$

Therefore, we have

$$dX = dG - \frac{\partial G}{\partial S} dS = \left(\frac{\partial G}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt \quad (21)$$

In contrast to all previous studies, we believe that the defaultable equity should grow at the risky rate of the equity including dividends, whereas the equity part of the convertible bond should earn the risky rate of the equity excluding dividends, i.e.,

$$(r + h(1 - \varphi_s))Gdt - (r - q + h(1 - \varphi_s))\frac{\partial G}{\partial S}Sdt = dX = \left(\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right)dt \quad (22)$$

So that the PDE of the equity component is given by

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + (r - q + h(1 - \varphi_s))S \frac{\partial G}{\partial S} - (r + h(1 - \varphi_s))G = 0 \quad (23)$$

Similarly applying Ito Lemma to the bond part of the convertible $B(S, t)$, we obtain

$$dB = \left(\mu S \frac{\partial B}{\partial S} + \frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} \right)dt + \sigma S \frac{\partial B}{\partial S} dW \quad (24)$$

Let us construct a portfolio so that we can eliminate the Wiener process as follows

$$Y = B - S \frac{\partial B}{\partial S} \quad (25)$$

Thus, we have

$$dY = dB - \frac{\partial B}{\partial S} dS = \left(\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} \right)dt \quad (26)$$

The defaultable equity should grow at the risky rate of the equity including dividends, while the bond part of the convertible bond grows at the risky rate of the bond. Consequently, we have

$$(r + h(1 - \varphi_b))Bdt - (r - q + h(1 - \varphi_s))\frac{\partial B}{\partial S}Sdt = dY = \left(\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} \right)dt \quad (27)$$

where φ_b is the recovery rate of the bond.

The PDE of the bond component is

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + (r - q + h(1 - \varphi_s))S \frac{\partial B}{\partial S} - (r + h(1 - \varphi_b))B = 0 \quad (28)$$

Equations (23) and (28) are coupled through appropriate final and boundary conditions reflecting the terms and conditions of each individual convertible and need to be solved simultaneously. Convertible bonds often incorporate various additional features, such as call and put provisions.

The final conditions at maturity T can be generalized as

$$G_T = \begin{cases} \eta S_T, & \text{if } \eta S_T > \min [P_c, \max (P_p, N + C)] \\ 0, & \text{otherwise} \end{cases} \quad (29)$$

$$B_T = \begin{cases} \min [P_c, \max (P_p, N + C)] & \text{if } \eta S_T \leq \min [P_c, \max (P_p, N + C)] \\ 0, & \text{otherwise} \end{cases} \quad (30)$$

where N denotes the bond principal, C denotes the coupon, P_c denotes the call price, P_p denotes the put price and η denotes the conversion ratio. The final conditions tell us that the convertible bond at the maturity is either a debt or an equity.

The upside constraints at time $t \in [0, T]$ are

$$\begin{cases} G_t = \eta S_t, B_t = 0 & \text{if } \eta S_t > \min [P_c, \max (P_p, \tilde{L}_t)] \\ G_t = 0, B_t = P_p & \text{else if } \tilde{L}_t \leq P_p \\ G_t = 0, B_t = P_c & \text{else if } \tilde{L}_t \geq P_c \\ G_t = \tilde{G}_t, B_t = \tilde{B}_t & \text{else} \end{cases} \quad (31)$$

where $\tilde{L}_t = \tilde{B}_t + \tilde{G}_t$ is the continuation value of the convertible bond, $\tilde{B}_t$ is the continuation value of the bond component and $\tilde{G}_t$ is the continuation value of the equity component.

4. Conclusion

This paper aims to value hybrid financial instruments (e.g., convertible bonds) whose values may simultaneously depend on different assets subject to credit risk in a proper and consistent way. The motivation for our model is that if a company goes bankrupt, all the securities (including the equity) of the company default. The recovery is realized in accordance with the priority established by the Bankruptcy Code. In other words, different securities have the same probability of default, but different recovery rates.

Our study shows that risky asset pricing is quite different from risk-free asset pricing. In fact, the expectation of a defaultable asset actually grows at a risky rate rather than the risk-free rate. This conclusion is very important for risky valuation.

We propose a hybrid framework to value risky equities and debts in a unified way. The model relies on the probability distribution of the default jump rather than the default jump itself. The model is quite accurate for pricing convertible bonds.

Empirically, we do not find evidence supporting a systematic underpricing hypothesis. We also find that convertible bonds have relatively large positive gammas, implying that convertible arbitrage can make a profit on a large upside and downside movement in the underlying stock price.

Appendix

A. A comparison of results

Let us now briefly turn to a comparison with previous works. We use a simple example described in Table C1, which is similar to the example used in Tsiveriotis and Fernandes (1998) and Ayache, et al. (2003). We assume that the interest rate, the bond spread, and the volatility are flat. The hazard rate is $0.02/(1-0)=0.02$. As the call and put prices are quoted using the clean prices, we need to convert them to the dirty prices as

$$P^{dirty}(t) = P^{clean}(t) + AI(t) \quad (C1)$$

where the accrued interest is give by

$$AI(t) = C \frac{\delta(t, t_s)}{\delta(t_e, t_s)} \quad (C2)$$

where C denotes the coupon, $\delta(t_s, s_e)$ denotes the accrual factor or day count fraction for period (t_s, t_e) where $t_s \leq t \leq t_e$, t_s denotes the start time of the accrual period, and t_e denotes the end time of the accrual period. The numerical results are shown in Table C2, from which we can see that our model generates lower results than AVF and TF models.

Table C1. A 5-year convertible bond

Maturity	5 years
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Payment frequency	Semiannual
Coupon payment	4
Notional	100
Conversion rule	1 share (ratio) in 0 – 5 years
Clean call price	110 in 2 – 5 years
Clean put price	105 at 3 years
Spot stock price	100
Implied volatility of the convertible	0.2
Interest rate	0.05
Bond spread	0.02
Bond recovery rate	0
Stock recovery rate	0

Table C2. Model comparison results

This table presents the numerical results for model comparison. The Tsiveriotis and Fernandes (1998) model is referred to as the TF model and the Ayache, et al. (2003) model is referred to as the AFV model. The convertible bond is described in Table C1.

Time steps	This model	AFV	TF
200	122.6921	122.7341	124.0025
400	122.6938	122.7333	123.9916
800	122.6961	122.7325	123.9821
1600	122.6953	122.7319	123.9754
3200	122.6952	122.7316	123.9714

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