On modules over the mod 2 Steenrod algebra and hit problems

ĐẶNG VÕ PHÚC

Faculty of Education Studies, University of Khanh Hoa, Vietnam

(This work is dedicated to the cherished memory of the esteemed Professor Reginald Wood (1938-2022), an exceptional algebraic topologist. His invaluable contributions to the field will forever be remembered and celebrated)

**** This article has undergone a thorough revision and is now considered to be the complete and final version. We welcome any comments or feedback from readers**

Abstract

Let us consider the prime field of two elements, $\mathbb{F}_2$. The classical "hit problem" in algebraic topology, which is widely considered to be an important and fascinating open problem that has yet to be resolved, asks for a minimal set of generators for the polynomial algebra, $\mathcal{P}_m := \mathbb{F}_2[x_1, x_2, \dots, x_m]$, on m variables $x_1, \dots, x_m$, each of which has degree one, regarded as a connected unstable $\mathcal{A}$ -module. The algebra $\mathcal{P}_m$ is the cohomology with $\mathbb{F}_2$ -coefficients of the product of m copies of the Eilenberg-MacLan complex $K(\mathbb{F}_2, 1)$. Despite extensive study over the past three decades, the hit problem remains unresolved for $m \geq 5$.

In this article, we develop our previous work [Commun. Korean Math. Soc. **35** (2020), 371-399] on the hit problem for $\mathcal{A}$ -module $\mathcal{P}_5$ in the generic degree $n_s = 5(2^s - 1) + 18 \cdot 2^s$ with s an arbitrary non-negative integer. As a consequence, a localized variation of the Kameko conjecture, which concerns the dimension of the cohit space $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_m$ in relation to parameter vectors, has been claimed to be veracious in the instance where $m = 5$ and the degree is n_s . Also, we demonstrate that this conjecture remains valid for all $m \geq 1$ and degrees ≤ 12 . This study has two important applications: (1) it establishes the dimension result for the cohit space $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_m$ for $m = 6$ in the generic degree $5(2^{s+4} - 1) + n_1 \cdot 2^{s+4}$ with $s > 0$; and (2) it describes the representations of the general linear group of rank 5 over $\mathbb{F}_2$. As a result, we prove that the algebraic transfer, defined by W. Singer [Math. Z. **202** (1989), 493-523], is an isomorphism in bidegrees $(5, 5 + n_s)$ with $s \geq 0$. Besides, we obtain new results on the behavior of this algebra transfer for all homological degrees m . Specifically, we demonstrate that Singer's transfer is a trivial isomorphism in bidegree $(m, m + 12)$ for any $m > 0$.

Keywords:

Adams spectral sequences; Steenrod algebra; Lambda algebra; Peterson hit problem; Algebraic transfer
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1. Context and Overview

Consider the set of all stable cohomology operations of degree i with coefficients in the prime field $\mathbb{F}_2$, denoted by $\mathcal{O}^S(i, \mathbb{F}_2, \mathbb{F}_2)$. The $\mathbb{F}_2$ -algebra $\mathcal{A} := \bigoplus_{i \geq 0} \mathcal{O}^S(i, \mathbb{F}_2, \mathbb{F}_2)$ is known as the mod 2 Steenrod algebra and has been extensively explored by mathematicians from a variety of fields, including algebraic topology, homotopy theory, manifold theory, combinatorics, representation theory, and more. It is remarkable that in several instances, the resulting $\mathcal{A}$ -module structure on $H^*(X; \mathbb{F}_2)$ gives additional insight into CW complexes X . For instance, to our knowledge, the cohomology rings of the CW complexes $\mathbb{C}P^4/\mathbb{C}P^2$ and $\mathbb{S}^6 \vee \mathbb{S}^8$ agree as graded commutative $\mathbb{F}_2$ -algebras, but differ as modules over $\mathcal{A}$ (see Subsect.3.1 for an explicit proof).

Regarding the structure of modules over the Steenrod ring, as it is known, the cohomology with $\mathbb{F}_2$ -coefficients of the Eilenberg-MacLan complex $K(\mathbb{F}_2, 1)$ is a polynomial ring $\mathbb{F}_2[x]$ in one variable, and thus, $H^*((K(\mathbb{F}_2, 1))^{\times m}; \mathbb{F}_2) = \underbrace{\mathbb{F}_2[x_1] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_2] \otimes_{\mathbb{F}_2} \cdots \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_m]}_{m \text{ times}} = \mathbb{F}_2[x_1, \dots, x_m]$, wherein

$|x_1| = |x_2| = \cdots = |x_m| = 1$. As the polynomial ring $\mathcal{P}_m := \mathbb{F}_2[x_1, \dots, x_m]$ is the cohomology of a CW-complex, it is equipped with a structure of unstable module over $\mathcal{A}$. The action of $\mathcal{A}$ on $\mathcal{P}_m$ is determined by the rule

$$Sq^k(x_j^n) = \binom{n}{k} x_j^{n+k},$$

extended to all of $\mathcal{P}_m$ by the Cartan formula and linearity. Denote by $(\mathcal{P}_m)_n$ the $\mathbb{F}_2$ -subspace of $\mathcal{P}_m$ consisting of all homogeneous polynomials of degree n in $\mathcal{P}_m$. Then $\mathcal{P}_m = \bigoplus_n (\mathcal{P}_m)_n$ where $(\mathcal{P}_m)_0 = \mathbb{F}_2$ and $(\mathcal{P}_m)_n = 0$ for all $n < 0$. Letting $G(m)$ for the general linear group of rank m over $\mathbb{F}_2$. This group acts on $\mathcal{P}_m$ by matrix substitution, and so, in addition to $\mathcal{A}$ -module structure, $\mathcal{P}_m$ is also a (right) $G(m)$ -module. The classical "hit problem" for the algebra $\mathcal{P}_m$, which is concerned with seeking a minimal set of $\mathcal{A}$ -generators for $\mathcal{P}_m$, has been initiated in a variety of contexts by Peterson [Pet87], Priddy [Pri90], Singer [Sin91], and Wood [Woo89]. The study of modules over the Steenrod algebra $\mathcal{A}$ and related hit problems is a central topic in algebraic topology and has received extensive attention from numerous authors, including Ault and Singer [AS11], Brunetti, Ciampella and Lomonaco [BCL12], Brunetti and Lomonaco [BL14], Crabb and Hubbuck [CH96], Janfada [Jan08], Kameko [Kam90], Mothebe et al. [MU15, Mot16, MKR16], Nam [Nam04], Repka and Selick [RS98], Walker and Wood [WW18a, WW18b], the present author [Phu16, Phu20a, Phu20b, Phu21a, Phu21b, Phu22a, Phu22b, Phu22c, Phu22d, Phu23a, Phu23d], etc. We believe that the best references here are the books by Walker and Wood [WW18a, WW18b], where the most popular articles on these subjects are collected.

When $\mathbb{F}_2$ is an $\mathcal{A}$ -module concentrated in degree 0, solving the hit problem is equivalent to determining an $\mathbb{F}_2$ -basis for the space of indecomposable elements, also known as "non-hit" elements, $Q^{\otimes m}$, in each positive degree n . $Q^{\otimes m}$ is defined as the quotient space of $\mathcal{P}_m$ modulo the submodule generated by the action of $\mathcal{A}$ on $\mathcal{P}_m$, i.e., $Q^{\otimes m} = \mathcal{P}_m / \overline{\mathcal{A}} \mathcal{P}_m \cong \mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_m$, where $\overline{\mathcal{A}}$ being the positive degree part of $\mathcal{A}$ (cf. [Pet87]). We can see that $Q^{\otimes m}$ is the largest quotient of $\mathcal{P}_m$ on which

$\mathcal{A}$ acts trivially. Here and subsequently, we will denote the homogeneous components of degree n in $Q^{\otimes m}$ by

$$Q_n^{\otimes m} := (Q^{\otimes m})_n = (\mathcal{P}_m)_n / (\mathcal{P}_m)_n \cap \overline{\mathcal{A}}\mathcal{P}_m.$$

At that time, there existed a decomposition $Q^{\otimes m} = \bigoplus_{n \geq 0} Q_n^{\otimes m}$. Since the action of the linear group

$G(m)$ and the action of $\mathcal{A}$ on $\mathcal{P}_m$ commute, there is an induced action of $G(m)$ on $Q^{\otimes m}$. The $\mathcal{A}$ -indecomposables are presently only ascertainable for $m \leq 4$ (cf. [Pet87, Boa93, Kam90, Jan08, Sum15]). The general case remains a stimulating and unsolved problem. It must be noted that an essential aspect of this hit problem is verifying whether a given polynomial in $\mathcal{P}_m$ indeed hit or not. The dearth of comprehensive information available for larger values of m has instigated a substantial number of researchers to undertake investigations into the $\mathcal{A}$ -decomposables $\overline{\mathcal{A}}\mathcal{P}_m$. In fact, research into $\overline{\mathcal{A}}\mathcal{P}_m$ may offer potential avenues for progress towards a more comprehensive understanding of the $\mathcal{A}$ -indecomposables for larger values of m . In 2008, Janfada [Jan08] accomplished the work by Kameko [Kam90] by introducing a criterion for monomials with generic degrees in $\mathcal{P}_3$ to be hit. Subsequently, in 2015, Sum conducted an extensive study on the hit problem for four variables in [Sum15]. However, this case still lacks complete resolution as no established criteria exist for hit elements of generic degrees in $\mathcal{P}_4$.

We will write $(\mathcal{P}_m)^* = H_*(K(\mathbb{F}_2, 1))^{\times m}; \mathbb{F}_2)$ as the dual of $\mathcal{P}_m$. This homology is, in fact, the divided power algebra in m generators $a_1^{(1)}, a_2^{(1)}, \dots, a_m^{(1)}$, wherein each $a_j^{(1)} \equiv a_j$ denotes the linear dual to x_j . It is widely recognized that $(\mathcal{P}_m)^*$ has a right $\mathcal{A}$ -module structure. The (right) action of $\mathcal{A}$ on $(\mathcal{P}_m)^*$ is given by

$$(a_j^{(t)})Sq^k = \binom{t-k}{k} a_j^{(t-k)}, \quad 1 \leq j \leq m,$$

and the standard Cartan formula. The dual of $Q^{\otimes m}$, denoted by $P_{\mathcal{A}}(\mathcal{P}_m)^*$, is a subspace of $(\mathcal{P}_m)^*$ consisting of all $\overline{\mathcal{A}}$ -annihilated elements. It is widely known that the determination of the cohit space $Q^{\otimes m}$ in each positive degree n is considered important in comprehending the E^2 -terms of the Adams Spectral Sequence (Adams SS, for short), $\text{Ext}_{\mathcal{A}}^{m, m+n}(\mathbb{F}_2, \mathbb{F}_2)$, through the m -th transfer homomorphism,

$$Tr_m^{\mathcal{A}} : (\mathbb{F}_2 \otimes_{G(m)} P_{\mathcal{A}}(\mathcal{P}_m)^*)_n \longrightarrow \text{Ext}_{\mathcal{A}}^{m, m+n}(\mathbb{F}_2, \mathbb{F}_2).$$

Here the domain of $Tr_m^{\mathcal{A}}$ is dual to the $G(m)$ -invariants space $(Q_n^{\otimes m})^{G(m)}$. The transfer $Tr_m^{\mathcal{A}}$, constructed by Singer [Sin89], has garnered significant attention from numerous mathematicians. Involving this, we also knew that one of the major goals of homotopy theory is to explicitly compute stable homotopy groups of sphere, $\pi_*^S(\mathbb{S}^0)$, and that the (mod 2) cohomology of the algebra $\mathcal{A}$, which forms the E^2 -term of the aforementioned Adams SS converging to the 2-primary component of $\pi_*^S(\mathbb{S}^0)$, is a tool that many algebraic topologists are interested in studying the groups $\pi_*^S(\mathbb{S}^0)$. Nevertheless, $\text{Ext}_{\mathcal{A}}^{m, m+n}(\mathbb{F}_2, \mathbb{F}_2)$ is itself very difficult to grasp. Hence, the transfer map $Tr_m^{\mathcal{A}}$ can be viewed as a valuable tool in gaining insight into these complex Ext groups. Indeed, $Tr_m^{\mathcal{A}}$ is an isomorphism for $m \leq 3$: see [Sin89] for $m = 1, 2$ and [Boa93] for $m = 3$. In addition, by Singer [Sin89], the "total" transfer $\bigoplus_{m \geq 0} Tr_m^{\mathcal{A}}$ is a homomorphism of algebras [Sin89]. It is noteworthy that

even though $Tr_m^{\mathcal{A}}$ may not be surjective for certain ranks m (as shown in references [Ha07, Phu23c, Sin89]), it still holds significant value in helping us understand Ext groups. In particular, the following conjecture remains unanswered, even for the homological degree five.

Conjecture 1.1 (cf. Singer [Sin89]). *$Tr_m^{\mathcal{A}}$ is a one-to-one homomorphism for arbitrary m .*

This is evident for ranks $m \leq 3$ due to the fact that, as per the above discussion, $Tr_m^{\mathcal{A}}$ is an isomorphism in these ranks. However, proving or refuting this conjecture in the general case remains a challenging task. Most recently, the work in [Phu23c], which required a great deal of effort on our part, has finally put an end to the conjecture of rank four that has lasted for over three decades. The precise statement is that we showed in [Phu23c] that Conjecture 1.1 holds true for the homological degree four. In the present study, we aim to investigate Singer's conjecture for arbitrary m in certain internal degrees by utilizing the techniques of the hit problem. The chief results achieved by us are formulated in the sequel.

2. Statement of results

Our contributions. As presented in Sect.1, the hit problem for the $\mathcal{A}$ -module $\mathcal{P}_m$ and Singer's Conjecture 1.1 are still open for any $m \geq 5$ and positive degrees n . The interaction between these two subjects, as we will discover, will be a crucial component in describing Ext groups. So, motivated by these contexts and numerous previous interesting results, the two principal objectives that we strive to achieve in this paper are as follows: we would first like to develop our previous work [Phu20b] on the hit problem for $\mathcal{P}_5$ in the generic degree $n := n_s = 5(2^s - 1) + 18 \cdot 2^s$ with s an arbitrary non-negative integer. As a direct result, a localized form of the Kameko conjecture that focuses on the dimension of the cohit space $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_m$ connected to parameter vectors has been demonstrated to be true for the specific case of $m = 5$ and degree n_s . As applications, we establish the dimension result for the cohit $\mathbb{F}_2$ -module $Q^{\otimes 6}$ in the generic degree $5(2^{s+4} - 1) + n_1 \cdot 2^{s+4}$ and examine Conjecture 1.1 for bidegrees $(5, 5 + n_s)$ with $s \geq 0$. Next, as a continuation of the previous works in [Boa93, Sum15, Phu23b, Phu22a, Phu22b, Phu22c, Phu22d], we shall investigate Conjecture 1.1 in bidegree $(m, m + n)$ for $n = 12$ and arbitrary $m > 0$. The cases of ranks $m \leq 3$ are relatively easy to compute (cf. [Sin89, Boa93]). It has also been studied by Sum [Sum18] for rank $m = 5$, and by the present author [Phu22d] for ranks $m = 4, 6, 7, 8$. It is important to note that understanding the specific degrees can be a useful step in solving the general case. The results established in this manuscript are essentially new.

A key tool in our approach to investigating the chief results of this paper is inevitably based on the Kameko morphism [Kam90],

$$\begin{aligned} (\widetilde{Sq}_*^0)_{(m,n)} : Q_n^{\otimes m} &\longrightarrow Q_{\frac{n-m}{2}}^{\otimes m} \quad (n - m \text{ even}) \\ [x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}] &\longmapsto \begin{cases} [x_1^{\frac{a_1-1}{2}} x_2^{\frac{a_2-1}{2}} \dots x_m^{\frac{a_m-1}{2}}] & \text{if } a_j \text{ odd, } j = 1, 2, \dots, m, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It shows that $(\widetilde{Sq}_*^0)_{(m,n)}$ is always an epimorphism of $\mathbb{F}_2 G(m)$ -modules. This characteristic suggests a descent approach for tackling the hit problem. Kameko utilized this extensively in his computation of $Q^{\otimes m}$ for $m = 3$. Due to the complexity of calculating $Q^{\otimes m}$, two useful approaches would be to either measure its global dimension by taking the maximum dimension of $Q_n^{\otimes m}$ for all positive values of n , or to examine $Q^{\otimes m}$ in small degrees of n , and then generalize the results obtained. In addition, there is also a homotopical approach, as seen in [AP93], but it will not be discussed in this paper. For related problems, readers can refer to [GHP94, GP95]. Besides, the well-established results of Wood [Woo89] (also referred to as Peterson's conjecture [Pet87]) and Kameko [Kam90] (Theorem 2.1) can help to streamline the calculation process for $Q^{\otimes m}$ in specific degrees.

Theorem 2.1. *Given the arithmetic function*

$$\begin{aligned} \beta : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto \min \left\{ k \in \mathbb{N} : n = \sum_{1 \leq i \leq k} (2^{d_i} - 1), d_i > 0 \right\} = \min \left\{ k \in \mathbb{N} : \alpha(n + k) \leq k \right\}, \end{aligned}$$

where the α function counts the number of ones in the binary expansion of its argument.

(I) $Q_n^{\otimes m}$ is trivial if $\beta(n) > m$. Consequently $Q^{\otimes m}$ is trivial in degrees n unless n is of the form $n = (2^{d_1} - 1) + (2^{d_2} - 1) + \dots + (2^{d_m} - 1)$ with $d_1 \geq d_2 \geq \dots \geq d_m > 0$ (cf. Wood [Woo89]).

(II) The map $(\widetilde{Sq}_*^0)_{(m,n)}$ is an isomorphism of $\mathbb{F}_2 G(m)$ -modules if $\beta(n) = m$ (cf. Kameko [Kam90]).

This result in point (I) serves as the foundation for ongoing research focused on determining the "hit" elements and the conditions that govern them. Since then, numerous authors have attempted to refine and extend Wood's approach, including Meyer and Silverman [MS00], Monks [Mon94], Silverman [Sil95, Sil98], Singer [Sin91], etc. Particularly, the application of Theorem 2.1 can reduce the calculation of $Q^{\otimes m}$ in positive degrees n for cases where $\beta(n) < m$. In these cases, n is of the "generic" form $d(2^s - 1) + k \cdot 2^s$, where d, s , and k are non-negative integers, and $\beta(k) < d < m$ (cf. [PS15]).

Main results and applications. It is imperative to note the following observations in regards to the generic degree $n_s = 5(2^s - 1) + 18 \cdot 2^s$. The degree n_s can be represented as

$$n_s = 2^{s+4} + 2^{s+2} + 2^{s+1} + 2^{s-1} + 2^{s-1} - 5.$$

This implies that $\beta(n_s) = 5$ for any $s > 1$. So, in view of Theorem 2.1(II), the iterated Kameko map $((\widetilde{Sq}_*^0)_{(5,n_s)})^{s-1} : Q_{n_s}^{\otimes 5} \longrightarrow Q_{n_1}^{\otimes 5}$ is an isomorphism of $\mathbb{F}_2 G(5)$ -modules for all $s \geq 1$. Thus, it is only necessary to calculate the dimension of $Q_{n_s}^{\otimes 5}$ for $s = 0$ and $s = 1$. The case where $s = 0$ has been explicitly computed in [Phu20b] and its dimension has been found to be 730. For $s = 1$, the Kameko map $(\widetilde{Sq}_*^0)_{(5,n_1)} : Q_{n_1}^{\otimes 5} \longrightarrow Q_{n_0}^{\otimes 5}$ is an epimorphism, which leads to an isomorphism

$$Q_{n_1}^{\otimes 5} \cong \text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)}) \oplus \text{Im}((\widetilde{Sq}_*^0)_{(5,n_1)}) \cong \text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)}) \oplus Q_{n_0}^{\otimes 5}.$$

Moreover, by using the monomorphism $\varphi : \mathcal{P}_5 \longrightarrow \mathcal{P}_5$, $u \longmapsto x_1 x_2 \dots x_5 u^2$, one derives

Corollary 2.2. *We have an isomorphism of $\mathbb{F}_2$ -vector spaces:*

$$Q_{n_0}^{\otimes 5} \cong \left\langle \{[\varphi(u)] \in Q_{n_1}^{\otimes 5} : u \text{ belongs to a minimal set of } \mathcal{A}\text{-generators for } \mathcal{P}_5 \text{ in degree } n_0\} \right\rangle \subset Q_{n_1}^{\otimes 5}.$$

By the above data, to find the dimension of $Q_{n_1}^{\otimes 5}$, we need to compute the dimension of $\text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)})$. So, we arrive at our first main result in the present article. More precisely, the following assertion is true.

Theorem 2.3. *The kernel of the Kameko $(\widetilde{Sq}_*^0)_{(5,n_1)}$ is an $\mathbb{F}_2$ -vector space of dimension 1900.*

As $Q_{n_s}^{\otimes 5} \cong Q_{n_1}^{\otimes 5}$ for all $s \geq 1$ and $\dim Q_{n_1}^{\otimes 5} = 730 + \dim \text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)})$, together with the aid of Theorem 2.3, we establish

Corollary 2.4. *The cohit space $Q_{n_s}^{\otimes 5}$ has dimension 2630 for every positive integer s .*

In [Phu23a, Remark 3.15], we have pointed out that with generic degrees $(m-1)(2^s-1) + k \cdot 2^s$, if there is a non-negative integer u such that $u < s$ and $1 \leq m-3 \leq \mu((m-1)(2^u-1) + k \cdot 2^u) = \alpha((m-1)(2^u-1) + k \cdot 2^u + \mu((m-1)(2^u-1) + k \cdot 2^u)) \leq m-2$, then

$$\dim Q_{(m-1)(2^s-u+m-1-1)+((m-1)(2^u-1)+k \cdot 2^u)2^{s-u+m-1}}^{\otimes m} = (2^m - 1) \dim Q_{(m-1)(2^s-1)+k \cdot 2^s}^{\otimes(m-1)},$$

for every $s \geq u$. Thanks to this, for $m = 6$, $k = 18$ and $u = 1$, we have $(m-1)(2^s-1) + k \cdot 2^s = n_s$ and

$$\beta(n_1) = \beta(41) = 3 = \alpha(41 + \beta(41)) = \alpha(n_1 + \beta(n_1)).$$

Therefrom, with the aid of Corollary 2.4, it may be claimed that

Corollary 2.5. *For each integer $s \geq 1$, $\dim Q_{5(2^{s+4}-1)+n_1 \cdot 2^{s+4}}^{\otimes 6} = (2^6 - 1) \dim Q_{n_s}^{\otimes 5} = 165690$.*

Applying Theorem 2.3, we obtain the following technical theorem, which is the second main result of the paper.

Theorem 2.6. *For each non-negative integer s , the domain of the fifth Singer algebraic transfer $Tr_5^{\mathcal{A}}$ in degree n_s is determined by $(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} = \langle [\zeta_s] \rangle$, where $\zeta_s \in P_{\mathcal{A}} H_{n_s}((K(\mathbb{F}_2, 1))^{\times 5}; \mathbb{F}_2)$ is the following sum:*

$$\begin{aligned} & a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}+2^{s+1}-1)} a_5^{(2^{s+1}+2^s-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}+2^{s+1}+2^s-1)} a_5^{(2^{s+1}-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+3}+2^s-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+3}+2^s-1)} a_5^{(2^{s+1}-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}+2^s-1)} a_4^{(2^{s+3}-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+3}-1)} a_4^{(2^{s+2}+2^s-1)} a_5^{(2^{s+1}-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}+2^s-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+3}+2^{s+1}-1)} a_4^{(2^{s+2}+2^s-1)} a_5^{(2^{s+1}-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}+2^s-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+3}+2^{s+1}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+1}+2^s-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+1}+2^s-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+3}+2^{s+1}-1)} a_4^{(2^{s+1}+2^s-1)} a_5^{(2^{s+1}-1)} \\ & + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+1}+2^s-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+3}+2^{s+1}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} \end{aligned}$$

$$\begin{aligned}
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+3}+2^{s+2}-1)} a_4^{(2^{s+1}+2^s-1)} a_5^{(2^{s+1}-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+2}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+1}+2^s-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+2}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}+2^{s+1}-1)} a_5^{(2^{s+2}+2^s-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}+2^{s+1}-1)} a_5^{(2^{s+2}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+3}+2^{s+2}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+2}-1)} a_3^{(2^{s+2}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}-1)} a_3^{(2^{s+3}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}-1)} a_3^{(2^{s+3}-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+1}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+3}+2^{s+2}+2^{s+1}+2^s-1)} a_3^{(2^{s+1}-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+1}-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}+2^s-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+2}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}+2^{s+1}+2^s-1)} a_5^{(2^{s+2}-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}+2^s-1)} a_4^{(2^{s+2}+2^{s+1}-1)} a_5^{(2^{s+2}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}+2^s-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+2}+2^{s+1}-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+2}+2^{s+1}+2^s-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+2}+2^{s+1}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}+2^{s+1}-1)} a_5^{(2^{s+2}+2^{s+1}+2^s-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+3}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+2}+2^s-1)} a_5^{(2^{s+3}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+1}+2^s-1)} a_5^{(2^{s+3}-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}+2^{s+1}+2^s-1)} a_4^{(2^{s+1}-1)} a_5^{(2^{s+3}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+3}+2^{s+1}-1)} a_5^{(2^{s+1}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+3}+2^{s+1}+2^s-1)} a_5^{(2^{s+1}-1)} + a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}-1)} a_4^{(2^{s+3}-1)} a_5^{(2^{s+1}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}+2^s-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+1}+2^s-1)} \\
& +a_1^{(2^{s+1}-1)} a_2^{(2^{s+2}+2^{s+1}-1)} a_3^{(2^{s+2}+2^{s+1}-1)} a_4^{(2^{s+2}-1)} a_5^{(2^{s+1}+2^s-1)}.
\end{aligned}$$

We shall give here a detailed outline of the proof of the theorem. As $Q_{n_s}^{\otimes 5} \cong Q_{n_1}^{\otimes 5}$ for any $s \geq 1$ and $Q_{n_1}^{\otimes 5} \cong \text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)}) \bigoplus Q_{n_0}^{\otimes 5}$, we have an estimate:

$$\dim(Q_{n_s}^{\otimes 5})^{G(5)} = \dim(Q_{n_1}^{\otimes 5})^{G(5)} \leq \dim(\text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)}))^{G(5)} + \dim(Q_{n_0}^{\otimes 5})^{G(5)}, \text{ for every } s \geq 1.$$

Using a monomial basis of $Q_{n_0}^{\otimes 5}$ in [Phu20b], we explicitly compute the action of $G(5)$ on $Q_{n_0}^{\otimes 5}$ and find that $\dim(Q_{n_0}^{\otimes 5})^{G(5)} = 1$ and $(Q_{n_0}^{\otimes 5})^{G(5)} = \langle [\mathcal{R}'_4] \rangle$, where

$$\begin{aligned}
\mathcal{R}'_4 &= x_1 x_2 x_3 x_4 x_5^{14} + x_1 x_2 x_3 x_4^4 x_5 + x_1 x_2 x_3^4 x_4 x_5 + x_1 x_2^3 x_3 x_4 x_5^{12} \\
&+ x_1 x_2^3 x_3 x_4^{12} x_5 + x_1 x_2^3 x_3^{12} x_4 x_5 + x_1^3 x_2 x_3 x_4 x_5^{12} + x_1^3 x_2 x_3 x_4^{12} x_5 \\
&+ x_1^3 x_2 x_3^{12} x_4 x_5 + x_1^3 x_2^5 x_3 x_4 x_5^8 + x_1^3 x_2^5 x_3 x_4^8 x_5 + x_1^3 x_2^5 x_3^8 x_4 x_5.
\end{aligned}$$

By combining Theorem 2.3 and the method analogous to that used above, we can also obtain that $\dim(\text{Ker}((\widetilde{Sq}_*^0)_{(5,n_1)}))^{G(5)} = 0$. Hence, for each integer $s \geq 1$, one derives:

$$\dim(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} = \dim(Q_{n_s}^{\otimes 5})^{G(5)} \leq 1, \quad (Q_{n_s}^{\otimes 5})^{G(5)} \subseteq \langle [\varphi^s(\mathcal{R}'_4)] \rangle,$$

where φ is the up Kameko map $\mathcal{P}_5 \longrightarrow \mathcal{P}_5$, $u \longmapsto x_1 x_2 \dots x_5 u^2$ and $\varphi^s(\mathcal{R}'_4)$ is the following polynomial:

$$\begin{aligned}
& x_1^{2^{s+1}-1} x_2^{2^{s+1}-1} x_3^{2^{s+1}-1} x_4^{2^{s+1}-1} x_5^{2^{s+3}+2^{s+2}+2^{s+1}+2^s-1} + x_1^{2^{s+1}-1} x_2^{2^{s+1}-1} x_3^{2^{s+1}-1} x_4^{2^{s+3}+2^{s+2}+2^{s+1}+2^s-1} x_5^{2^{s+1}-1} \\
& + x_1^{2^{s+1}-1} x_2^{2^{s+1}-1} x_3^{2^{s+3}+2^{s+2}+2^{s+1}+2^s-1} x_4^{2^{s+1}-1} x_5^{2^{s+1}-1} + x_1^{2^{s+1}-1} x_2^{2^{s+2}-1} x_3^{2^{s+1}-1} x_4^{2^{s+1}-1} x_5^{2^{s+3}+2^{s+2}+2^s-1} \\
& + x_1^{2^{s+1}-1} x_2^{2^{s+2}-1} x_3^{2^{s+1}-1} x_4^{2^{s+3}+2^{s+2}+2^s-1} x_5^{2^{s+1}-1} + x_1^{2^{s+1}-1} x_2^{2^{s+2}-1} x_3^{2^{s+3}+2^{s+2}+2^s-1} x_4^{2^{s+1}-1} x_5^{2^{s+1}-1} \\
& + x_1^{2^{s+2}-1} x_2^{2^{s+1}-1} x_3^{2^{s+1}-1} x_4^{2^{s+1}-1} x_5^{2^{s+3}+2^{s+2}+2^s-1} + x_1^{2^{s+2}-1} x_2^{2^{s+1}-1} x_3^{2^{s+1}-1} x_4^{2^{s+3}+2^{s+2}+2^s-1} x_5^{2^{s+1}-1} \\
& + x_1^{2^{s+2}-1} x_2^{2^{s+1}-1} x_3^{2^{s+3}+2^{s+2}+2^s-1} x_4^{2^{s+1}-1} x_5^{2^{s+1}-1} + x_1^{2^{s+2}-1} x_2^{2^{s+2}+2^{s+1}-1} x_3^{2^{s+1}-1} x_4^{2^{s+1}-1} x_5^{2^{s+3}+2^s-1} \\
& + x_1^{2^{s+2}-1} x_2^{2^{s+2}+2^{s+1}-1} x_3^{2^{s+1}-1} x_4^{2^{s+3}+2^s-1} x_5^{2^{s+1}-1} + x_1^{2^{s+2}-1} x_2^{2^{s+2}+2^{s+1}-1} x_3^{2^{s+3}+2^s-1} x_4^{2^{s+1}-1} x_5^{2^{s+1}-1}.
\end{aligned}$$

Let $\langle -, - \rangle$ denote the dual the canonical non-singular pairing $(\mathcal{P}_5)^* \times \mathcal{P}_5 \longrightarrow \mathbb{F}_2$. By an elementary computation, one sees that $\langle \zeta_s, \varphi^s(\mathcal{R}'_4) \rangle = 1$ and ζ_s are $\mathcal{A}$ -annihilated elements

(in actual fact, by the unstable condition and $(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} \cong (\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_1}$ for all $s \geq 1$, to check that $\zeta_s \in P_{\mathcal{A}}(\mathcal{P}_5)^*$, we only need to consider the actions of the Steenrod squares Sq^1 , Sq^2 and Sq^4 on ζ_0 and ζ_1 . These calculations can be easily done). So, $[\zeta_s]$ is dual to $[\varphi^s(\mathcal{R}'_4)]$ for every s .

By using the chain-level representation of the fifth Singer transfer via the lambda algebra Λ [BCK⁺66], we will show that $\dim(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} \geq 1$ for any s . It should be recalled that the algebra Λ , which is viewed as the E^1 -term of the Adams SS, is a differential bigraded associative algebra, with generators $\lambda_n \in \Lambda^{1,n}$ for $n \geq 0$, and the defining Adem relations given by:

$$(2.1) \quad \lambda_i \lambda_{2i+n+1} = \sum_{j \geq 0} \binom{n-j-1}{j} \lambda_{i+n-j} \lambda_{2i+1+j} \quad (i \geq 0, n \geq 0),$$

and with a differential defined as follows:

$$(2.2) \quad d(\lambda_{n-1}) = \sum_{j \geq 1} \binom{n-j-1}{j} \lambda_{n-j-1} \lambda_{j-1} \quad (n \geq 1), \quad d(\lambda_0) = 0.$$

Let $\Lambda^{m,*}$ be an $\mathbb{F}_2$ -vector subspace of Λ that is spanned by all monomials in λ_i of length m . We consider a homomorphism

$$\delta = \bigoplus_{m \geq 0} \delta_m : \bigoplus_{m \geq 0} (\mathcal{P}_m)^* \longrightarrow \bigoplus_{m \geq 0} \Lambda^{m,*} = \Lambda,$$

as constructed by Chơn and Hà [CH11]. The $\mathbb{F}_2$ -linear map $\delta_m : (\mathcal{P}_m)^* \longrightarrow \Lambda^{m,*}$ is considered as the E^1 -level of the transfer $Tr_m^{\mathcal{A}}$ and is defined by the following inductive formula:

$$\delta_m \left(\prod_{1 \leq k \leq m} a_k^{(j_k)} \right) = \begin{cases} \lambda_{j_1} & \text{if } m = 1, \\ \sum_{i \geq j_m} \delta_{m-1} \left(\prod_{1 \leq k \leq m-1} (a_k^{(j_k)}) Sq^{i-j_m} \right) \lambda_i & \text{if } m > 1, \end{cases}$$

for any $\prod_{1 \leq k \leq m} a_k^{(j_k)} \in (\mathcal{P}_m)^*$. It is also noteworthy to mention that δ_m is not an $\mathcal{A}$ -homomorphism.

The authors [CH11] stated that if an element θ in $(\mathcal{P}_m)^*$ is $\overline{\mathcal{A}}$ -annihilated, then $[\delta_m(\theta)] = Tr_m^{\mathcal{A}}([\theta])$. This result allows us to apply the case where $m = 5$ and the $\overline{\mathcal{A}}$ -annihilated element is $\theta = \zeta_s$. By combining the equalities (2.1) and (2.2), we get

$$\begin{aligned} Tr_5^{\mathcal{A}}([\zeta_s]) &= [\delta_5(\zeta_s)] = [\lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1} \lambda_{2^{s+2}+2^{s+1}-1} \lambda_{2^{s+2}+2^s-1} \\ &\quad + \lambda_{2^{s+1}-1} \lambda_{2^{s+3}-1} \lambda_{2^{s+2}+2^{s+1}-1} \lambda_{2^{s+2}-1} \lambda_{2^{s+1}+2^s-1} + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1}^2 \lambda_{2^{s+3}-1} \lambda_{2^{s+2}+2^s-1} \\ &\quad + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1}^2 \lambda_{2^{s+2}+2^{s+1}-1} \lambda_{2^{s+2}+2^{s+1}+2^s-1} + \lambda_{2^{s+2}-1}^3 \lambda_{2^{s+3}+2^s-1} \\ &\quad + d(\lambda_{2^{s+1}-1} \lambda_{2^{s+2}+2^s-1} \lambda_{2^{s+3}-1}^2 + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1} \lambda_{2^{s+2}+2^{s+1}-1} \lambda_{2^{s+3}+2^{s+1}+2^s-1} \\ &\quad + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1} \lambda_{2^{s+3}+2^{s+2}+2^s-1} \lambda_{2^{s+2}-1} + \lambda_{2^{s+1}-1} \lambda_{2^s-1} \lambda_{2^{s+3}+2^{s+2}-1} \lambda_{2^{s+3}-1})] \\ &= [\lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1} \lambda_{2^{s+2}+2^{s+1}-1}^2 \lambda_{2^{s+2}+2^s-1} + \lambda_{2^{s+1}-1} \lambda_{2^{s+3}-1} \lambda_{2^{s+2}+2^{s+1}-1} \lambda_{2^{s+2}-1} \lambda_{2^{s+1}+2^s-1} \\ &\quad + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1}^2 \lambda_{2^{s+3}-1} \lambda_{2^{s+2}+2^s-1} + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1}^2 \lambda_{2^{s+2}+2^{s+1}-1} \lambda_{2^{s+2}+2^{s+1}+2^s-1} \\ &\quad + \lambda_{2^{s+1}-1} \lambda_{2^{s+2}-1}^3 \lambda_{2^{s+3}+2^s-1}] = h_{s+1} e_s \in \text{Ext}_{\mathcal{A}}^{5,5+n_s}(\mathbb{F}_2, \mathbb{F}_2), \text{ for any } s \geq 0. \end{aligned}$$

Here $[\lambda_{2^{s+1}-1}] = h_{s+1} = (Sq^0)^{s+1}(h_0) \in \text{Ext}_{\mathcal{A}}^{1,2^{s+1}}(\mathbb{F}_2, \mathbb{F}_2)$ and $e_s = (Sq^0)^s(e_0) \in \text{Ext}_{\mathcal{A}}^{4,21 \cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$ for all s , where Sq^0 is the classical squaring operation. Thus every element in the Sq^0 -family $\{h_{s+1}e_s : s \geq 0\}$ belongs to the image of $Tr_5^{\mathcal{A}}$. This together with the facts that $\text{Ext}_{\mathcal{A}}^{5,5+n_0}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2 h_1 e_0 = \mathbb{F}_2 h_0 f_0$ and $\text{Ext}_{\mathcal{A}}^{5,5+n_s}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2 h_{s+1} e_s = \mathbb{F}_2 h_s f_s = \mathbb{F}_2 h_{s-1}^2 c_{s+1}$ for any $s > 0$ (cf. [Lin08, Che11]) show that $\dim(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} \geq 1$, for every $s \geq 0$. By what we have already shown, it may be concluded that the $G(5)$ -coinvariant $(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s}$ is one-dimensional and $(\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} = \mathbb{F}_2[\zeta_s]$ for all s . The theorem follows.

Thus, by the above setting, in order to successfully establish Theorem 2.6, it is only necessary to demonstrate the following key outcomes:

$$\begin{cases} (Q_{n_0}^{\otimes 5})^{G(5)} = \langle [\mathcal{R}'_4] \rangle; \\ (\text{Ker}((\widehat{Sq}_*^0)_{(5, n_1)}))^{G(5)} = 0. \end{cases}$$

The following comment is closely related to the above strategy.

Remark 2.7. As alluded to earlier, all decomposable elements $h_{s+1}e_s$ ($s \geq 0$) are detected by the transfer map $Tr_5^{\mathcal{A}}$. This conclusion can also be drawn from the following observations: firstly, every element in the Sq^0 -families $\{h_s \in \text{Ext}_{\mathcal{A}}^{1,2^s}(\mathbb{F}_2, \mathbb{F}_2)\}_{s \geq 0}$ and $\{e_s \in \text{Ext}_{\mathcal{A}}^{4,21 \cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)\}_{s \geq 0}$ belongs to the image of the transfers $Tr_1^{\mathcal{A}}$ and $Tr_4^{\mathcal{A}}$, respectively, as demonstrated in [Sin89] and [Ha07]. Additionally, it is important to consider the previous work of the author, which provides further insights on the indecomposable elements e_s [Phu23b, Remark 2.1.6]. Secondly, the "total" transfer $\bigoplus_{m \geq 0} Tr_m^{\mathcal{A}}$ is an algebra homomorphism. However, the aforementioned technique is highly valuable and significant, as it offers us with a means of explicitly determining the generators of the domain of $Tr_5^{\mathcal{A}}$. It is crucial to emphasize that under specific circumstances, this technique can also be extended to higher homological degrees. (We refer the reader to [Phu22d] for a few various results of such a technique). Accordingly, a more thorough investigation of this technique is also planned.

As an immediate consequence of Theorem 2.6, it can be easily seen that

Corollary 2.8. *The fifth algebraic transfer*

$$Tr_5^{\mathcal{A}} : (\mathbb{F}_2 \otimes_{G(5)} P_{\mathcal{A}}(\mathcal{P}_5)^*)_{n_s} \longrightarrow \text{Ext}_{\mathcal{A}}^{5,5+n_s}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism for arbitrary $s \geq 0$. Consequently, Singer's Conjecture 1.1 holds for bidegrees $(5, 5 + n_s)$ with $s \geq 0$.

We end our work by confirming Conjecture 1.1 for bidegrees $(m, m + 12)$. What may not be immediately obvious is the following statement.

Theorem 2.9. *For each rank $m > 0$, the algebraic transfer*

$$Tr_m^{\mathcal{A}} : (\mathbb{F}_2 \otimes_{G(m)} P_{\mathcal{A}}(\mathcal{P}_m)^*)_{12} \longrightarrow \text{Ext}_{\mathcal{A}}^{m,m+12}(\mathbb{F}_2, \mathbb{F}_2)$$

is a trivial isomorphism. As a result, Conjecture 1.1 holds true in bidegree $(m, m + 12)$ for arbitrary $m > 0$.

The theorem has been proven by Singer [Sin89] for $m = 1, 2$, by Boardman [Boa93] for $m = 3$, by Sum [Sum18] for $m = 5$, and by the present writer [Phu22d] for $4 \leq m \leq 8, m \neq 5$.

Some practical notes. This paper utilized the **MAGMA** computer algebra system [BCP97], in conjunction with other software tools, to execute multiple calculations. Moreover, the complexity of certain calculations in this work necessitated methods distinct from those utilized in previous studies. (It should be brought to the attention of the readers that for higher variables, only some generic degrees are known and the general hit problem remains unsolved. With the increase of variables and the limitations of current tools, it becomes challenging to calculate the dimension; finding the calculation algorithm for this work is crucial. That is why we became interested in programming and exploring easier ways to find a solution to the general problem. On another note, inspired by some noteworthy findings in the literature [Sil98, WW18a, WW18b], Sum [Sum23] has recently proposed an alternative approach to determine the dimension of the kernel of the Kameko $(\widetilde{Sq}_*^0)_{(m,n)}$ in a specific type of generic degrees n . Honestly, his technique is very complicated, and not everyone will be able to grasp it. However, it offers us a more multi-dimensional viewpoint and the prospect of discovering a comprehensive solution to the hit problem.) To sum up, in light of the outcomes achieved in this research, as well as recent works [Phu22b, Phu23a, Sum23], we are of the opinion that further exploration of these captivating topics could potentially yield useful tools for addressing classical problems in algebraic topology.

The remainder of this manuscript is organized in the following way: Sect.3 covers some of the previously established preliminary results. In Sect.4 we utilise the tools assembled in Sect.3 to demonstrate our primary results. In Sect.5, we compile a comprehensive list of all the allowable monomials of degree $n_0 = 5(2^0 - 1) + 18 \cdot 2^0$, and also enumerate the strictly nonallowable monomials of degree $n_1 = 5(2^1 - 1) + 18 \cdot 2^1$ contained within the $\mathcal{A}$ -module $\mathcal{P}_5$. Additionally, we also show in the Appendix that a local version of Kameko's conjecture as informed in [WW18b] holds true for any $m \geq 1$ and parameter vectors of degrees ≤ 12 .

3. The required preparations

This preliminary section starts with a brief overview of the Steenrod algebra over $\mathbb{F}_2$ and concludes with a concise summary of some well-known homomorphisms in [PS15, Sum15]. The majority of the results presented in this section serve as basic building blocks that will be utilized throughout the draft. In fact, the material presented here is not novel, so readers familiar with these subjects may opt to quickly review the section for notation or proceed directly to succeeding portions.

3.1. The mod 2 Steenrod algebra and its related examples

The (mod 2) Steenrod algebra $\mathcal{A}$ is recognized as the algebra of stable cohomology operations from $\mathbb{F}_2$ cohomology to itself, generated by the Steenrod squares $Sq^i : H^\bullet(\mathcal{U}; \mathbb{F}_2) \longrightarrow H^{\bullet+i}(\mathcal{U}; \mathbb{F}_2)$ in grading $i \geq 0$, subject to the Adem relations and the condition $Sq^0 = 1$ (for more information, refer to [SE62]). Here $H^\bullet(\mathcal{U}; \mathbb{F}_2)$ denotes the $\mathbb{F}_2$ -singular cohomology group of a topological space $\mathcal{U}$. One can define the cup product $\smile$, which takes the form $H^n(\mathcal{U}; \mathbb{F}_2) \times H^i(\mathcal{U}; \mathbb{F}_2) \longrightarrow H^{n+i}(\mathcal{U}; \mathbb{F}_2)$, $(u, v) \longmapsto u \smile v$. This cup product also gives a multiplication on the graded cohomology ring $H^*(\mathcal{U}; \mathbb{F}_2) = \bigoplus_{n \geq 0} H^n(\mathcal{U}; \mathbb{F}_2)$. It is worth noticing that the structure of $H^*(\mathcal{U}; \mathbb{F}_2)$ is not only as a graded commutative $\mathbb{F}_2$ -algebra, but also as an $\mathcal{A}$ -module. The $\mathcal{A}$ -module structure on $H^*(\mathcal{U}; \mathbb{F}_2)$ is often found to yield useful insights into the nature of $\mathcal{U}$ in many cases. As evidenced by the example below.

Example 3.1.1. *The spaces $\mathbb{C}P^4/\mathbb{C}P^2$ and $\mathbb{S}^6 \vee \mathbb{S}^8$ have cohomology rings that agree as graded commutative $\mathbb{F}_2$ -algebras, but are different as modules over $\mathcal{A}$. So, $\mathbb{C}P^4/\mathbb{C}P^2 \neq \mathbb{S}^6 \vee \mathbb{S}^8$, that is, these spaces are not homotopy equivalent.*

As there is no available reference for a proof of this event, it would be advantageous for the paper to be self-contained by presenting a relatively straightforward proof of it. Indeed, by utilizing the tools of Mayer-Vietoris sequence and Universal Coefficient Theorem, we can derive the result that

$$H^k(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2) \cong H^k(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } k = 0, 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

It is pertinent to mention that this conclusion holds true even when $\mathbb{F}_2$ is substituted by any ring.

* More generally, for any coefficient ring $\mathcal{R}$, we obtain

$$H^k(\mathbb{C}P^n/\mathbb{C}P^{n-2}; \mathcal{R}) \cong H^k(\mathbb{S}^{2n-2} \vee \mathbb{S}^{2n}; \mathcal{R}) = \begin{cases} \mathcal{R} & \text{if } k = 0, 2n-2, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

The inclusion map $\mathbb{C}P^2 \hookrightarrow \mathbb{C}P^4$ induces a restriction map $H^\bullet(\mathbb{C}P^4; \mathbb{F}_2) \longrightarrow H^\bullet(\mathbb{C}P^2; \mathbb{F}_2)$. Then, in any even degree, we have a short exact sequence

$$0 \longrightarrow H^{2m}(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2) \longrightarrow H^{2m}(\mathbb{C}P^4; \mathbb{F}_2) \longrightarrow H^{2m}(\mathbb{C}P^2; \mathbb{F}_2) \longrightarrow 0.$$

On the other hand, as it is known, $H^*(\mathbb{C}P^4; \mathbb{F}_2) = \bigoplus_{k \geq 0} H^k(\mathbb{C}P^4; \mathbb{F}_2) \cong \mathbb{F}_2[x]/\langle x^5 \rangle$, $x \in H^2(\mathbb{C}P^4; \mathbb{F}_2)$, can be calculated using the Gysin sequence of the fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^9 \longrightarrow \mathbb{C}P^4$. Moreover,

$$H^k(\mathbb{C}P^4; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } k = 0, 2, 4, 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

* More generally, for any coefficient ring $\mathcal{R}$, we have $H^*(\mathbb{C}P^n; \mathcal{R}) \cong \mathcal{R}[x]/\langle x^{n+1} \rangle$, $x \in H^2(\mathbb{C}P^n; \mathcal{R})$, and

$$H^k(\mathbb{C}P^n; \mathcal{R}) = \begin{cases} \mathcal{R} & \text{if } 0 \leq k \leq 2n, \text{ and } k \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence by the above short exact sequence, one has the isomorphisms: $H^6(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2) \cong H^6(\mathbb{C}P^4; \mathbb{F}_2)$ and $H^8(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2) \cong H^8(\mathbb{C}P^4; \mathbb{F}_2)$. Thus, under these isomorphisms in combination with the fact that x is the generator of $H^2(\mathbb{C}P^4; \mathbb{F}_2)$, it follows that x^3 and x^4 are sent to the non-zero

elements of $H^6(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2)$ and $H^8(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2)$, respectively. Owing to $x \in H^2(\mathbb{C}P^4; \mathbb{F}_2)$, the Cartan formula yields the following expression:

$$\begin{aligned} Sq^2(x^3) &= Sq^0(x^2) \smile Sq^2(x) + Sq^1(x^2) \smile Sq^1(x) + Sq^0(x) \smile Sq^2(x^2) \\ &= Sq^0(x^2) \smile Sq^2(x) + Sq^1(x^2) \smile Sq^1(x) + Sq^0(x) \smile (Sq^1(x))^2 \\ &= x^2 \smile x^2 = x^4 \neq 0 \quad (\text{since } Sq^1(x) \in H^3(\mathbb{C}P^4; \mathbb{F}_2) = 0). \end{aligned}$$

* More generally, given the total Steenrod square $Sq = \sum_{r \geq 0} Sq^r$ as an endomorphism of $H^*(\mathbb{C}P^n; \mathbb{F}_2)$

and the fact that $Sq(x^k) = (x + x^2)^k$, $k \geq 1$, we get $Sq^{2r}(x^k) = \binom{k}{r} x^{k+r}$, where $x \in H^2(\mathbb{C}P^n; \mathbb{F}_2)$,

for any $n \geq 2$, and $\binom{k}{r} \pmod{2}$. In particular, when $n = 1$, then $\mathbb{C}P^1 \cong \mathbb{S}^2$, i.e., $\mathbb{C}P^1$ is homeomorphic to $\mathbb{S}^2$; and so $Sq^2(x) = 0$. (Note that $\mathbb{S}^2 \cong \tilde{\mathbb{C}}$, where $\tilde{\mathbb{C}}$ is the one-point compactification of $\mathbb{C}$. Further, since the mapping $\mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \tilde{\mathbb{C}}$, $(u, v) \mapsto u/v$, is continuous, it induces a continuous bijection $\mathbb{C}P^1 \rightarrow \tilde{\mathbb{C}}$. As $\mathbb{C}P^1$ is compact and $\tilde{\mathbb{C}}$ is Hausdorff, $\mathbb{C}P^1 \cong \tilde{\mathbb{C}} \cong \mathbb{S}^2$.) When $n > 1$, then $Sq^2(x) = x^2$ and $Sq^2(x^k) = \binom{k}{1} x^{k+1}$ if $1 \leq k \leq n-1$, and $Sq^2(x^k) = 0$ if $k = n$ (since $H^{2n+2}(\mathbb{C}P^n; \mathbb{F}_2) = 0$). Therefrom, by pulling back via the quotient map, if a generates $H^{2n-2}(\mathbb{C}P^n/\mathbb{C}P^{n-2}; \mathbb{F}_2)$ and b generates $H^{2n}(\mathbb{C}P^n/\mathbb{C}P^{n-2}; \mathbb{F}_2)$, then

$$Sq^2(a) = (n-1)b = \begin{cases} b & \text{if } n \equiv 0 \pmod{2}, \\ 0 & \text{if } n \equiv 1 \pmod{2}, \end{cases}$$

for any $n \geq 3$.

Thus, we must have a non-trivial homomorphism $Sq^2 : H^6(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2) \rightarrow H^8(\mathbb{C}P^4/\mathbb{C}P^2; \mathbb{F}_2)$.

However, the homomorphism $Sq^2 : H^6(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) \rightarrow H^8(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2)$ is trivial, since the following commutative diagram:

$$\begin{array}{ccc} H^6(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) & \xrightarrow{Sq^2} & H^8(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) \\ i^* \downarrow \cong & & \downarrow i^* \\ H^6(\mathbb{S}^6; \mathbb{F}_2) = \mathbb{F}_2 & \xrightarrow{Sq^2} & H^8(\mathbb{S}^6; \mathbb{F}_2) = 0. \end{array}$$

Here, $i : \mathbb{S}^6 \hookrightarrow \mathbb{S}^6 \vee \mathbb{S}^8$ is the inclusion map, and $i^* : H^\bullet(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) \rightarrow H^\bullet(\mathbb{S}^6; \mathbb{F}_2)$ is the induced map on cohomology groups. (The readers should also take note that, by the Kunnetth formula, one has the isomorphisms: $H^6(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) \cong H^6(\mathbb{S}^6; \mathbb{F}_2) \oplus H^6(\mathbb{S}^8; \mathbb{F}_2) \cong H^6(\mathbb{S}^6; \mathbb{F}_2) = \mathbb{F}_2$ and $H^8(\mathbb{S}^6 \vee \mathbb{S}^8; \mathbb{F}_2) \cong H^8(\mathbb{S}^6; \mathbb{F}_2) \oplus H^8(\mathbb{S}^8; \mathbb{F}_2) \cong H^8(\mathbb{S}^8; \mathbb{F}_2) = \mathbb{F}_2$.) These facts lead to the spaces $\mathbb{C}P^4/\mathbb{C}P^2$ and $\mathbb{S}^6 \vee \mathbb{S}^8$ being different as modules over $\mathcal{A}$.

Remark. The above example can be generalized as follows: Let $\mathbb{C}P^n$ denotes the complex projective space with real manifold dimension $2n$. Then, $\mathbb{C}P^n/\mathbb{C}P^{n-2} \not\cong \mathbb{S}^{2n-2} \vee \mathbb{S}^{2n}$ if n even, while $\mathbb{C}P^n/\mathbb{C}P^{n-2} \simeq \mathbb{S}^{2n-2} \vee \mathbb{S}^{2n}$ if n odd.

Another noteworthy aspect concerning Steenrod squares is their stability property, which, when combined with the Freudenthal suspension theorem [Fre38], allows us to establish the following well-known result.

Example 3.1.2 (see also [Phu23d]). The homotopy groups $\pi_{k+1}(\mathbb{S}^k)$ are non-trivial for $k \geq 2$.

3.2. Some fundamental information about the classical hit problem

In this subsection, we hereby present fundamental facts concerning the classical hit problem for the Steenrod algebra, which has an extensive historical background. We also review some essential results whose proofs are widely available in the existing literature. (In addition, we would like to direct the reader to the monographs written by Walker and Wood [WW18a, WW18b] for a considerable amount of information on this subject.) Recall that the graded polynomial algebra

$\mathcal{P}_m = \mathbb{F}_2[x_1, \dots, x_m] = \bigoplus_{n \geq 0} (\mathcal{P}_m)_n$, which realizes the (mod 2) cohomology of the product of m copies of the Eilenberg-MacLan complex $K(\mathbb{F}_2, 1) = \mathbb{R}P^\infty = \varinjlim \mathbb{R}P^n$ (infinite-dimensional real projective space), viewed as an $\mathcal{A}[G(m)]$ -module. The grading is by the homogeneous components $(\mathcal{P}_m)_n = H^n((\mathbb{R}P^\infty)^{\times m}; \mathbb{F}_2)$ of degree n in the m variables $x_1, \dots, x_m$ of grading 1. We also note that $\prod_{1 \leq i \leq m} K(\mathbb{F}_2, 1) = B\mathbb{F}_2^m$, the classifying space of the elementary Abelian group $\mathbb{F}_2^m$ of rank m .

Dual to the hit problem is the " $\overline{\mathcal{A}}$ -annihilated problem", which involves finding a basis for the space of $\overline{\mathcal{A}}$ -annihilateds, $P_{\overline{\mathcal{A}}}(\mathcal{P}_m)^* = \left\{ u \in (\mathcal{P}_m)^* \mid (u)Sq^k = 0, \forall k \geq 1 \right\}$. Studying the hit problem and its dual is important because they lead to an understanding of the Singer transfer $Tr_m^{\mathcal{A}}$, as mentioned in Sect. 1. This transfer is one of the tools that helps us gain insight into the stable homotopy groups of spheres: $E_2^{*,*} = \text{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*^S / \text{odd torsion}$.

• **Parameter and exponent vector.** If $n \geq 0$ is an integer, we may represent it by its binary expansion $n = \sum_{t \geq 0} \alpha_t(n) 2^t$. For a monomial $X \in \mathcal{P}_m$, that is, $X = x_1^{u_1} x_2^{u_2} \dots x_m^{u_m}$, we define

$\text{Param}_i(X)$ to be an integer $\sum_{1 \leq j \leq m} \alpha_{i-1}(u_j)$, wherein $i \geq 1$. We also associate two sequences with

X : $u(X) = (u_1, u_2, \dots, u_m)$ and $\text{Param}(X) = (\text{Param}_1(X), \dots, \text{Param}_i(X), \dots)$. These sequences are referred to as the *exponent vector* and the *parameter vector* of X , respectively. Let $\text{Param} = (\text{Param}_1, \dots, \text{Param}_i, \dots)$ be a sequence of non-negative integers. This sequence is called a *parameter vector* if $\text{Param}_i = 0$ for $i \gg 0$. One also defines $\deg(\text{Param}) = \sum_{i \geq 1} 2^{i-1} \text{Param}_i$. We hereby establish

the convention that *the sets of all the parameter vectors and the exponent vectors are given the left lexicographical order*.

• **Linear order on $\mathcal{P}_m$.** Let us consider the monomials $X = x_1^{u_1} x_2^{u_2} \dots x_m^{u_m}$ and $Y = x_1^{v_1} x_2^{v_2} \dots x_m^{v_m}$ in $\mathcal{P}_m$ that have the same degree. We denote by $u(X)$ and $v(Y)$ the exponent vectors of X and Y , respectively. We define $u < v$ if there exists a positive integer d such that $u_j = v_j$ for all $j < d$ and $u_d < v_d$. We say that $X < Y$ if and only if either $\text{Param}(X) < \text{Param}(Y)$ or $\text{Param}(X) = \text{Param}(Y)$ and $u(X) < v(Y)$.

• **Binary relations on $\mathcal{P}_m$.** Let Param be a parameter vector. We define two subspaces of $\mathcal{P}_m$ associated with Param as follows: $\mathcal{P}_m^{\leq \text{Param}} = \{ \{X \in \mathcal{P}_m \mid \deg(X) = \deg(\text{Param}), \text{Param}(X) \leq \text{Param}\} \}$, and $\mathcal{P}_m^{< \text{Param}} = \{ \{X \in \mathcal{P}_m \mid \deg(X) = \deg(\text{Param}), \text{Param}(X) < \text{Param}\} \}$. Given homogeneous polynomials F and G in $\mathcal{P}_m$, $\deg(F) = \deg(G)$, one defines the following binary relations " $\sim$ " and " $\sim_{\text{Param}}$ " on $\mathcal{P}_m$:

- (i) $F \sim G$ if and only if $(F - G) \in \overline{\mathcal{A}}\mathcal{P}_m$. If $F \sim 0$ then F is "hit", that is, F is in the image of the action $\overline{\mathcal{A}} \otimes_{\mathbb{F}_2} \mathcal{P}_m \longrightarrow \mathcal{P}_m$;
- (ii) $F \sim_{\text{Param}} G$ if and only if $F, G \in \mathcal{P}_m^{\leq \text{Param}}$ and $(F - G) \in (\overline{\mathcal{A}}\mathcal{P}_m + \mathcal{P}_m^{< \text{Param}})$.

It is manifest that the binary relations " $\sim$ " and " $\sim_{\text{Param}}$ " fulfill the criteria for an equivalence relation. Let $(Q^{\otimes m})^{\text{Param}}$ denote the factor space of $\mathcal{P}_m$ by the equivalence relation " $\sim_{\text{Param}}$ ". According to the references [WW18a, Sum21, Phu20a], it follows that $(Q^{\otimes m})^{\text{Param}}$ is also a $G(m)$ -module, and $Q_n^{\otimes m}$ is isomorphic to $\bigoplus_{\deg(\text{Param})=n} (Q^{\otimes m})^{\text{Param}}$.

In [Kam90], Kameko put forth a conjecture stating that the cardinality of a minimal set of generators for the $\mathcal{A}$ -module $\mathcal{P}_m$ is dominated by an explicit quantity depending on the number of the polynomial algebra's variables m . By way of equivalence, Kameko's conjecture implies that $\dim Q_n^{\otimes m} \leq \prod_{1 \leq j \leq m} (2^j - 1)$ for all n . While the statement is valid for $m \leq 4$, it does not hold in general, as there are illustrative counterexamples (see [Sum15, WW18b]). Nonetheless, the local version of this conjecture is still unresolved and can be formulated as follows.

Conjecture 3.2.1 (cf. [WW18b]). *For each parameter vector Param of degree n , we have*

$$\dim(Q^{\otimes m})^{\text{Param}} \leq \prod_{1 \leq j \leq m} (2^j - 1).$$

Referring to the works [Pet87, Kam90, Sum15], it is known that the conjecture holds for $m \leq 4$. Drawing on our prior research [Phu20b] and the proof of Theorem 2.3, we have established that the conjecture is true for $m = 5$ and the generic degree $n = 5(2^s - 1) + 18 \cdot 2^s$, as elaborated in the concluding remarks of Subsect. 4.1.

- **(Non)allowable monomial.** A monomial $X \in \mathcal{P}_m$ is said to be *nonallowable* if there exist monomials $Y_1, Y_2, \dots, Y_k$ such that $Y_j < X$ for $1 \leq j \leq k$ and $X \sim \sum_{1 \leq j \leq k} Y_j$. In particular, if

$X \sim_{\text{Param}(X)} \sum_{1 \leq j \leq k} Y_j$, then we say that X is *Param(X)-nonallowable*. Therefrom X is said to be *allowable* when it is not nonallowable.

Given any non-negative integer r , let $\mathcal{A}_r = \{\{Sq^i : 0 \leq i \leq 2^r\}\}$ denote a sub-Hopf algebra of $\mathcal{A}$. We put $\overline{\mathcal{A}}_r = \mathcal{A} \cap \mathcal{A}_r$. Given polynomials F and G in $\mathcal{P}_m$, where $\deg(F) = \deg(G)$, let Param be a parameter vector such that $\deg(\text{Param}) = \deg(F) = \deg(G)$. We say that $F \sim_{(r, \text{Param})} G$ if and only if $(F - G) \in \overline{\mathcal{A}}_r \mathcal{P}_m + \mathcal{P}_m^{<\text{Param}}$. It is also straightforward to check that $\sim_{(r, \text{Param})}$ is an equivalence relation on $\mathcal{P}_m$.

- **Strictly nonallowable monomial.** A monomial $X \in \mathcal{P}_m$ is said to be *strictly nonallowable* if and only if there exist monomials $Y_1, Y_2, \dots, Y_k$ such that $Y_j < X$ for $1 \leq j \leq k$ and

$$X \sim_{(r-1, \text{Param}(X))} Y_1 + Y_2 + \dots + Y_k, \text{ where } r = \max\{i \in \mathbb{Z} : \text{Param}_i(X) > 0\}.$$

Thus the set of all the allowable monomials of degree n in $\mathcal{P}_m$ is a *minimal set of $\mathcal{A}$ -generators* for $\mathcal{P}_m$ in degree n . Hereafter, we write $\mathcal{C}_n^{\otimes m}$ for the set of all allowable monomials of degree n in the $\mathcal{A}$ -module $\mathcal{P}_m$.

Theorem 3.2.2 (cf. [Kam90]). *Let X be a monomial in $\mathcal{P}_m$. We consider a monomial Z , which assigns to a $s \times m$ -matrix $(\epsilon_{ij}(Z))$ such that for some non-negative integer r , $\epsilon_{ij}(Z) = \epsilon_{(i+r)j}(X)$ for $1 \leq i \leq s$ and $1 \leq j \leq m$. If Z is nonallowable, then so is, X .*

Theorem 3.2.3 (cf. [Sum15]). *Let X, Y and Z be monomials in $\mathcal{P}_m$ such that $\text{Param}_i(Z) = 0$ for $i > t \geq 1$. If Z is strictly nonallowable, then so is, ZY^{2^t} .*

Definition 3.2.4. A monomial $Z = \prod_{1 \leq j \leq m} x_j^{u_j}$ in $\mathcal{P}_m$ is called a *spike* if the powers u_j can be written as $2^{v_j} - 1$ for $v_j \in \mathbb{Z}$, $1 \leq j \leq m$. If Z is a spike with $u_1 > u_2 > \dots > u_{s-1} \geq u_s \geq 1$ and $u_j = 0$ for $j \geq s + 1$, then it is called a *minimal spike*.

It is well-established that spikes cannot appear in the image of any Steenrod square, making them an inseparable part of any generating set of the $\mathcal{A}$ -module $\mathcal{P}_m$. In addition, a spike of a certain positive degree is the minimal spike if its parameter vector order is minimal with respect to other spikes of that positive degree (cf. [PS15]). Particularly, the following key theorem regarding spikes will play a significant role in identifying hit monomials.

Theorem 3.2.5 (cf. [Sin91]). *Suppose that $X \in \mathcal{P}_m$ is a monomial of degree n , where $\beta(n) \leq m$. Let Z be the minimal spike of degree n in $\mathcal{P}_m$. If $\text{Param}(X) < \text{Param}(Z)$, then X is a hit monomial.*

In general, the converse of this theorem does not hold. As an example, consider $Z = x_1^{31} x_2^3 x_3^3 x_4^0 x_5^0 \in (\mathcal{P}_5)_{37}$ and $X = x_1(x_2 x_3 x_4 x_5)^9 \in (\mathcal{P}_5)_{37}$. It is evident that X can be expressed as $f g^{2^3}$, where $f = x_1 x_2 x_3 x_4 x_5$ and $g = x_2 x_3 x_4 x_5$. It follows that $\deg(f) = 5 < (2^3 - 1)\beta(\deg(g))$, and thus, as per Silverman [Sil98, Theorem 1.2], X is a hit monomial. Despite the fact that Z is the minimal spike, it can be observed that $\text{Param}(X) = (5, 0, 0, 4, 0) > \text{Param}(Z) = (3, 3, 1, 1, 1)$. For further details about a basis of $Q_{37}^{\otimes 5}$, we refer the reader to [Phu22c].

3.3. A review of some well-known homomorphisms

In this subsection, we review some useful homomorphisms that will appear a number of times in the proofs of our main results.

First, we consider the following homomorphism: for each $1 \leq l \leq m$, the $\mathcal{A}$ -homomorphism $q_{(l, m)} : \mathcal{P}_{m-1} \longrightarrow \mathcal{P}_m$ is described as follows:

$$q_{(l, m)}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j \leq l-1, \\ x_{j+1} & \text{if } l \leq j \leq m-1. \end{cases}$$

In connection with this $\mathbf{q}_{(l,m)}$, one has a result due to Moetele and Mothebe [MU15], which is rather interesting and very helpful for the sequel.

Theorem 3.3.1. *Let l and d be two positive integers with $1 \leq l \leq m$. If $X \in \mathcal{C}_n^{\otimes(m-1)}$, then $x_l^{2^d-1} \mathbf{q}_{(l,m)}(X) \in \mathcal{C}_{n+2^d-1}^{\otimes m}$.*

Next, we set

$$\mathcal{N}_m := \{(l, \mathcal{L}) \mid \mathcal{L} = (l_1, l_2, \dots, l_r), 1 \leq l < l_1 < l_2 < \dots < l_r \leq m, 0 \leq r \leq m-1\},$$

where by convention, $\mathcal{L} = \emptyset$, if $r = 0$. Denote by $r = \ell(\mathcal{L})$ the length of $\mathcal{L}$. For each $(l, \mathcal{L}) \in \mathcal{N}_m$, one defines the homomorphism $\mathbf{p}_{(l, \mathcal{L})} : \mathcal{P}_m \longrightarrow \mathcal{P}_{m-1}$, which is determined by

$$\mathbf{p}_{(l, \mathcal{L})}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j \leq l-1, \\ \sum_{p \in \mathcal{L}} x_{p-1} & \text{if } j = l, \\ x_{j-1} & \text{if } l+1 \leq j \leq m. \end{cases}$$

It is worthwhile noticing that $\mathbf{q}_{(l,m)}$ and $\mathbf{p}_{(l, \mathcal{L})}$ are also the homomorphisms of $\mathcal{A}$ -modules. In particular, it is easily observed that $\mathbf{p}_{(l, \emptyset)}(x_l) = 0$ for $1 \leq l \leq m$ and $\mathbf{p}_{(l, \mathcal{L})}(\mathbf{q}_{(l,m)}(X)) = X$ for any $X \in \mathcal{P}_{m-1}$.

Now, let $(l, \mathcal{L}) \in \mathcal{N}_m$, $1 \leq r \leq m-1$, and let

$$X_{(\mathcal{L}, u)} = x_{l_u}^{2^{r-1}+2^{r-2}+\dots+2^{r-u}} \prod_{u < d \leq r} x_{l_d}^{2^{r-d}} \text{ for } 1 \leq u \leq r, \quad X_{(\emptyset, 1)} = 1.$$

We consider the following $\mathbb{F}_2$ -linear map, due to [Sum15]:

$$\psi_{(l, \mathcal{L})} : \mathcal{P}_{m-1} \longrightarrow \mathcal{P}_m$$

$$\prod_{1 \leq s \leq m-1} x_s^{t_s} \longmapsto \begin{cases} \frac{x_l^{2^r-1} \mathbf{q}_{(l,m)}(\prod_{1 \leq s \leq m-1} x_s^{t_s})}{X_{(\mathcal{L}, u)}} & \text{if there exist } u \text{ such that:} \\ & t_{l_{1-1}} + 1 = \dots = t_{l_{(u-1)-1}} + 1 = 2^r, \\ & t_{l_u-1} + 1 > 2^r, \\ & \alpha_{r-d}(t_{l_u-1}) - 1 = 0, \quad 1 \leq d \leq u, \\ & \alpha_{r-d}(t_{l_d-1}) - 1 = 0, \quad u+1 \leq d \leq r, \\ & \text{otherwise.} \\ 0 \end{cases}$$

It should be emphasized that this $\psi_{(l, \mathcal{L})}$ is not a homomorphism of $\mathcal{A}$ -modules in general. For instance, with $m = 4$, $\mathcal{L} = (2, 3, 4) \neq \emptyset$, and the monomial $x_1^{12} x_2^6 x_3^9 \in (\mathcal{P}_3)_{27}$, we have

$$\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9) = \frac{x_1^{2^3-1} \mathbf{q}_{(1,4)}(x_1^{12} x_2^6 x_3^9)}{X_{(\mathcal{L}, 1)}} = \frac{x_1^7 x_2^{12} x_3^6 x_4^9}{x_2^4 x_3^2 x_4} = x_1^7 x_2^8 x_3^4 x_4^8 \in (\mathcal{P}_4)_{27}$$

and $\text{Param}(x_1^{12} x_2^6 x_3^9) = (1, 1, 2, 2) = \text{Param}(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9))$. So, a straightforward calculation yields that

$$Sq^2(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9)) = x_1^9 x_2^8 x_3^4 x_4^8 \neq x_1^7 x_2^8 x_3^6 x_4^8 = \psi_{(1, \mathcal{L})}(Sq^2(x_1^{12} x_2^6 x_3^9)).$$

Moreover, we demonstrated in [PS15] the following technical finding.

Theorem 3.3.2. *Let X be a monomial in $\mathcal{P}_m$. Then, $\mathbf{p}_{(l, \mathcal{L})}(X) \in \mathcal{P}_{m-1}^{\leq \text{Param}(X)}$.*

It follows from this observation that when Param is a parameter vector, the homomorphism induced by $\mathbf{p}_{(l, \mathcal{L})}$ from $(Q^{\otimes m})^{\text{Param}}$ to $(Q^{\otimes(m-1)})^{\text{Param}}$ can serve as effective tools for establishing the linear independence of certain subsets of $(Q^{\otimes m})^{\text{Param}}$.

We end this subsection with a few pivotal rules for our work proofs in the next sections. Let $\mathcal{P}_m^0$ and $\mathcal{P}_m^{>0}$ denote the $\mathcal{A}$ -submodules of $\mathcal{P}_m$ spanned by all the monomials $x_1^{t_1} x_2^{t_2} \dots x_m^{t_m}$ such that $\prod_{1 \leq j \leq m} t_j = 0$ and $\prod_{1 \leq j \leq m} t_j > 0$, respectively. By setting $(Q^{\otimes m})^0 := \mathcal{P}_m^0 / \mathcal{A} \mathcal{P}_m^0$, and $(Q^{\otimes m})^{>0} :=$

$\mathcal{P}_m^{>0}/\overline{\mathcal{A}}\mathcal{P}_m^{>0}$, one has an isomorphism: $Q^{\otimes m} \cong (Q^{\otimes m})^0 \bigoplus (Q^{\otimes m})^{>0}$. For a subset $\mathcal{V} \subset \mathcal{P}_{m-1}$, we put

$$\widetilde{\Phi}^0(\mathcal{V}) = \bigcup_{1 \leq l \leq m} \psi_{(l, \emptyset)}(\mathcal{V}) = \bigcup_{1 \leq l \leq m} \mathbf{q}_{(l, m)}(\mathcal{V}), \quad \widetilde{\Phi}^{>0}(\mathcal{V}) = \bigcup_{(l, \mathcal{L}) \in \mathcal{N}_m, 1 \leq \ell(\mathcal{L}) \leq m-1} (\psi_{(l, \mathcal{L})}(\mathcal{V}) \setminus \mathcal{P}_m^0),$$

and $\widetilde{\Phi}_*(\mathcal{V}) = \widetilde{\Phi}^0(\mathcal{V}) \bigcup \widetilde{\Phi}^{>0}(\mathcal{V})$. Since $\mathbf{q}_{(l, m)}$ is a homomorphism of the $\mathcal{A}$ -modules, if $\mathcal{V}$ is a minimal set of generators for the $\mathcal{A}$ -module $\mathcal{P}_{m-1}$ in a certain positive degree, then $\widetilde{\Phi}^0(\mathcal{V})$ is also a minimal set of generators for the $\mathcal{A}$ -module $\mathcal{P}_m^0$ in that positive degree.

4. Proofs of our chief results

The aim of this section is to prove each of the chief results (namely, Theorems 2.3, 2.6 and 2.9) that were presented in Sect. 2. To facilitate the reader's understanding, we provide a brief summary of the notational conventions that we will employ throughout this text.

Notation 4.1. (i) For a polynomial $F \in \mathcal{P}_m$, we denote by $[F]$ the classes in $Q^{\otimes m}$ represented by F . If $\mathbf{Param}$ is a parameter vector and $F \in \mathcal{P}^{\leq \mathbf{Param}}$, then we denote by $[F]_{\mathbf{Param}}$ the classes in $(Q^{\otimes m})^{\mathbf{Param}}$ represented by F . For a subset $\mathcal{C} \subset \mathcal{P}_m$, as usual, we write $|\mathcal{C}|$ for the cardinal of $\mathcal{C}$; at the same time, we put $[\mathcal{C}] = \{[F] : F \in \mathcal{C}\}$. If $\mathcal{C} \subset \mathcal{P}_m^{\leq \mathbf{Param}}$, then we set $[\mathcal{C}]_{\mathbf{Param}} = \{[F]_{\mathbf{Param}} : F \in \mathcal{C}\}$.

(ii) Let $\mathbf{Param}$ be a parameter vector of degree n . We set

$$(\mathcal{C}_n^{\otimes m})^{\mathbf{Param}} := \mathcal{C}_n^{\otimes m} \cap \mathcal{P}_m^{\leq \mathbf{Param}}, \quad (\mathcal{C}_n^{\otimes m})^{\mathbf{Param}^0} := (\mathcal{C}_n^{\otimes m})^{\mathbf{Param}} \cap \mathcal{P}_m^0, \quad (\mathcal{C}_n^{\otimes m})^{\mathbf{Param}^{>0}} := (\mathcal{C}_n^{\otimes m})^{\mathbf{Param}} \cap \mathcal{P}_m^{>0},$$

$$(Q_n^{\otimes m})^{\mathbf{Param}^0} := (Q^{\otimes m})^{\mathbf{Param}} \cap (Q_n^{\otimes m})^0, \text{ and } (Q_n^{\otimes m})^{\mathbf{Param}^{>0}} := (Q^{\otimes m})^{\mathbf{Param}} \cap (Q_n^{\otimes m})^{>0}.$$

Then it is clear that the sets $[(\mathcal{C}_n^{\otimes m})^{\mathbf{Param}}]_{\mathbf{Param}}$, $[(\mathcal{C}_n^{\otimes m})^{\mathbf{Param}^0}]_{\mathbf{Param}}$ and $[(\mathcal{C}_n^{\otimes m})^{\mathbf{Param}^{>0}}]_{\mathbf{Param}}$ are respectively the bases of the $\mathbb{F}_2$ -vector spaces $(Q_n^{\otimes m})^{\mathbf{Param}}$, $(Q_n^{\otimes m})^{\mathbf{Param}^0}$ and $(Q_n^{\otimes m})^{\mathbf{Param}^{>0}}$.

(iii) Putting $U_m = \{1, 2, \dots, m\}$, $\mathcal{X}_{(V, m)} = \prod_{u \in U_m \setminus V} x_u$, where $V \subseteq U_m$. In particular,

$$\mathcal{X}_{(U_m, m)} = 1, \quad \mathcal{X}_{(\emptyset, m)} = x_1 x_2 \dots x_m,$$

$$\mathcal{X}_{(\{u\}, m)} = x_1 \dots \hat{x}_u \dots x_m, \text{ for } 1 \leq u \leq m.$$

Given any $X = x_1^{t_1} x_2^{t_2} \dots x_m^{t_m} \in \mathcal{P}_m$, let $V_k(X) = \{j \in U_m : \alpha_k(t_j) = 0\}$ for all $k \geq 0$. Then, $X = \prod_{k \geq 0} \mathcal{X}_{(V_k(X), m)}^{2^k}$ and $\deg(\mathbf{Param}_k(X)) = \deg(\mathcal{X}_{(V_{k-1}(X), m)})$ for $k \geq 1$. Noting also that due

to [Kam90], one has a identify $X = \prod_{k \geq 0} \mathcal{X}_{(V_k(X), m)}^{2^k} = \prod_{k, j} x_j^{2^{k-1} \epsilon_{kj}(X)}$ with $\epsilon_{kj}(X) \in \{0, 1\}$. For instance, with $m = 2$,

$$\begin{array}{lcl} \text{the monomials in } \mathcal{P}_2: & x_1^2 x_2^2 & x_1^2 & x_2^4 \\ \text{matrix:} & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(iv) The subsequent homomorphism is of significant interest and proves to be extremely valuable in achieving our objectives. Let $\mathcal{V}_m$ be a rank m elementary abelian 2-group. We may equally well view $\mathcal{V}_m$ as an m -dimensional $\mathbb{F}_2$ -vector space. It is well-known that $H^*(\mathcal{V}_m; \mathbb{F}_2) \cong S(\mathcal{V}_m^*)$, the symmetric algebra over the dual space $\mathcal{V}_m^* = H^1(\mathcal{V}_m; \mathbb{F}_2)$. Pick $x_1, \dots, x_m$ to be a basis of $H^1(\mathcal{V}_m; \mathbb{F}_2)$. Then, we have $H^*(\mathcal{V}_m; \mathbb{F}_2) \cong \mathcal{P}_m$. Note that $\mathcal{V}_m \cong \langle \{x_1, \dots, x_m\} \rangle \subset \mathcal{P}_m$. For $1 \leq t \leq m$, define the $\mathbb{F}_2$ -linear map $\mathfrak{S}_t : \mathcal{V}_m \longrightarrow \mathcal{V}_m$, which is given by $\mathfrak{S}_t(x_t) = x_{t+1}$, $\mathfrak{S}_t(x_{t+1}) = x_t$, $\mathfrak{S}_t(x_j) = x_j$ ($j \neq t, t+1$, $1 \leq t \leq m-1$), and $\mathfrak{S}_m(x_1) = x_1 + x_2$, $\mathfrak{S}_m(x_j) = x_j$ ($2 \leq j \leq m$). Denote by $\Sigma_m \subset G(m)$ the symmetric group of degree m . Then, Σ_m is generated by the ones associated with $\mathfrak{S}_t$, $1 \leq t \leq m-1$. For each permutation in Σ_m , consider corresponding

permutation matrix; these form a group of matrices isomorphic to Σ_m . So, $G(m)$ is generated by the matrices associated with $\mathfrak{S}_t$, $1 \leq t \leq m$.

We note that as the parameter vector of a monomial is invariant under the permutation of the generators x_j , $1 \leq j \leq m$, $(Q^{\otimes m})^{\text{Param}}$ also has a Σ_m -module structure. The map $\mathfrak{S}_t$ above induces a homomorphism $\mathfrak{S}_t : \mathcal{P}_m \longrightarrow \mathcal{P}_m$ of algebras. Then we say that a class $[F]_{\text{Param}} \in (Q^{\otimes m})^{\text{Param}}$ is an $G(m)$ -invariant if $\mathfrak{S}_t(F) \sim_{\text{Param}} F$ for $1 \leq t \leq m$. If $\mathfrak{S}_t(F) \sim_{\text{Param}} F$ for $1 \leq t \leq m-1$, then $[F]_{\text{Param}}$ is an Σ_m -invariant.

4.1. The proof of Theorem 2.3

Let X be an allowable monomial of degree n_1 in the $\mathcal{A}$ -module $\mathcal{P}_5$ such that $[X]$ belongs to $\text{Ker}((\widetilde{Sq}_*)_{(5,n_1)})$. It is clear that $Y = x_1^{25-1}x_2^{23-1}x_3^{22-1}x_4^{20-1}x_5^{20-1} = x_1^{31}x_2^7x_3^3 \in \mathcal{P}_5$ is the minimal spike monomial of degree n_1 , and $\text{Param}(Y) = (3, 3, 2, 1, 1)$. Hence Y belongs to $\mathcal{C}_{n_1}^{\otimes 5}$. Since $[X] \neq [0]$ and $\deg(X)$ is odd, in view of Theorem 3.2.5, either $\text{Param}_1(X) = 3$ or $\text{Param}_1(X) = 5$. If $\text{Param}_1(X) = 5$, then $X = \mathcal{X}_{(\emptyset, 5)}Z^2$ with $Z \in (\mathcal{P}_5)_{n_0}$. Since $X \in \mathcal{C}_{n_1}^{\otimes 5}$ and $\text{Param}_i(\mathcal{X}_{(\emptyset, 5)}) = 0$ for all $i > 1$, by Theorem 3.2.2, one derives $Z \in \mathcal{C}_{n_0}^{\otimes 5}$, and so, $(\widetilde{Sq}_*)_{(5,n_1)}([X]) = [Z] \neq [0]$. This contradicts the fact that $[X] \in \text{Ker}((\widetilde{Sq}_*)_{(5,n_1)})$. Thus, $\text{Param}_1(X)$ is equal to 3. Due to this and Theorem 3.2.3, we must have that $X = \mathcal{X}_{\{i,j\}, 5}Z_1^2$ with $1 \leq i < j \leq 5$ and $Z_1 \in \mathcal{C}_{19}^{\otimes 5}$. Owing to Tin's thesis [Tin17], $\text{Param}(Z_1) \in \{(3, 2, 1, 1), (3, 2, 3), (3, 4, 2)\}$. Now, by the usage of Theorem 3.2.3, we shall show that $[X]_{\text{Param}(X)} = [0]_{\text{Param}(X)}$, if either $\text{Param}(Z_1) = (3, 2, 3)$ or $\text{Param}(Z_1) = (3, 4, 2)$.

Case $\text{Param}(Z_1) = (3, 2, 3)$. The lemma below is a direct corollary of the preceding outcomes established in [Sum15] and [Phu16].

Lemma 4.1.1. *Let i, j, k, l and m be five distinct integers and $1 \leq i, j, k, l, m \leq 5$. Then, the following monomials of degree 17 are nonallowable:*

- i) $x_i^7x_j^2x_kx_l^7$, $x_i^3x_j^6x_kx_l^7$, for $j < k$,
 $x_i^3x_j^2x_k^5x_l^5$, $x_i^7x_j^3x_k^4x_l^3$, $x_i^3x_j^3x_k^6x_l^5$, for $j < k < l$,
 $x_i^3x_j^4x_k^2x_lx_m^7$, $x_i^3x_j^2x_kx_l^4x_m^7$, $x_i^3x_j^2x_k^4x_lx_m^7$, $x_i^3x_j^4x_k^2x_l^5x_m^3$, for $j < k < l$;
- ii) $x_1^7x_2^6x_kx_lx_m^2$, $x_1^7x_2^2x_kx_l^6x_m$, $x_1^3x_2^6x_kx_lx_m^6$, $x_1^7x_2^2x_kx_l^2x_m^5$, $x_1^3x_2^4x_k^2x_lx_m^7$,
 $x_1^3x_2^2x_kx_l^4x_m^7$, $x_1^3x_2^2x_k^4x_lx_m^7$, $x_1^3x_2^6x_kx_l^2x_m^5$, $x_1^3x_2^6x_kx_l^4x_m^3$, $x_1^3x_2^4x_k^3x_l^4x_m^3$;
- iii) $x_1^3x_2^4x_3x_4^6x_5^3$, $x_1^3x_2^4x_3^3x_4^6x_5$, $x_1^3x_2^4x_3^6x_4x_5^3$, $x_1^3x_2^4x_3^6x_4^3x_5$.

As an illustration, let us examine the monomials $X = x_1^3x_2^4x_k^3x_l^4x_m^3$ and $Y = x_i^3x_j^4x_k^6x_lx_m^3$. Using the Cartan formula, one can derive the following equalities:

$$\begin{aligned} X &\sim_{(2, \text{Param}(X))} x_1^3x_2^3x_k^4x_l^4x_m^3 + x_1^3x_2^3x_k^3x_l^4x_m^4 + x_1^2x_2^5x_k^3x_l^4x_m^3 + x_1^2x_2^3x_k^5x_l^4x_m^3 + x_1^2x_2^3x_k^3x_l^4x_m^5, \\ Y &\sim_{\text{Param}(Y)} x_i^3x_j^4x_k^5x_l^2x_m^3, \end{aligned}$$

which consequently establish that X and Y are strictly nonallowable and $\text{Param}(Y)$ -nonallowable, respectively. Hence, X and Y are nonallowable monomials.

Lemma 4.1.2. *All permutations of the following monomials are strictly nonallowable :*

$$\begin{array}{llll} x_1^3x_2^4x_3^9x_4^{10}x_5^{15}, & x_1^3x_2^4x_3^9x_4^{11}x_5^{14}, & x_1^3x_2^4x_3^{10}x_4^{11}x_5^{13}, & x_1^3x_2^5x_3^8x_4^{10}x_5^{15}, \\ x_1^3x_2^5x_3^8x_4^{11}x_5^{14}, & x_1^3x_2^5x_3^9x_4^{10}x_5^{14}, & x_1^3x_2^5x_3^{10}x_4^{10}x_5^{13}, & x_1^3x_2^5x_3^{10}x_4^{11}x_5^{12}, \\ x_1^3x_2^7x_3^8x_4^{15}, & x_1^3x_2^7x_3^8x_4^{14}, & x_1^3x_2^7x_3^8x_4^{13}, & x_1^3x_2^7x_3^8x_4^{12}, \\ x_1^3x_2^7x_3^9x_4^{12}, & X := x_1^7x_2^7x_3^8x_4^{11}, & Y := x_1^7x_2^7x_3^8x_4^{10}, & Z := x_1^7x_2^{11}x_3^4x_4^9x_5^{10}, \\ x_1^7x_2^{11}x_3^5x_4^{10}, & x_1^7x_2^{11}x_3^{11}x_4^8x_5, & & \end{array}$$

Proof. Each monomial in the lemma is of the parameter vector $\text{Param}^* := (3, 3, 2, 3)$. The proof of the lemma for the given set of monomials $x_1^3x_2^4x_3^9x_4^{10}x_5^{15}$, $x_1^3x_2^4x_3^9x_4^{11}x_5^{14}$, $\dots$, $x_1^3x_2^7x_3^8x_4^{10}x_5^{12}$, $x_1^7x_2^{11}x_3^5x_4^8x_5^{10}$ and $x_1^7x_2^{11}x_3^{11}x_4^8x_5$ is rather straightforward. We thus inspect that the monomials X , Y , and Z are

strictly nonallowable. Indeed, through the use of the Cartan formula, we obtain the following expression:

$$\begin{aligned}
X = & Sq^1 \left(x_1^7 x_2^3 x_3 x_4^3 x_5^{26} + x_1^7 x_2^3 x_3 x_4^{10} x_5^{19} + x_1^7 x_2^7 x_3^8 x_4^5 x_5^{13} + x_1^7 x_2^9 x_3 x_4^{10} x_5^{13} \right. \\
& \left. + x_1^7 x_2^9 x_3 x_4^{12} x_5^{11} + x_1^{11} x_2^3 x_3 x_4^3 x_5^{22} + x_1^{11} x_2^3 x_3 x_4^6 x_5^{19} \right) \\
& + Sq^2 \left(x_1^7 x_2^3 x_3 x_4^6 x_5^{22} + x_1^7 x_2^3 x_3^2 x_4^5 x_5^{22} + x_1^7 x_2^3 x_3^2 x_4^6 x_5^{21} + x_1^7 x_2^6 x_3 x_4^6 x_5^{19} \right. \\
& \left. + x_1^7 x_2^6 x_3 x_4^{18} x_5^7 + x_1^7 x_2^7 x_3^8 x_4^5 x_5^{14} + x_1^7 x_2^7 x_3^8 x_4^6 x_5^{11} + x_1^7 x_2^9 x_3^2 x_4^{10} x_5^{11} \right) \\
& + Sq^4 \left(x_1^5 x_2^7 x_3^8 x_4^3 x_5^{14} + x_1^5 x_2^7 x_3^8 x_4^6 x_5^{11} + x_1^7 x_2^3 x_3^2 x_4^3 x_5^{22} + x_1^7 x_2^3 x_3^2 x_4^6 x_5^{19} + x_1^{11} x_2^5 x_3 x_4^6 x_5^{14} \right. \\
& \left. + x_1^{11} x_2^5 x_3^4 x_4^3 x_5^{14} + x_1^{11} x_2^5 x_3^4 x_4^6 x_5^{11} + x_1^{11} x_2^6 x_3 x_4^6 x_5^{13} + x_1^{11} x_2^6 x_3 x_4^{12} x_5^7 \right) \\
& + Sq^8 \left(x_1^7 x_2^5 x_3 x_4^6 x_5^{14} + x_1^7 x_2^5 x_3^4 x_4^3 x_5^{14} + x_1^7 x_2^5 x_3^4 x_4^6 x_5^{11} + x_1^7 x_2^6 x_3 x_4^6 x_5^{13} + x_1^7 x_2^6 x_3 x_4^{12} x_5^7 \right) \\
& + \sum_{1 \leq i \leq 4} X_i \mod (\mathcal{P}_5^{<\text{Param}^*}),
\end{aligned}$$

where $X_1 = x_1^5 x_2^7 x_3^8 x_4^{10} x_5^{11} < X$, $X_2 = x_1^5 x_2^{11} x_3^8 x_4^3 x_5^{14} < X$, $X_3 = x_1^5 x_2^{11} x_3^8 x_4^6 x_5^{11} < X$, $X_4 = x_1^7 x_2^5 x_3^8 x_4^{10} x_5^{11} < X$. Next, we have

$$\begin{aligned}
Y = & \sum_{1 \leq i \leq 11} Y_i + Sq^1 \left(x_1^7 x_2^3 x_3 x_4^{11} x_5^{18} + x_1^7 x_2^5 x_3^8 x_4^7 x_5^{13} + x_1^7 x_2^5 x_3^8 x_4^9 x_5^{11} + x_1^7 x_2^7 x_3^8 x_4^7 x_5^{11} \right. \\
& \left. + x_1^7 x_2^9 x_3 x_4^{11} x_5^{12} + x_1^7 x_2^9 x_3 x_4^{13} x_5^{10} + x_1^{11} x_2^3 x_3 x_4^7 x_5^{18} \right) \\
& + Sq^2 \left(x_1^7 x_2^3 x_3 x_4^{22} x_5^6 + x_1^7 x_2^3 x_3^2 x_4^7 x_5^{20} + x_1^7 x_2^3 x_3^8 x_4^7 x_5^{14} + x_1^7 x_2^6 x_3 x_4^7 x_5^{18} \right. \\
& \left. + x_1^7 x_2^6 x_3 x_4^{19} x_5^6 + x_1^7 x_2^6 x_3^8 x_4^7 x_5^{11} + x_1^7 x_2^7 x_3^8 x_4^7 x_5^{10} + x_1^7 x_2^9 x_3^2 x_4^{11} x_5^{10} \right) \\
& + Sq^4 \left(x_1^4 x_2^7 x_3^8 x_4^7 x_5^{11} + x_1^5 x_2^3 x_3^8 x_4^7 x_5^{14} + x_1^5 x_2^6 x_3^8 x_4^7 x_5^{11} + x_1^5 x_2^7 x_3^8 x_4^7 x_5^{10} \right. \\
& \left. + x_1^7 x_2^3 x_3^2 x_4^7 x_5^{18} + x_1^{11} x_2^5 x_3 x_4^{14} x_5^6 + x_1^{11} x_2^5 x_3^4 x_4^7 x_5^{10} + x_1^{11} x_2^6 x_3 x_4^7 x_5^{12} + x_1^{11} x_2^6 x_3 x_4^{13} x_5^6 \right) \\
& + Sq^8 \left(x_1^7 x_2^5 x_3 x_4^{14} x_5^6 + x_1^7 x_2^5 x_3^4 x_4^7 x_5^{10} + x_1^7 x_2^6 x_3 x_4^7 x_5^{12} + x_1^7 x_2^6 x_3 x_4^{13} x_5^6 \right) \mod (\mathcal{P}_5^{<\text{Param}^*}),
\end{aligned}$$

wherein the monomials $Y_i < X$, $1 \leq i \leq 11$, are determined as follows:

$$\begin{aligned}
Y_1 &= x_1^4 x_2^7 x_3^8 x_4^{11} x_5^{11}, & Y_2 &= x_1^4 x_2^{11} x_3^8 x_4^7 x_5^{11}, & Y_3 &= x_1^5 x_2^3 x_3^8 x_4^{11} x_5^{14}, \\
Y_4 &= x_1^5 x_2^6 x_3^8 x_4^{11} x_5^{11}, & Y_5 &= x_1^5 x_2^7 x_3^8 x_4^{11} x_5^{10}, & Y_6 &= x_1^5 x_2^{10} x_3^8 x_4^7 x_5^{11}, \\
Y_7 &= x_1^5 x_2^{11} x_3^8 x_4^7 x_5^{10}, & Y_8 &= x_1^7 x_2^3 x_3^8 x_4^9 x_5^{14}, & Y_9 &= x_1^7 x_2^5 x_3^8 x_4^{10} x_5^{11}, \\
Y_{10} &= x_1^7 x_2^5 x_3^8 x_4^{11} x_5^{10}, & Y_{11} &= x_1^7 x_2^7 x_3^8 x_4^8 x_5^{11}.
\end{aligned}$$

Lastly, through a straightforward calculation, we obtain

$$\begin{aligned}
Z = & Sq^1 \left(x_1^7 x_2^9 x_3^4 x_4^7 x_5^{13} + x_1^7 x_2^{11} x_3^4 x_4^7 x_5^{11} + x_1^7 x_2^{13} x_3 x_4^7 x_5^{12} + x_1^7 x_2^{13} x_3 x_4^9 x_5^{10} \right) \\
& + Sq^2 \left(x_1^3 x_2^9 x_3^2 x_4^{11} x_5^{14} + x_1^3 x_2^{10} x_3^4 x_4^{11} x_5^{11} + x_1^3 x_2^{10} x_3^8 x_4^7 x_5^{11} + x_1^3 x_2^{11} x_3^2 x_4^{11} x_5^{12} \right. \\
& \left. + x_1^3 x_2^{14} x_3 x_4^{11} x_5^{10} + x_1^7 x_2^9 x_3^2 x_4^7 x_5^{14} + x_1^7 x_2^{10} x_3^4 x_4^7 x_5^{11} + x_1^7 x_2^{11} x_3 x_4^{10} x_5^{10} \right. \\
& \left. + x_1^7 x_2^{11} x_3^2 x_4^7 x_5^{12} + x_1^7 x_2^{11} x_3^2 x_4^9 x_5^{10} + x_1^7 x_2^{14} x_3 x_4^7 x_5^{10} \right) \\
& + Sq^4 \left(x_1^4 x_2^{11} x_3^4 x_4^7 x_5^{11} + x_1^5 x_2^9 x_3^2 x_4^7 x_5^{14} + x_1^5 x_2^{10} x_3^4 x_4^7 x_5^{11} + x_1^5 x_2^{11} x_3^2 x_4^7 x_5^{12} + x_1^5 x_2^{14} x_3 x_4^7 x_5^{10} \right) \\
& + Z_1 := x_1^3 x_2^9 x_3^2 x_4^{13} x_5^{14} + Z_2 := x_1^3 x_2^9 x_3^4 x_4^{11} x_5^{14} + Z_3 := x_1^3 x_2^{10} x_3^4 x_4^{11} x_5^{13} \\
& + Z_4 := x_1^3 x_2^{10} x_3^4 x_4^{13} x_5^{11} + Z_5 := x_1^3 x_2^{10} x_3^8 x_4^7 x_5^{13} + Z_6 := x_1^3 x_2^{11} x_3^2 x_4^{13} x_5^{12} \\
& + Z_7 := x_1^3 x_2^{11} x_3^4 x_4^{11} x_5^{12} + Z_8 := x_1^3 x_2^{12} x_3^4 x_4^{11} x_5^{11} + Z_9 := x_1^3 x_2^{12} x_3^8 x_4^7 x_5^{11} \\
& + Z_{10} := x_1^3 x_2^{13} x_3^2 x_4^{11} x_5^{12} + Z_{11} := x_1^3 x_2^{14} x_3 x_4^{11} x_5^{12} + Z_{12} := x_1^3 x_2^{14} x_3 x_4^{13} x_5^{10} \\
& + Z_{13} := x_1^4 x_2^{11} x_3^4 x_4^{11} x_5^{11} + Z_{14} := x_1^4 x_2^{11} x_3^8 x_4^7 x_5^{11} + Z_{15} := x_1^7 x_2^9 x_3^2 x_4^9 x_5^{14} \\
& + Z_{16} := x_1^7 x_2^{10} x_3^4 x_4^{11} x_5^{11} + Z_{17} := x_1^7 x_2^{11} x_3 x_4^{10} x_5^{12} + Z_{18} := x_1^7 x_2^{11} x_3 x_4^{12} x_5^{10} \\
& + Z_{19} := x_1^7 x_2^{11} x_3^4 x_4^8 x_5^{11} \mod (\mathcal{P}_5^{<\text{Param}^*}), \text{ where } Z_i < Z \text{ for every } i.
\end{aligned}$$

It can be seen from the above equalities that

$$X \sim_{(3, \text{Param}^*)} \sum_{1 \leq i \leq 4} X_i, \quad Y \sim_{(3, \text{Param}^*)} \sum_{1 \leq i \leq 11} Y_i, \quad \text{and} \quad Z \sim_{(3, \text{Param}^*)} \sum_{1 \leq i \leq 19} Z_i.$$

Consequently, the nonallowability of X , Y , Z can be established. Hence, the lemma is proven to be true. $\square$

We can observe from a direct computation that there exists a monomial W as given in Lemmas 4.1.1 and 4.1.2, such that X can be expressed as $X = \mathcal{X}_{(\emptyset, 5)} Z_1^2 = WS^{2^l}$, where S is a suitable monomial in $\mathcal{P}_5$ and $l = \max\{j : \text{Param}_j(W) > 0\}$. As per Theorem 3.2.3, it can be deduced that X is strictly nonallowable.

Case $\text{Param}(Z_1) = (3, 4, 2)$. Our first observation is that the statement below follows directly from a result in [Sum19, Theorem 1.1].

Lemma 4.1.3. *The following monomials are strictly nonallowable :*

$$\begin{array}{cccc} x_1^2 x_2^4 x_3^5 x_4^7 x_5^7, & x_1^2 x_2^4 x_3^7 x_4^5 x_5^7, & x_1^2 x_2^5 x_3^5 x_4^6 x_5^7, & x_1^2 x_2^5 x_3^6 x_4^5 x_5^7, \\ x_1^2 x_2^5 x_3^4 x_4^7 x_5^7, & x_1^2 x_2^5 x_3^7 x_4^4 x_5^7, & x_1^2 x_2^6 x_3^7 x_4^5 x_5^7, & x_1^2 x_2^7 x_3^4 x_4^5 x_5^7, \\ x_1^3 x_2^4 x_3^4 x_4^7 x_5^7, & x_1^3 x_2^4 x_3^5 x_4^6 x_5^7, & x_1^3 x_2^4 x_3^6 x_4^5 x_5^7, & x_1^3 x_2^4 x_3^7 x_4^4 x_5^7, \\ x_1^3 x_2^4 x_3^7 x_4^5 x_5^7, & x_1^3 x_2^5 x_3^5 x_4^6 x_5^7, & x_1^3 x_2^5 x_3^6 x_4^5 x_5^7, & x_1^3 x_2^5 x_3^7 x_4^4 x_5^7, \\ x_1^3 x_2^5 x_3^4 x_4^7 x_5^7, & x_1^3 x_2^6 x_3^6 x_4^5 x_5^7, & x_1^3 x_2^6 x_3^7 x_4^4 x_5^7, & x_1^3 x_2^7 x_3^4 x_4^5 x_5^7, \\ x_1^4 x_2^4 x_3^4 x_4^7 x_5^7, & x_1^4 x_2^5 x_3^5 x_4^6 x_5^7, & x_1^4 x_2^5 x_3^6 x_4^5 x_5^7, & x_1^4 x_2^5 x_3^7 x_4^4 x_5^7, \\ x_1^4 x_2^6 x_3^4 x_4^7 x_5^7, & x_1^4 x_2^6 x_3^5 x_4^6 x_5^7, & x_1^4 x_2^6 x_3^6 x_4^5 x_5^7, & x_1^4 x_2^6 x_3^7 x_4^4 x_5^7, \\ x_1^5 x_2^4 x_3^4 x_4^7 x_5^7, & x_1^5 x_2^4 x_3^5 x_4^6 x_5^7, & x_1^5 x_2^4 x_3^6 x_4^5 x_5^7, & x_1^5 x_2^4 x_3^7 x_4^4 x_5^7. \end{array}$$

After a straightforward computation, we observe that the monomials of the form $X = \mathcal{X}_{(\emptyset, 5)} Z_1^2$ can be expressed as $X = Y \mathcal{X}_{(1, 2, 3, 5)}^8$, where Y is a suitable monomial in Lemma 4.1.3. It is clear that $\text{Param}_3(Y) = 4 \neq 0$ and $\text{Param}_l(Y) = 0$ for any $l > 3$. Thus, we can apply Theorem 3.2.3 to conclude that X is strictly nonallowable.

Summing up, from the above cases, we must have that the intersection of $\text{Ker}(\widetilde{Sq}_*^0)_{(5, n_1)}$ with $(Q_{n_1}^{\otimes 5})^{>0}$ must be equal to $(Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$, where $\widetilde{\text{Param}} := (3, 3, 2, 1, 1)$. As a result, we obtain an isomorphism: $\text{Ker}(\widetilde{Sq}_*^0)_{(5, n_1)} \cong (Q_{n_1}^{\otimes 5})^0 \oplus (Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$. We proceed to perform explicit computations to determine the dimensions of the subspaces $(Q_{n_1}^{\otimes 5})^0$ and $(Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$.

Calculation of $(Q_{n_1}^{\otimes 5})^0$. For $\mathcal{J} = (j_1, j_2, \dots, j_r) : 1 \leq j_1 < j_2 < \dots < j_r \leq 4$, we define a monomorphism $\theta_{\mathcal{J}} : \mathcal{P}_r \longrightarrow \mathcal{P}_5$ of $\mathcal{A}$ -algebras by substituting $\theta_{\mathcal{J}}(x_k) = x_{j_k}$, for $1 \leq k \leq r$. It can then be readily deduced that $(\mathbb{F}_2 \otimes_{\mathcal{A}} \theta_{\mathcal{J}}(\mathcal{P}_r^{>0}))_{n_1} \cong (Q_{n_1}^{\otimes r})^{>0}$ for $1 \leq r \leq 4$. Therefore, by utilizing Theorem 2.1(I) and performing a simple computation, we obtain

$$\dim(Q_{n_1}^{\otimes 5})^0 = \sum_{3 \leq r \leq 4} \binom{5}{r} \dim(\mathbb{F}_2 \otimes_{\mathcal{A}} \theta_{\mathcal{J}}(\mathcal{P}_r^{>0}))_{n_1} = \sum_{3 \leq r \leq 4} \binom{5}{r} \dim(Q_{n_1}^{\otimes r})^{>0}.$$

On the other side, in accordance with the works [Kam90] and [Sum15], the dimensions of $(Q_{n_1}^{\otimes 3})^{>0}$ and $(Q_{n_1}^{\otimes 4})^{>0}$ are determined to be 15 and 165, respectively. Applying these results and the aforementioned formula yields the conclusion that the dimension of $(Q_{n_1}^{\otimes 5})^0$ is 975.

Calculation of $(Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$. For any natural numbers s and l satisfying $1 \leq l \leq 5$, let us define the set $\mathcal{C}_{(l, n_1)}$ as the collection of elements of the form $x_l^{2^s-1} \mathbf{q}_{(l, 5)}(Y)$ in $(\mathcal{P}_5)_{n_1}$, where $Y \in \mathcal{C}_{39-2^s}^{\otimes 4}$ and $\alpha(46-2^s) \leq 4$. We also introduce the notation $\mathcal{C}_{(l, \widetilde{\text{Param}})}$ to represent the intersection of $\mathcal{C}_{(l, n_1)}$ with $\mathcal{P}_5^{\leq \widetilde{\text{Param}}}$, and $\mathcal{C}_{(l, \widetilde{\text{Param}})}^{>0}$ to denote the intersection of $\mathcal{C}_{(l, \widetilde{\text{Param}})}$ with $(\mathcal{P}_5^{>0})_{n_1}$. According to Theorem 3.3.1, it follows that $\mathcal{C}_{(l, \widetilde{\text{Param}})} \subset (Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}}$. We hereby conclude the demonstration of the theorem by showing that $(Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}} = \mathcal{D} \cup \widetilde{\Phi}^{>0}((\mathcal{C}_{n_1}^{\otimes 4})^{\widetilde{\text{Param}}}) \cup \left(\bigcup_{1 \leq l \leq 5} \mathcal{C}_{(l, \widetilde{\text{Param}})}^{>0} \right)$, wherein $\mathcal{D} = \{X_t \mid 1 \leq t \leq 199\}$, $\widetilde{\Phi}^{>0}((\mathcal{C}_{n_1}^{\otimes 4})^{\widetilde{\text{Param}}}) = \{X_t \mid 200 \leq t \leq 620\}$ and $\bigcup_{1 \leq l \leq 5} \mathcal{C}_{(l, \widetilde{\text{Param}})}^{>0} = \{X_t \mid 621 \leq t \leq 925\}$.

Here the monomials X_t , $1 \leq t \leq 925$, are described as in Subsect. 5.3. Indeed, we begin by recalling a known result in [SE62].

Proposition 4.1.4 (cf. [SE62]). *For any $X \in (\mathcal{P}_m)_1$, the Steenrod squares Sq^i act on X^d as $\binom{d}{i} X^{d+i}$. In particular, if d is odd, then $Sq^1(X^d) = X^{d+1}$, while if d is even, then $Sq^1(X^d) = 0$.*

The following technical claim serves as a crucial step in the proof of the theorem. The approach taken to establish this result involves the Cartan formula and Proposition 4.1.4.

Proposition 4.1.5. *We have a set of strictly nonallowable monomials $\{T_j \in (\mathcal{P}_5^{>0})_{n_1} \mid \text{Param}(T_j) = \widetilde{\text{Param}}, 1 \leq j \leq 144\}$, as described in Subsect. 5.4 of the appendix.*

Proof. We establish the proposition by demonstrating its validity for the monomials $T_1 = x_1 x_2^{19} x_3^{24} x_4^3 x_5^6$, $T_2 = x_1 x_2^{14} x_3^{19} x_4^5 x_5^2$, $T_3 = x_1 x_2^{15} x_3^{18} \underbrace{x_4^5 x_5^2}$, while other monomials are proven analogously. A noteworthy observation that $\text{Param}(T_i) = \text{Param}$ for every i . Computing these monomials is rather intricate. Indeed, the application of the Cartan formula and Proposition 4.1.4 leads to the following equality:

$$T_1 = \sum_{1 \leq j \leq 63} X_j + Sq^1(W) + Sq^2(X) + Sq^4(Y) + Sq^8(Z) \mod (\mathcal{P}_5^{<\text{Param}}),$$

where

$$\begin{aligned}
W &= x_1^5 x_2^3 x_3^{15} x_4^{12} + x_1^7 x_2^3 x_3^{11} x_4^{13} + x_1^2 x_2^3 x_3^{11} x_4^{11} + x_1^2 x_2^3 x_3^{13} x_4^{11} + x_1^2 x_2^3 x_3^{11} x_4^{11}, \\
X &= x_1 x_2^2 x_3^7 x_4^{22} + x_1 x_2^2 x_3^{22} x_4^7 x_5^7 + x_1 x_2^3 x_3^6 x_4^7 x_5^{22} + x_1 x_2^3 x_3^{22} x_4^7 x_5^6 \\
&\quad + x_1 x_2^6 x_3^7 x_4^7 x_5^{18} + x_1 x_2^6 x_3^{18} x_4^7 x_5^7 + x_1 x_2^7 x_3^6 x_4^3 x_5^{22} + x_1 x_2^7 x_3^7 x_4^2 x_5^{22} \\
&\quad + x_1 x_2^7 x_3^7 x_4^6 x_5^{18} + x_1 x_2^7 x_3^{10} x_4^7 x_5^{14} + x_1 x_2^7 x_3^{11} x_4^7 x_5^{13} + x_1 x_2^7 x_3^{11} x_4^9 x_5^{11} \\
&\quad + x_1 x_2^7 x_3^{13} x_4^7 x_5^{11} + x_1 x_2^7 x_3^{14} x_4^7 x_5^{10} + x_1 x_2^7 x_3^{18} x_4^7 x_5^7 + x_1 x_2^7 x_3^{22} x_4^2 x_5^7 \\
&\quad + x_1 x_2^7 x_3^{22} x_4^3 x_5^6 + x_1 x_2^9 x_3^{11} x_4^7 x_5^{11}, \\
Y &= x_1 x_2^4 x_3^7 x_4^7 x_5^{18} + x_1 x_2^4 x_3^7 x_4^7 x_5^{18} + x_1 x_2^4 x_3^7 x_4^{11} x_5^{14} + x_1 x_2^4 x_3^{14} x_4^7 x_5^7 \\
&\quad + x_1 x_2^4 x_3^{18} x_4^7 x_5^7 + x_1 x_2^4 x_3^{18} x_4^7 x_5^7 + x_1 x_2^5 x_3^6 x_4^3 x_5^{22} + x_1 x_2^5 x_3^6 x_4^7 x_5^{18} \\
&\quad + x_1 x_2^5 x_3^6 x_4^{11} x_5^{14} + x_1 x_2^5 x_3^7 x_4^2 x_5^{22} + x_1 x_2^5 x_3^7 x_4^6 x_5^{18} + x_1 x_2^5 x_3^{14} x_4^{11} x_5^6 \\
&\quad + x_1 x_2^5 x_3^{18} x_4^6 x_5^7 + x_1 x_2^5 x_3^{18} x_4^7 x_5^6 + x_1 x_2^5 x_3^{22} x_4^5 x_5^7 + x_1 x_2^5 x_3^{22} x_4^6 x_5^6 \\
&\quad + x_1 x_2^6 x_3^7 x_4^7 x_5^{16} + x_1 x_2^6 x_3^7 x_4^{11} x_5^{12} + x_1 x_2^6 x_3^{12} x_4^{11} x_5^7 + x_1 x_2^6 x_3^{16} x_4^7 x_5^7 \\
&\quad + x_1 x_2^{11} x_3^6 x_4^5 x_5^{14} + x_1 x_2^{11} x_3^7 x_4^4 x_5^{14} + x_1 x_2^{11} x_3^7 x_4^6 x_5^{12} + x_1 x_2^{11} x_3^{12} x_4^6 x_5^7 \\
&\quad + x_1 x_2^{11} x_3^{14} x_4^4 x_5^7 + x_1 x_2^{11} x_3^{14} x_4^5 x_5^6, \\
Z &= x_1 x_2^4 x_3^7 x_4^7 x_5^{14} + x_1 x_2^4 x_3^{14} x_4^7 x_5^7 + x_1 x_2^5 x_3^6 x_4^7 x_5^{14} + x_1 x_2^5 x_3^{14} x_4^7 x_5^6 \\
&\quad + x_1 x_2^6 x_3^7 x_4^7 x_5^{12} + x_1 x_2^6 x_3^{12} x_4^7 x_5^7 + x_1 x_2^6 x_3^6 x_4^5 x_5^{14} + x_1 x_2^7 x_3^7 x_4^4 x_5^{14} \\
&\quad + x_1 x_2^7 x_3^6 x_4^6 x_5^{12} + x_1 x_2^7 x_3^{12} x_4^6 x_5^7 + x_1 x_2^7 x_3^{14} x_4^4 x_5^7 + x_1 x_2^7 x_3^{14} x_4^5 x_5^6, \\
\sum_{1 \leq j \leq 63} X_j &= x_1 x_2^2 x_3^7 x_4^7 x_5^{24} + x_1 x_2^2 x_3^7 x_4^9 x_5^{22} + x_1 x_2^2 x_3^9 x_4^7 x_5^{22} + x_1 x_2^2 x_3^{22} x_4^7 x_5^9 \\
&\quad + x_1 x_2^2 x_3^{22} x_4^9 x_5^7 + x_1 x_2^2 x_3^{24} x_4^7 x_5^7 + x_1 x_2^3 x_3^6 x_4^7 x_5^{24} + x_1 x_2^3 x_3^6 x_4^9 x_5^{22} \\
&\quad + x_1 x_2^3 x_3^8 x_4^7 x_5^{22} + x_1 x_2^3 x_3^{22} x_4^7 x_5^8 + x_1 x_2^3 x_3^{22} x_4^9 x_5^6 + x_1 x_2^3 x_3^{24} x_4^7 x_5^6 \\
&\quad + x_1 x_2^4 x_3^7 x_4^{11} x_5^{18} + x_1 x_2^4 x_3^7 x_4^{11} x_5^{18} + x_1 x_2^4 x_3^{11} x_4^7 x_5^{18} + x_1 x_2^4 x_3^{18} x_4^7 x_5^{11} \\
&\quad + x_1 x_2^4 x_3^{18} x_4^{11} x_5^7 + x_1 x_2^4 x_3^{18} x_4^{11} x_5^7 + x_1 x_2^5 x_3^6 x_4^3 x_5^{26} + x_1 x_2^5 x_3^6 x_4^{11} x_5^{18} \\
&\quad + x_1 x_2^5 x_3^7 x_4^2 x_5^{26} + x_1 x_2^5 x_3^7 x_4^{10} x_5^{18} + x_1 x_2^5 x_3^{10} x_4^3 x_5^{22} + x_1 x_2^5 x_3^{11} x_4^2 x_5^{22} \\
&\quad + x_1 x_2^5 x_3^{11} x_4^6 x_5^{18} + x_1 x_2^5 x_3^{18} x_4^6 x_5^{11} + x_1 x_2^5 x_3^{18} x_4^{10} x_5^7 + x_1 x_2^5 x_3^{18} x_4^{11} x_5^6 \\
&\quad + x_1 x_2^5 x_3^{22} x_4^2 x_5^{11} + x_1 x_2^5 x_3^{22} x_4^3 x_5^{10} + x_1 x_2^5 x_3^{26} x_4^2 x_5^7 + x_1 x_2^5 x_3^{26} x_4^3 x_5^6 \\
&\quad + x_1 x_2^6 x_3^7 x_4^9 x_5^{18} + x_1 x_2^6 x_3^7 x_4^{11} x_5^{16} + x_1 x_2^6 x_3^9 x_4^7 x_5^{18} + x_1 x_2^6 x_3^{16} x_4^{11} x_5^7 \\
&\quad + x_1 x_2^6 x_3^{18} x_4^7 x_5^9 + x_1 x_2^6 x_3^{18} x_4^9 x_5^7 + x_1 x_2^7 x_3^6 x_4^3 x_5^{24} + x_1 x_2^7 x_3^6 x_4^9 x_5^{18} \\
&\quad + x_1 x_2^7 x_3^7 x_4^2 x_5^{24} + x_1 x_2^7 x_3^7 x_4^8 x_5^{18} + x_1 x_2^7 x_3^7 x_4^8 x_5^{18} + x_1 x_2^7 x_3^7 x_4^{10} x_5^{16} \\
&\quad + x_1 x_2^7 x_3^8 x_4^3 x_5^{22} + x_1 x_2^7 x_3^9 x_4^2 x_5^{22} + x_1 x_2^7 x_3^9 x_4^6 x_5^{18} + x_1 x_2^7 x_3^{10} x_4^5 x_5^{18} \\
&\quad + x_1 x_2^7 x_3^{10} x_4^7 x_5^{16} + x_1 x_2^7 x_3^{11} x_4^4 x_5^{18} + x_1 x_2^7 x_3^{11} x_4^6 x_5^{16} + x_1 x_2^7 x_3^{16} x_4^6 x_5^{11} \\
&\quad + x_1 x_2^7 x_3^{16} x_4^{10} x_5^7 + x_1 x_2^7 x_3^{18} x_4^{10} x_5^7 + x_1 x_2^7 x_3^{18} x_4^{11} x_5^{11} + x_1 x_2^7 x_3^{18} x_4^{15} x_5^{10} \\
&\quad + x_1 x_2^7 x_3^{18} x_4^6 x_5^9 + x_1 x_2^7 x_3^{18} x_4^8 x_5^7 + x_1 x_2^7 x_3^{18} x_4^8 x_5^7 + x_1 x_2^7 x_3^{18} x_4^9 x_5^6 \\
&\quad + x_1 x_2^7 x_3^{22} x_4^2 x_5^9 + x_1 x_2^7 x_3^{22} x_4^3 x_5^8 + x_1 x_2^7 x_3^{24} x_4^2 x_5^7,
\end{aligned}$$

It can be observed that $X_j < T_1$ for all j , $1 \leq j \leq 63$.

Next, with the monomial T_2 , we have

$$\begin{aligned}
T_2 = & \left[Sq^1 \left(x_1^2 x_2^{11} x_3^{15} x_4^5 x_5^5 + x_1^2 x_2^{11} x_3^{15} x_4^9 x_5^3 + x_1^2 x_2^{11} x_3^{17} x_4^7 x_5^3 + x_1^2 x_2^{13} x_3^{15} x_4^7 x_5^3 \right) \right. \\
& + Sq^2 \left(x_1 x_2^6 x_3^{23} x_4^3 x_5^6 + x_1 x_2^6 x_3^{23} x_4^6 x_5^3 + x_1 x_2^6 x_3^{23} x_4^7 x_5^2 + x_1 x_2^7 x_3^{23} x_4^2 x_5^6 \right. \\
& \quad + x_1 x_2^7 x_3^{23} x_4^6 x_5^2 + x_1 x_2^{10} x_3^{15} x_4^7 x_5^6 + x_1 x_2^{10} x_3^{15} x_4^{10} x_5^3 + x_1 x_2^{11} x_3^{15} x_4^7 x_5^5 \\
& \quad + x_1 x_2^{11} x_3^{15} x_4^9 x_5^3 + x_1 x_2^{11} x_3^{17} x_4^7 x_5^3 + x_1 x_2^{13} x_3^{15} x_4^7 x_5^3 + x_1 x_2^{14} x_3^{15} x_4^7 x_5^2 \left. \right) \\
& + Sq^4 \left(x_1 x_2^6 x_3^{15} x_4^{12} x_5^3 + x_1 x_2^7 x_3^{15} x_4^8 x_5^6 + x_1 x_2^7 x_3^{15} x_4^{10} x_5^4 + x_1 x_2^{10} x_3^{15} x_4^5 x_5^6 \right. \\
& \quad + x_1 x_2^{10} x_3^{15} x_4^6 x_5^5 + x_1 x_2^{10} x_3^{15} x_4^7 x_5^4 + x_1 x_2^{11} x_3^{15} x_4^4 x_5^6 + x_1 x_2^{11} x_3^{15} x_4^6 x_5^4 \\
& \quad + x_1 x_2^{14} x_3^{15} x_4^4 x_5^3 + x_1 x_2^{14} x_3^{15} x_4^5 x_5^2 \left. \right) \\
& + Sq^8 \left(x_1 x_2^6 x_3^{15} x_4^5 x_5^6 + x_1 x_2^6 x_3^{15} x_4^6 x_5^5 + x_1 x_2^6 x_3^{15} x_4^7 x_5^4 + x_1 x_2^7 x_3^{15} x_4^4 x_5^6 \right. \\
& \quad + x_1 x_2^7 x_3^{15} x_4^6 x_5^4 + x_1 x_2^8 x_3^{15} x_4^4 x_5^2 + x_1 x_2^{10} x_3^{15} x_4^4 x_5^3 + x_1 x_2^{10} x_3^{15} x_4^5 x_5^2 \left. \right) \\
& \left. + \sum_{1 \leq j \leq 33} Y_j \right] \mod (\mathcal{P}_5^{<\text{Param}}),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{1 \leq j \leq 33} Y_j = & x_1 x_2^6 x_3^{15} x_4^{16} x_5^3 + x_1 x_2^6 x_3^{19} x_4^5 x_5^{10} + x_1 x_2^6 x_3^{19} x_4^6 x_5^9 + x_1 x_2^6 x_3^{19} x_4^7 x_5^8 \\
& + x_1 x_2^6 x_3^{19} x_4^9 x_5^6 + x_1 x_2^6 x_3^{19} x_4^{10} x_5^5 + x_1 x_2^6 x_3^{19} x_4^{11} x_5^4 + x_1 x_2^6 x_3^{19} x_4^{12} x_5^3 \\
& + x_1 x_2^6 x_3^{23} x_4^3 x_5^8 + x_1 x_2^6 x_3^{23} x_4^8 x_5^3 + x_1 x_2^6 x_3^{23} x_4^9 x_5^2 + x_1 x_2^6 x_3^{25} x_4^3 x_5^6 \\
& + x_1 x_2^6 x_3^{25} x_4^5 x_5^4 + x_1 x_2^6 x_3^{25} x_4^7 x_5^2 + x_1 x_2^7 x_3^{19} x_4^4 x_5^{10} + x_1 x_2^7 x_3^{19} x_4^5 x_5^8 \\
& + x_1 x_2^7 x_3^{23} x_4^2 x_5^8 + x_1 x_2^7 x_3^{23} x_4^8 x_5^2 + x_1 x_2^7 x_3^{25} x_4^2 x_5^6 + x_1 x_2^7 x_3^{25} x_4^6 x_5^2 \\
& + x_1 x_2^8 x_3^{23} x_4^3 x_5^6 + x_1 x_2^8 x_3^{23} x_4^6 x_5^3 + x_1 x_2^9 x_3^{23} x_4^2 x_5^6 + x_1 x_2^9 x_3^{23} x_4^6 x_5^2 \\
& + x_1 x_2^{10} x_3^{17} x_4^7 x_5^6 + x_1 x_2^{10} x_3^{23} x_4^4 x_5^3 + x_1 x_2^{10} x_3^{23} x_4^5 x_5^2 + x_1 x_2^{11} x_3^{16} x_4^7 x_5^6 \\
& + x_1 x_2^{11} x_3^{18} x_4^7 x_5^4 + x_1 x_2^{12} x_3^{18} x_4^7 x_5^3 + x_1 x_2^{14} x_3^{16} x_4^7 x_5^3 + x_1 x_2^{14} x_3^{17} x_4^7 x_5^2 \\
& + x_1 x_2^{14} x_3^{19} x_4^4 x_5^3.
\end{aligned}$$

It is evident that $Y_j < T_2$ for each j . Finally, with the monomial T_3 , we have

$$\begin{aligned}
T_3 = & \left[Sq^1 \left(x_1^2 x_2^{15} x_3^{11} x_4^7 x_5^5 + x_1^2 x_2^{15} x_3^{11} x_4^9 x_5^3 + x_1^2 x_2^{15} x_3^{13} x_4^7 x_5^3 + x_1^2 x_2^{17} x_3^{11} x_4^7 x_5^3 \right) \right. \\
& + Sq^2 \left(x_1 x_2^{15} x_3^{10} x_4^7 x_5^6 + x_1 x_2^{15} x_3^{10} x_4^{10} x_5^3 + x_1 x_2^{15} x_3^{11} x_4^7 x_5^5 + x_1 x_2^{15} x_3^{11} x_4^9 x_5^3 \right. \\
& \quad + x_1 x_2^{15} x_3^{13} x_4^7 x_5^3 + x_1 x_2^{15} x_3^{14} x_4^7 x_5^2 + x_1 x_2^{17} x_3^{11} x_4^7 x_5^3 + x_1 x_2^{23} x_3^6 x_4^3 x_5^6 \\
& \quad + x_1 x_2^{23} x_3^6 x_4^6 x_5^3 + x_1 x_2^{23} x_3^6 x_4^7 x_5^2 + x_1 x_2^{23} x_3^7 x_4^2 x_5^6 + x_1 x_2^{23} x_3^7 x_4^6 x_5^2 \left. \right) \\
& + Sq^4 \left(x_1 x_2^{15} x_3^6 x_4^5 x_5^{10} + x_1 x_2^{15} x_3^6 x_4^6 x_5^9 + x_1 x_2^{15} x_3^6 x_4^{11} x_5^4 + x_1 x_2^{15} x_3^6 x_4^{12} x_5^3 \right. \\
& \quad + x_1 x_2^{15} x_3^7 x_4^8 x_5^6 + x_1 x_2^{15} x_3^7 x_4^{10} x_5^4 + x_1 x_2^{15} x_3^{11} x_4^4 x_5^6 + x_1 x_2^{15} x_3^{11} x_4^6 x_5^4 \\
& \quad + x_1 x_2^{15} x_3^{14} x_4^4 x_5^3 + x_1 x_2^{15} x_3^{14} x_4^5 x_5^2 + x_1 x_2^{21} x_3^6 x_4^3 x_5^6 + x_1 x_2^{21} x_3^6 x_4^6 x_5^3 \\
& \quad + x_1 x_2^{21} x_3^6 x_4^7 x_5^2 + x_1 x_2^{21} x_3^7 x_4^2 x_5^6 + x_1 x_2^{21} x_3^7 x_4^6 x_5^2 \left. \right) \\
& + Sq^8 \left(x_1 x_2^8 x_3^{11} x_4^7 x_5^6 + x_1 x_2^8 x_3^{14} x_4^7 x_5^3 + x_1 x_2^9 x_3^{10} x_4^7 x_5^6 + x_1 x_2^9 x_3^{14} x_4^7 x_5^2 \right. \\
& \quad + x_1 x_2^{10} x_3^{11} x_4^4 x_5^4 + x_1 x_2^{10} x_3^{12} x_4^7 x_5^3 + x_1 x_2^{11} x_3^6 x_4^7 x_5^8 + x_1 x_2^{11} x_3^6 x_4^9 x_5^6 \\
& \quad + x_1 x_2^{11} x_3^6 x_4^{10} x_5^5 + x_1 x_2^{11} x_3^6 x_4^{12} x_5^3 + x_1 x_2^{11} x_3^7 x_4^4 x_5^{10} + x_1 x_2^{11} x_3^7 x_4^6 x_5^8 \\
& \quad + x_1 x_2^{11} x_3^{10} x_4^5 x_5^6 + x_1 x_2^{11} x_3^{10} x_4^6 x_5^5 + x_1 x_2^{11} x_3^{10} x_4^7 x_5^4 + x_1 x_2^{11} x_3^{14} x_4^4 x_5^3 \\
& \quad + x_1 x_2^{11} x_3^{14} x_4^5 x_5^2 + x_1 x_2^{13} x_3^6 x_4^3 x_5^{10} + x_1 x_2^{13} x_3^6 x_4^{10} x_5^3 + x_1 x_2^{13} x_3^6 x_4^{11} x_5^2 \\
& \quad + x_1 x_2^{13} x_3^7 x_4^2 x_5^{10} + x_1 x_2^{13} x_3^7 x_4^{10} x_5^2 + x_1 x_2^{13} x_3^{10} x_4^3 x_5^6 + x_1 x_2^{13} x_3^{10} x_4^6 x_5^3 \\
& \quad + x_1 x_2^{13} x_3^{10} x_4^7 x_5^2 + x_1 x_2^{13} x_3^{11} x_4^2 x_5^6 + x_1 x_2^{13} x_3^{11} x_4^6 x_5^2 + x_1 x_2^{15} x_3^6 x_4^3 x_5^8 \\
& \quad + x_1 x_2^{15} x_3^6 x_4^5 x_5^6 + x_1 x_2^{15} x_3^6 x_4^6 x_5^5 + x_1 x_2^{15} x_3^6 x_4^7 x_5^4 + x_1 x_2^{15} x_3^6 x_4^8 x_5^3 \\
& \quad + x_1 x_2^{15} x_3^6 x_4^9 x_5^2 + x_1 x_2^{15} x_3^7 x_4^2 x_5^8 + x_1 x_2^{15} x_3^7 x_4^4 x_5^6 + x_1 x_2^{15} x_3^7 x_4^6 x_5^4 \\
& \quad + x_1 x_2^{15} x_3^7 x_4^8 x_5^2 + x_1 x_2^{15} x_3^8 x_4^3 x_5^6 + x_1 x_2^{15} x_3^8 x_4^6 x_5^3 + x_1 x_2^{15} x_3^8 x_4^7 x_5^2 \\
& \quad + x_1 x_2^{15} x_3^9 x_4^2 x_5^6 + x_1 x_2^{15} x_3^9 x_4^6 x_5^2 \left. \right) + \sum_{1 \leq j \leq 36} Z_j \Big] \mod (\mathcal{P}_5^{<\text{Param}}),
\end{aligned}$$

wherein

$$\begin{aligned}
\sum_{1 \leq j \leq 36} Z_j = & x_1 x_2^8 x_3^{19} x_4^7 x_5^6 + x_1 x_2^8 x_3^{22} x_4^7 x_5^3 + x_1 x_2^9 x_3^{18} x_4^7 x_5^6 + x_1 x_2^9 x_3^{22} x_4^7 x_5^2 \\
& + x_1 x_2^{10} x_3^{19} x_4^7 x_5^4 + x_1 x_2^{10} x_3^{20} x_4^7 x_5^3 + x_1 x_2^{11} x_3^6 x_4^7 x_5^{16} + x_1 x_2^{11} x_3^6 x_4^{17} x_5^6 \\
& + x_1 x_2^{11} x_3^6 x_4^{18} x_5^5 + x_1 x_2^{11} x_3^6 x_4^{20} x_5^3 + x_1 x_2^{11} x_3^7 x_4^4 x_5^{18} + x_1 x_2^{11} x_3^7 x_4^6 x_5^{16} \\
& + x_1 x_2^{11} x_3^{18} x_4^5 x_5^6 + x_1 x_2^{11} x_3^{18} x_4^6 x_5^5 + x_1 x_2^{11} x_3^{18} x_4^7 x_5^4 + x_1 x_2^{11} x_3^{22} x_4^4 x_5^3 \\
& + x_1 x_2^{11} x_3^{22} x_4^5 x_5^2 + x_1 x_2^{13} x_3^6 x_4^3 x_5^{18} + x_1 x_2^{13} x_3^6 x_4^{18} x_5^3 + x_1 x_2^{13} x_3^6 x_4^{19} x_5^2 \\
& + x_1 x_2^{13} x_3^7 x_4^2 x_5^{18} + x_1 x_2^{13} x_3^7 x_4^{18} x_5^2 + x_1 x_2^{13} x_3^{18} x_4^3 x_5^6 + x_1 x_2^{13} x_3^{18} x_4^6 x_5^3 \\
& + x_1 x_2^{13} x_3^{18} x_4^7 x_5^2 + x_1 x_2^{13} x_3^{19} x_4^2 x_5^6 + x_1 x_2^{13} x_3^{19} x_4^6 x_5^2 + x_1 x_2^{15} x_3^6 x_4^3 x_5^{16} \\
& + x_1 x_2^{15} x_3^6 x_4^{17} x_5^2 + x_1 x_2^{15} x_3^7 x_4^2 x_5^{16} + x_1 x_2^{15} x_3^7 x_4^{16} x_5^2 + x_1 x_2^{15} x_3^{16} x_4^3 x_5^6 \\
& + x_1 x_2^{15} x_3^{16} x_4^6 x_5^3 + x_1 x_2^{15} x_3^{17} x_4^2 x_5^6 + x_1 x_2^{15} x_3^{17} x_4^6 x_5^2 + x_1 x_2^{15} x_3^{18} x_4^4 x_5^3.
\end{aligned}$$

Evidently, Z_j is less than T_3 for any j .

Thus, we have successfully established that $T_1 \sim_{(4, \widetilde{\text{Param}})} \sum_{1 \leq j \leq 63} X_j$, $T_2 \sim_{(4, \widetilde{\text{Param}})} \sum_{1 \leq j \leq 33} Y_j$, and $T_3 \sim_{(4, \widetilde{\text{Param}})} \sum_{1 \leq j \leq 36} Z_j$. This, in turn, confirms the validity of the proposition. $\square$

Let $X \in (\mathcal{P}_5^{>0})_{n_1}$ denote an allowable monomial such that $\text{Param}(X) = \widetilde{\text{Param}}$. Let Param^* be defined as the parameter vector $(3, 2, 1, 1)$. We have $\text{Param}_1(X) = 3$ and can write $X = \mathcal{X}_{(\{i, j\}, 5)} Y^2$, wherein $1 \leq i < j \leq 5$ and Y is a monomial of degree 19 in $\mathcal{P}_5$. Since X belongs to $(\mathcal{C}_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$, we can apply Theorem 3.2.2 to conclude that Y must be an element of $(\mathcal{C}_{19}^{\otimes 5})^{\text{Param}^*}$. Upon performing a direct computation, we observe that for any $Z \in (\mathcal{C}_{19}^{\otimes 5})^{\text{Param}^*}$ and $1 \leq i < j \leq 5$, if $\mathcal{X}_{(\{i, j\}, 5)} Z^2 \neq X_t$ for $1 \leq t \leq 925$, then either $\mathcal{X}_{(\{i, j\}, 5)} Z^2$ belongs to the set of monomials delineated in Proposition 4.1.5, or it has the form FG^{16} , wherein G is a suitable monomial in $\mathcal{P}_5$, and F is a nonallowable monomial of degree 25 in the $\mathcal{A}$ -module $\mathcal{P}_5$. Thus, in light of Theorem 3.2.3, we infer that $[X_{(\{i, j\}, 5)} Z^2]_{\widetilde{\text{Param}}} = [X_{(\{i, j\}, 5)} Z^2] = [0]$. Since $X = \mathcal{X}_{(\{i, j\}, 5)} Y^2$ is allowable, $X = X_t$ for some t , $1 \leq t \leq 925$. This implies $(\mathcal{C}_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}} \subseteq \{X_t \mid 1 \leq t \leq 925\}$. Further, the set $\{X_t \mid 1 \leq t \leq 925\}$ is linearly independent in the space $(Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$. To verify this, we exploit a result in [Sum15] and Theorems 3.2.5 and 3.3.2, which are considered as useful support tools. Indeed, assume there is a linear relation $\mathcal{S} = \sum_{1 \leq t \leq 925} \gamma_t X_t \sim 0$, on which γ_t belongs to $\mathbb{F}_2$ for every t . Consider the homomorphisms $\mathfrak{p}_{(\ell, \mathcal{L})} : \mathcal{P}_5 \longrightarrow \mathcal{P}_4$, which are defined in Subsect. 3.3 for $m = 5$. According to Theorem 3.3.2, $\mathfrak{p}_{(\ell, \mathcal{L})}$ passes to a homomorphism from $(Q_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}}$ to $(Q_{n_1}^{\otimes 4})^{\widetilde{\text{Param}}}$. Moreover, $|(\mathcal{C}_{n_1}^{\otimes 4})^{\widetilde{\text{Param}}^{>0}}| = |(\mathcal{C}_{n_1}^{\otimes 4})^{>0}| = 165$ (cf. [Sum15]). By the usage of Theorems 3.2.5 and 3.3.2, we explicitly compute $\mathfrak{p}_{(\ell, \mathcal{L})}(\mathcal{S})$ in terms of allowable monomials in $(\mathcal{C}_{n_1}^{\otimes 4})^{\widetilde{\text{Param}}^{>0}}$ (modulo $\overline{\mathcal{A}}\mathcal{P}_4$). By means of explicit computations using the relations $\mathfrak{p}_{(\ell, \mathcal{L})}(\mathcal{S}) \sim 0$ where $\ell(\mathcal{L}) > 0$, we can deduce that $\gamma_t = 0$ for all t , $1 \leq t \leq 925$. This ultimately concludes the verification of the aforementioned claim and, hence, the theorem.

FINAL REMARK

In our prior research [Phu20b], we have demonstrated:

Proposition 4.1.6. *The following claims are each true:*

i) If $Y \in \mathcal{C}_{n_0}^{\otimes 5}$, then $\overline{\text{Param}} := \text{Param}(Y)$ is one of the following sequences:

$$\begin{aligned}
\overline{\text{Param}}_{[1]} &:= (2, 2, 1, 1), & \overline{\text{Param}}_{[2]} &:= (2, 2, 3), & \overline{\text{Param}}_{[3]} &:= (2, 4, 2), \\
\overline{\text{Param}}_{[4]} &:= (4, 1, 1, 1), & \overline{\text{Param}}_{[5]} &:= (4, 1, 3), & \overline{\text{Param}}_{[6]} &:= (4, 3, 2).
\end{aligned}$$

$$\text{ii) } |(\mathcal{C}_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[k]}}| = \begin{cases} 300 & \text{if } k = 1, \\ 15 & \text{if } k = 2, 5, \\ 10 & \text{if } k = 3, \\ 110 & \text{if } k = 4, \\ 280 & \text{if } k = 6. \end{cases}$$

It is relevant to note that $|(\mathcal{C}_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}}| = |(\mathcal{C}_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}^{>0}}|$ for $k = 2, 3$, and $|(\mathcal{C}_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[2]}^0}| = 0 = |(\mathcal{C}_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[3]}^0}|$. Moreover, $\dim(Q_{n_0}^{\otimes 5}) = \sum_{1 \leq k \leq 6} |(\mathcal{C}_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}}| = 730$. Using Corollary 2.2 (refer to Sect.2), one gets $Q_{n_0}^{\otimes 5} \cong \varphi(Q_{n_0}^{\otimes 5}) \cong \bigoplus_{1 \leq k \leq 6} (Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}}$, wherein $\varphi : Q_{n_0}^{\otimes 5} \longrightarrow Q_{n_1}^{\otimes 5}$, $[u] \longmapsto [x_1 \dots x_5 u^2]$, is a monomorphism and

$$\begin{aligned} \widehat{\text{Param}}_{[1]} &:= (5, 2, 2, 1, 1), & \widehat{\text{Param}}_{[2]} &:= (5, 2, 2, 2, 3), & \widehat{\text{Param}}_{[3]} &:= (5, 2, 4, 2), \\ \widehat{\text{Param}}_{[4]} &:= (5, 4, 1, 1, 1), & \widehat{\text{Param}}_{[5]} &:= (5, 4, 1, 3), & \widehat{\text{Param}}_{[6]} &:= (5, 4, 3, 2) \end{aligned}$$

are parameter vectors of degree n_1 . So, as demonstrated in the proof of Theorem 2.3, we obtain

$$\begin{aligned} Q_{n_1}^{\otimes 5} &\cong \text{Ker}((\widetilde{Sq_*^0})_{(5, n_1)}) \bigoplus \varphi(Q_{n_0}^{\otimes 5}) \cong (Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}^0} \bigoplus (Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}^{>0}} \bigoplus \left(\bigoplus_{1 \leq k \leq 6} (Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}} \right) \\ &\cong (Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}} \bigoplus \left(\bigoplus_{1 \leq k \leq 6} (Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}} \right), \end{aligned}$$

wherein $\dim(Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}} = \dim \text{Ker}((\widetilde{Sq_*^0})_{(5, n_1)}) = 1900$ and $\dim(Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}} = \dim(Q_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}}$ for every k . Drawing on the information presented and the fact that $Q_{n_s}^{\otimes 5}$ is isomorphic to $Q_{n_1}^{\otimes 5}$ for all s greater than or equal to 1, we have immediately

Corollary 4.1.7. *Conjecture 3.2.1 is valid for $m = 5$ in the generic degree $n_s = 5(2^s - 1) + 18 \cdot 2^s$ for any $s \geq 0$.*

4.2. The proof of Theorem 2.6

As already presented in Sect.2, in order to carry out the proof of Theorem 2.6, one must show: $(Q_{n_0}^{\otimes 5})^{G(5)} = \langle [\mathcal{R}'_4] \rangle$ and $(\text{Ker}((\widetilde{Sq_*^0})_{(5, n_1)}))^{G(5)} = 0$. These suffice to prove the technical proposition (Proposition 4.2.1) below.

Proposition 4.2.1. *Let $\widehat{\text{Param}}$ and $\widehat{\text{Param}}_{[k]}$ be the parameter vectors as in the proof of Theorem 2.3 and Proposition 4.1.6, respectively. Then, the following assertions each hold:*

- i) $((Q_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[k]}})^{G(5)} = 0$ with $k \in \{1, 2, 3, 5, 6\}$;
- ii) $((Q_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[4]}})^{G(5)} = \langle [\mathcal{R}'_4]_{\widehat{\text{Param}}_{[4]}} \rangle$, where

$$\begin{aligned} \mathcal{R}'_4 &= x_1 x_2 x_3 x_4 x_5^{14} + x_1 x_2 x_3 x_4^{14} x_5 + x_1 x_2 x_3^{14} x_4 x_5 + x_1 x_2^3 x_3 x_4 x_5^{12} \\ &\quad + x_1 x_2^3 x_3 x_4^{12} x_5 + x_1 x_2^3 x_3^{12} x_4 x_5 + x_1^3 x_2 x_3 x_4 x_5^{12} + x_1^3 x_2 x_3 x_4^{12} x_5 \\ &\quad + x_1^3 x_2 x_3^{12} x_4 x_5 + x_1^3 x_2^5 x_3 x_4 x_5^8 + x_1^3 x_2^5 x_3 x_4^8 x_5 + x_1^3 x_2^5 x_3^8 x_4 x_5; \end{aligned}$$

- iii) $((Q_{n_1}^{\otimes 5})^{\widehat{\text{Param}}})^{G(5)} = 0$.

Proof. We shall demonstrate the validity of item i) when $k = 1$, and provide a detailed proof for item ii). The remaining items can be established utilizing similar methodologies. Let $\widehat{\text{Param}}$ be a parameter vector of degree 5 and let $T_1, T_2, \dots, T_s$ be the monomials in $\mathcal{P}_5^{\leq \widehat{\text{Param}}_{[k]}}$ for $s \geq 1$. We set

$$\Sigma_5(T_1, T_2, \dots, T_s) = \{\sigma(T_j) : \sigma \in \Sigma_5, 1 \leq j \leq s\} \subset \mathcal{P}_5^{\leq \widehat{\text{Param}}_{[k]}}.$$

Proposition 4.1.6 establishes that the dimension of the cohit module $(Q_{n_0}^{\otimes 5})^{\widehat{\text{Param}}_{[1]}}$ is 300. Further, a basis for this space can be found in the set $\{[X_j] : 1 \leq j \leq 300\}$, where the monomials X_j , $1 \leq j \leq 300$, are described as in Subsect.5.2. Perhaps it should be noted again that $\widehat{\text{Param}}_{[1]}$ is a parameter vector of the minimal spike $x_1^{15} x_2^3 = x_1^{15} x_2^3 \prod_{3 \leq i \leq 5} x_i^0 \in (\mathcal{P}_5)_{n_0}$, and so, $[X]_{\widehat{\text{Param}}_{[1]}} = [X]$ for all $X \in (\mathcal{P}_5)_{n_0}$.

Lemma 4.2.2. We have $((Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}})^{\Sigma_5} = \langle [\mathcal{R}_i] : 1 \leq i \leq 5 \rangle$, where

$$\begin{aligned}\mathcal{R}_1 &:= \sum_{1 \leq j \leq 20} X_j, \quad \mathcal{R}_2 := \sum_{21 \leq j \leq 30} X_j, \quad \mathcal{R}_3 := \sum_{31 \leq j \leq 60} X_j, \\ \mathcal{R}_4 &:= \sum_{61 \leq j \leq 69} X_j + \sum_{73 \leq j \leq 84} X_j + \sum_{141 \leq j \leq 150} X_j, \quad \mathcal{R}_5 := X_{276} + X_{277} + X_{279} + X_{280}.\end{aligned}$$

Proof. Based on an admissible monomial basis for $(Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}}$, verifying the following isomorphism is not an overly arduous task:

$$(Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}} \cong \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \mathbb{U}_3 \oplus \mathbb{U}_4 \oplus \mathbb{U}_5 \oplus \mathbb{U}_6,$$

on which

$$\begin{aligned}\mathbb{U}_1 &:= \langle [\Sigma_5(X_1) \cup \Sigma_5(X_{21}) \cup \Sigma_5(X_{31})] \rangle &= \langle \{[X_j] : 1 \leq j \leq 60\} \rangle \subset (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}^0}, \\ \mathbb{U}_2 &:= \langle [\Sigma_5(X_{61}, X_{91}, X_{101}, X_{121}, X_{131}, X_{141})] \rangle &= \langle \{[X_j] : 61 \leq j \leq 150\} \rangle \subset (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}^0}, \\ \mathbb{U}_3 &:= \langle [\Sigma_5(X_{151}, X_{171}, X_{176}, X_{181})] \rangle &= \langle \{[X_j] : 151 \leq j \leq 195\} \rangle \subset (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}^0}, \\ \mathbb{U}_4 &:= \langle [\Sigma_5(X_{196}, X_{206}, X_{236}, X_{241}, X_{251}, X_{261}, X_{271})] \rangle &= \langle \{[X_j] : 196 \leq j \leq 275\} \rangle \subset (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}^0}, \\ \mathbb{U}_5 &:= \langle [\Sigma_5(X_{276}, X_{281}, X_{284}, X_{288}, X_{289})] \rangle &= \langle \{[X_j] : 276 \leq j \leq 290\} \rangle \subset (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}^{>0}}, \\ \mathbb{U}_6 &:= \langle [\Sigma_5(X_{291})] \rangle &= \langle \{[X_j] : 291 \leq j \leq 300\} \rangle \subset (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}^{>0}}\end{aligned}$$

are Σ_5 -submodules of $(Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}}$. We then can obtain the following description by performing direct calculations:

- $\mathbb{U}_1^{\Sigma_5} = \langle [\mathcal{R}_1], [\mathcal{R}_2], [\mathcal{R}_3] \rangle$, with $\mathcal{R}_1 := \sum_{1 \leq j \leq 20} X_j$, $\mathcal{R}_2 := \sum_{21 \leq j \leq 30} X_j$, and $\mathcal{R}_3 := \sum_{31 \leq j \leq 60} X_j$;
- $\mathbb{U}_2^{\Sigma_5} = \langle [\mathcal{R}_4 := \sum_{61 \leq j \leq 69} X_j + \sum_{73 \leq j \leq 84} X_j + \sum_{141 \leq j \leq 150} X_j] \rangle$;
- $\mathbb{U}_5^{\Sigma_5} = \langle [\mathcal{R}_5 := X_{276} + X_{277} + X_{279} + X_{280}] \rangle$;
- $\mathbb{U}_j^{\Sigma_5} = 0$, for $j = 3, 4, 6$.

The calculation of Σ_5 -invariant $\mathbb{U}_1^{\Sigma_5}$ will be detailed, and other invariants are computed similarly. Suppose $d = \sum_{1 \leq j \leq 60} \xi_j X_j$ with $\xi_j \in \mathbb{F}_2$, $1 \leq j \leq 60$, and $[d] \in \mathbb{U}_1^{\Sigma_5}$. Elementary calculations show:

$$\begin{aligned}\mathfrak{S}_1(d) + d &\sim \left((\xi_7 + \xi_{13})X_7 + (\xi_8 + \xi_{14})X_8 + (\xi_9 + \xi_{15})X_9 + (\xi_{10} + \xi_{17})X_{10} + (\xi_{11} + \xi_{18})X_{11} \right. \\ &\quad + (\xi_{16} + \xi_{20})X_{16} + (\xi_{24} + \xi_{27})X_{24} + (\xi_{25} + \xi_{28})X_{25} + (\xi_{26} + \xi_{29})X_{26} \\ &\quad + (\xi_{34} + \xi_{43})X_{43} + (\xi_{35} + \xi_{44})X_{35} + (\xi_{36} + \xi_{45})X_{36} + (\xi_{37} + \xi_{46})X_{37} \\ &\quad + (\xi_{38} + \xi_{47})X_{38} + (\xi_{39} + \xi_{48})X_{39} + (\xi_{40} + \xi_{55})X_{40} + (\xi_{41} + \xi_{56})X_{41} \\ &\quad \left. + (\xi_{42} + \xi_{57})X_{42} + (\xi_{49} + \xi_{58})X_{49} + (\xi_{50} + \xi_{59})X_{50} + (\xi_{51} + \xi_{60})X_{51} \right), \\ \mathfrak{S}_2(d) + d &\sim \left((\xi_3 + \xi_7)X_3 + (\xi_4 + \xi_8)X_4 + (\xi_5 + \xi_{10})X_5 + (\xi_6 + \xi_{11})X_6 + (\xi_9 + \xi_{12})X_9 \right. \\ &\quad + (\xi_5 + \xi_{10})X_5 + (\xi_6 + \xi_{11})X_6 + (\xi_9 + \xi_{12})X_9 + (\xi_{15} + \xi_{16})X_{15} \\ &\quad + (\xi_{19} + \xi_{20})X_{19} + (\xi_{22} + \xi_{24})X_{22} + (\xi_{23} + \xi_{25})X_{23} + (\xi_{29} + \xi_{30})X_{29} \\ &\quad + (\xi_{31} + \xi_{34})X_{31} + (\xi_{32} + \xi_{35})X_{32} + (\xi_{33} + \xi_{40})X_{33} + (\xi_{38} + \xi_{41})X_{38} \\ &\quad + (\xi_{39} + \xi_{42})X_{39} + (\xi_{45} + \xi_{52})X_{45} + (\xi_{46} + \xi_{53})X_{46} + (\xi_{47} + \xi_{49})X_{47} \\ &\quad \left. + (\xi_{48} + \xi_{50})X_{48} + (\xi_{51} + \xi_{54})X_{51} + (\xi_{56} + \xi_{58})X_{56} + (\xi_{57} + \xi_{59})X_{57} \right),\end{aligned}$$

$$\begin{aligned}
\mathfrak{S}_3(d) + d &\sim \left((\xi_1 + \xi_3)X_1 + (\xi_2 + \xi_5)X_2 + (\xi_4 + \xi_6)X_4 + (\xi_8 + \xi_9)X_8 + (\xi_{11} + \xi_{12})X_{11} \right. \\
&\quad + (\xi_{14} + \xi_{15})X_{14} + (\xi_{18} + \xi_{19})X_{18} + (\xi_{21} + \xi_{22})X_{21} + (\xi_{25} + \xi_{26})X_{25} \\
&\quad + (\xi_{28} + \xi_{29})X_{28} + (\xi_{32} + \xi_{33})X_{32} + (\xi_{34} + \xi_{36})X_{34} + (\xi_{35} + \xi_{38})X_{35} \\
&\quad + (\xi_{40} + \xi_{41})X_{40} + (\xi_{43} + \xi_{45})X_{43} + (\xi_{44} + \xi_{47})X_{44} + (\xi_{46} + \xi_{48})X_{46} \\
&\quad \left. + (\xi_{50} + \xi_{51})X_{50} + (\xi_{53} + \xi_{54})X_{53} + (\xi_{55} + \xi_{56})X_{55} + (\xi_{59} + \xi_{60})X_{59} \right), \\
\mathfrak{S}_4(d) + d &\sim \left((\xi_1 + \xi_2)X_1 + (\xi_3 + \xi_4)X_3 + (\xi_5 + \xi_6)X_5 + (\xi_7 + \xi_8)X_7 + (\xi_{10} + \xi_{11})X_{10} \right. \\
&\quad + (\xi_{13} + \xi_{14})X_{13} + (\xi_{17} + \xi_{18})X_{17} + (\xi_{22} + \xi_{23})X_{22} + (\xi_{24} + \xi_{25})X_{24} \\
&\quad + (\xi_{27} + \xi_{28})X_{27} + (\xi_{31} + \xi_{32})X_{31} + (\xi_{34} + \xi_{35})X_{34} + (\xi_{36} + \xi_{37})X_{36} \\
&\quad + (\xi_{38} + \xi_{39})X_{38} + (\xi_{41} + \xi_{42})X_{41} + (\xi_{43} + \xi_{44})X_{43} + (\xi_{45} + \xi_{46})X_{45} \\
&\quad + (\xi_{47} + \xi_{48})X_{47} + (\xi_{49} + \xi_{50})X_{49} + (\xi_{52} + \xi_{53})X_{52} + (\xi_{56} + \xi_{57})X_{56} \\
&\quad \left. + (\xi_{58} + \xi_{59})X_{58} \right),
\end{aligned}$$

where the $\mathcal{A}$ -homomorphisms $\mathfrak{S}_t : \mathcal{P}_5 \longrightarrow \mathcal{P}_5$ are given as in Notation 4.1(iv). Combining the above equalities and the relations $\mathfrak{S}_t(d) \sim d$, $1 \leq t \leq 4$, we immediately deduce that

$$\xi_t = \begin{cases} \xi_1 & \text{if } 2 \leq t \leq 20, \\ \xi_{21} & \text{if } 22 \leq t \leq 30, \\ \xi_{31} & \text{if } 32 \leq t \leq 60. \end{cases}$$

Thus, the aforementioned calculations lead us to the lemma we aimed to establish. $\square$

Now, let $[T] \in ((Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}})^{G(5)}$. By Lemma 4.2.2, we have $T \sim \sum_{1 \leq j \leq 5} \beta_j \mathcal{R}_j$ with $\beta_j \in \mathbb{F}_2$ for every j . A simple computation indicates:

$$\begin{aligned}
\mathfrak{S}_5(T) + T &\sim \left((\beta_1 + \beta_3)x_2^3x_5^{15} + (\beta_1 + \beta_4)x_2^{15}x_5^3 + (\beta_1 + \beta_2)x_1^3x_2^{15} + \beta_2(x_2^7x_5^{11}) \right. \\
&\quad \left. + \beta_5(x_2x_3^{14}x_4x_5^2) + \text{other terms} \right) \sim 0, \text{ which leads to } \beta_j = 0, j = 1, 2, \dots, 5.
\end{aligned}$$

This means that the invariant space $((Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[1]}})^{G(5)}$ vanishes, thereby concluding the proof of item *i*).

We proceed by introducing a lemma that will play a crucial role in establishing the proof of item *ii*).

Lemma 4.2.3. (I) We obtain a direct summand decomposition of the Σ_5 -module: $(Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[4]}} = \mathbb{V}_1 \oplus \mathbb{V}_2$, wherein

$$\begin{aligned}
\mathbb{V}_1 &:= \langle [\Sigma_5(Y_1) \cup \Sigma_5(Y_{21}) \cup \Sigma_5(Y_{51})]_{\overline{\text{Param}}_{[4]}} \rangle = \langle \{[Y_j]_{\overline{\text{Param}}_{[4]}} : 1 \leq j \leq 70\} \rangle, \\
\mathbb{V}_2 &:= \langle [\Sigma_5(Y_{71}, Y_{74}) \cup \Sigma_5(Y_{77}, Y_{78}, Y_{81}, Y_{82})]_{\overline{\text{Param}}_{[4]}} \rangle = \langle \{[Y_j]_{\overline{\text{Param}}_{[4]}} : 71 \leq j \leq 110\} \rangle.
\end{aligned}$$

(II) We have that $((Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[4]}})^{\Sigma_5} = \langle \{[\mathcal{R}'_i]_{\overline{\text{Param}}_{[4]}} : 1 \leq i \leq 4\} \rangle$, wherein

$$\mathcal{R}'_1 := \sum_{1 \leq j \leq 20} Y_j, \mathcal{R}'_2 := \sum_{21 \leq j \leq 50} Y_j, \mathcal{R}'_3 := \sum_{51 \leq j \leq 70} Y_j, \mathcal{R}'_4 := \sum_{96 \leq j \leq 107} Y_j.$$

Here, the monomials Y_j , $1 \leq j \leq 110$, are given as in Subsect. 5.2.

Outline of the proof. It is an elementary computation to prove the point (I). The next step is to establish the claim that $\mathbb{V}_1^{\Sigma_5} = \langle \{[\mathcal{R}'_1]_{\overline{\text{Param}}_{[4]}} , [\mathcal{R}'_2]_{\overline{\text{Param}}_{[4]}} , [\mathcal{R}'_3]_{\overline{\text{Param}}_{[4]}} \} \rangle$. This can be proven in a way similar to that of Lemma 4.2.2, which was presented earlier. (As a matter of fact, we will derive the desired description through an elementary but rather tedious computation, which we leave to the interested reader to perform.) Finally, we obtain $\mathbb{V}_2^{\Sigma_5} = \langle [\mathcal{R}'_4]_{\overline{\text{Param}}_{[4]}} \rangle$. Indeed, as previously observed, $\mathbb{V}_2$ is an $\mathbb{F}_2$ -vector space with dimension 40 and basis $\{[Y_j]_{\overline{\text{Param}}_{[4]}} : 71 \leq j \leq 110\}$.

By the usage of the relations $\mathfrak{S}_t(e) \sim_{\overline{\text{Param}}_{[4]}} e$, where $1 \leq t \leq 4$, $e = \sum_{71 \leq j \leq 110} \gamma_j Y_j$ with $\gamma_j \in \mathbb{F}_2$, $j = 71, \dots, 110$, and $[e]_{\overline{\text{Param}}_{[4]}} \in \mathbb{V}_2^{\Sigma^5}$, one can derive $\gamma_{96} = \gamma_{97} = \dots = \gamma_{107}$ and $\gamma_j = 0$ for $j \notin \{96, 97, \dots, 107\}$. This completes the desired statement. $\square$

Now, for any $[H]_{\overline{\text{Param}}_{[4]}} \in ((Q_{n_0}^{\otimes 5})^{\overline{\text{Param}}_{[4]}})^{G(5)}$, due to Lemma 4.2.3, one has

$$H \sim_{\overline{\text{Param}}_{[4]}} \gamma_1 \mathcal{R}'_1 + \gamma_2 \mathcal{R}'_2 + \gamma_3 \mathcal{R}'_3 + \gamma_4 \mathcal{R}'_4, \quad \gamma_j \in \mathbb{F}_2, \quad 1 \leq j \leq 4.$$

From Theorem 3.2.5 and the $\mathcal{A}$ -homomorphism $\mathfrak{S}_5 : \mathcal{P}_5 \longrightarrow \mathcal{P}_5$ mentioned in Notation 4.1(iv), we explicitly compute $\mathfrak{S}_5(H) + H$ in allowable terms Y_j (modulo $((\overline{\mathcal{A}}\mathcal{P}_5 \cap \mathcal{P}_5^{\leq \overline{\text{Param}}_{[4]}}) + \mathcal{P}_5^{< \overline{\text{Param}}_{[4]}}))$, and obtain

$$\begin{aligned} \mathfrak{S}_5(H) + H &\sim_{\overline{\text{Param}}_{[4]}} \left(\gamma_1 (x_1^3 x_2 x_3^{12} x_4 x_5) + (\gamma_2 + \gamma_3) (x_1 x_2 x_3^3 x_4^{12} x_5) + \gamma_3 (x_2 x_3^3 x_4^5 x_5^9) + \text{other terms} \right) \\ &\sim_{\overline{\text{Param}}_{[4]}} 0. \end{aligned}$$

The equality implies that $\gamma_1 = \gamma_2 = \gamma_3 = 0$, which in turn establishes the proposition. $\square$

4.3. The proof of Theorem 2.9

The methodology used to prove this theorem is identical to that employed in the proofs of Theorems 2.3 and 2.6. As such, those readers desiring to attain a more profound understanding of this subject matter may do so by explicating any details that have been omitted in this section.

We observe that as $\beta(12) = 4$, by Theorem 2.1(I), $Q_{12}^{\otimes m} = 0$ for $m \leq 3$. As it is known, $Q_{12}^{\otimes m} \cong (Q_{12}^{\otimes m})^0 \bigoplus (Q_{12}^{\otimes m})^{>0}$, and so, we need to determine the dimensions of $(Q_{12}^{\otimes m})^0$ and $(Q_{12}^{\otimes m})^{>0}$. Thanks to the results by [MKR16], we obtain

Corollary 4.3.1. *For each integer $m \geq 4$, suppose X is an allowable monomial in $(\mathcal{P}_m)_{12}$. Then the parameter vector of X is one of the following sequences:*

$$\begin{aligned} \text{Param}_{(1)} &:= (4, 2, 1), \quad \text{Param}_{(2)} := (4, 4), \quad \text{Param}_{(3)} := (6, 1, 1), \quad \text{Param}_{(4)} := (6, 3), \\ \text{Param}_{(5)} &:= (8, 2), \quad \text{Param}_{(6)} := (10, 1), \quad \text{Param}_{(7)} := (12, 0). \end{aligned}$$

Indeed, as is well-known, $(Q_{12}^{\otimes m})^0 = \langle [\widetilde{\Phi}^0(\mathcal{C}_{12}^{\otimes(m-1)})] \rangle$, and for every $m > 12$, we have $Q_{12}^{\otimes m} = (Q_{12}^{\otimes m})^0$. So, we only need to determine parameter vectors of $X \in (\mathcal{P}_m)_{12}$ with $4 \leq m \leq 12$. A direct calculation shows:

$$\text{Param}(X) \in \begin{cases} \{\text{Param}_{(j)} : 1 \leq j \leq 2\} & \text{if } 4 \leq m \leq 5 \text{ (see also [Sum15, PS17])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 4\} & \text{if } m = 6 \text{ (see also [Tin22])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 4\} & \text{if } m = 7, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 5\} & \text{if } 8 \leq m \leq 9, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 6\} & \text{if } 10 \leq m \leq 11, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 7\} & \text{if } m = 12. \end{cases}$$

For instance, consider $m = 7$, since $\beta(12) = 4$ and $\deg(X) = 12$ is an even number, either $\text{Param}_1(X) = 4$ or $\text{Param}_1(X) = 6$. If $\text{Param}_1(X) = 4$, then X is of the form $x_a x_b x_c x_d u^2$ where $1 \leq a < b < c < d \leq 7$ and $u \in (\mathcal{P}_7)_4 = (\mathcal{P}_7)_4^0$. As X is allowable, by Theorem 3.2.2, u is also an allowable monomial. According to [MKR16], we have $\dim Q_4^{\otimes 7} = 2 \cdot \binom{7}{2} + 2 \cdot \binom{7}{3} + \binom{7}{4} = 147$.

A monomial basis for $Q_4^{\otimes 7}$ is a set consisting of all classes represented by the allowable monomials u of the following forms:

$$\begin{aligned} &x_i x_j^3, \quad \text{for } 1 \leq i, j \leq 7, i \neq j, \\ &x_i x_j x_k x_\ell, \quad \text{for } 1 \leq i < j < k < \ell \leq 7, \\ &x_i x_j^2 x_k, \quad \text{for } i < j, 1 \leq i, j, k \leq 7, i \neq k, j \neq k. \end{aligned}$$

Obviously $\text{Param}(x_i x_j^3) = \text{Param}(x_i x_j^2 x_k) = (2, 1)$ and $\text{Param}(x_i x_j x_k x_\ell) = (4)$, which imply that either $\text{Param}(u) = (2, 1)$ or $\text{Param}(u) = (4)$. So, either $\text{Param}(X) = \text{Param}_{(1)}$ or $\text{Param}(X) = \text{Param}_{(2)}$. Similarly, if $\text{Param}_1(X) = 6$, then X can be represented as $x_a x_b x_c x_d x_e x_f v^2$ where v is an allowable monomial of degree 3 in $\mathcal{P}_7$, and $1 \leq a < b < c < d < e < f \leq 7$. According to the calculations carried out in [MKR16], the dimension of $Q_3^{\otimes 7}$ is determined to be equal to $\sum_{1 \leq j \leq 3} \binom{7}{j} = 63$. Then the parameter vector of v is either $(1, 1)$ or (3) . This leads to either $\text{Param}(X) = \text{Param}_{(3)}$ or $\text{Param}(X) = \text{Param}_{(4)}$. In summary, if $X \in \mathcal{C}_{12}^{\otimes 7}$, then $\text{Param}(X) \in \{\text{Param}_{(j)} : 1 \leq j \leq 4\}$. As with previous results (cf. Theorem 2.3, Corollary 4.3.1 and the papers [MKR16, PS17, Phu23a, Phu22b, Sum15]), by direct calculations, one obtains the descriptions of the indecomposables $(Q_{12}^{\otimes m})^0$ and $(Q_{12}^{\otimes m})^{>0}$ below.

Corollary 4.3.2. *For any $m \geq 4$, we have the following isomorphisms:*

$$(Q_{12}^{\otimes m})^0 \cong \begin{cases} 0 & \text{if } m = 4, \\ \bigoplus_{1 \leq j \leq 2} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} & \text{if } 5 \leq m \leq 6 \text{ (see also [PS17, Tin22])}, \\ \bigoplus_{1 \leq j \leq 4} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} & \text{if } 7 \leq m \leq 8, \\ \bigoplus_{1 \leq j \leq 5} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} & \text{if } 9 \leq m \leq 10, \\ \bigoplus_{1 \leq j \leq 6} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} & \text{if } 11 \leq m \leq 12, \\ \bigoplus_{1 \leq j \leq 7} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} & \text{if } m \geq 13, \end{cases}$$

and

$$(Q_{12}^{\otimes m})^{>0} \cong \begin{cases} \bigoplus_{1 \leq j \leq 2} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} & \text{if } 4 \leq m \leq 5 \text{ (see also [Sum15, PS17])}, \\ \bigoplus_{1 \leq j \leq 4} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} & \text{if } m = 6 \text{ (see also [Tin22])}, \\ \bigoplus_{1 \leq j \leq 4} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} & \text{if } m = 7, \\ \bigoplus_{2 \leq j \leq 5} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} & \text{if } m = 8, \\ \bigoplus_{4 \leq j \leq 5} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} & \text{if } m = 9, \\ \bigoplus_{5 \leq j \leq 6} (Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} & \text{if } m = 10, \\ (Q_{12}^{\otimes m})^{\text{Param}_{(6)}^{>0}} & \text{if } m = 11, \\ (Q_{12}^{\otimes m})^{\text{Param}_{(7)}^{>0}} & \text{if } m = 12, \\ 0 & \text{if } m \geq 13, \end{cases}$$

Consequently, the dimensions of the cohit modules $(Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0}$ and $(Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}}$ are determined as follows:

$$\dim(Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} = \begin{cases} 20 \cdot \binom{5}{4} & \text{if } m = 5 \text{ and } j = 1 \text{ (cf. [PS17])}, \\ \binom{5}{4} & \text{if } m = 5 \text{ and } j = 2 \text{ (cf. [PS17])}, \\ 20 \cdot \binom{6}{4} + 75 \cdot \binom{6}{5} & \text{if } m = 6 \text{ and } j = 1, \\ \binom{6}{4} + 10 \cdot \binom{6}{5} & \text{if } m = 6 \text{ and } j = 2, \end{cases}$$

$$\dim(Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} = \left\{ \begin{array}{ll} 20. \binom{7}{4} + 75. \binom{7}{5} + 90. \binom{7}{6} & \text{if } m = 7 \text{ and } j = 1, \\ \binom{7}{4} + 10. \binom{7}{5} + 45. \binom{7}{6} & \text{if } m = 7 \text{ and } j = 2, \\ 21. \binom{7}{6} & \text{if } m = 7 \text{ and } j = 3, \\ 20. \binom{7}{6} & \text{if } m = 7 \text{ and } j = 4, \\ 20. \binom{8}{4} + 75. \binom{8}{5} + 90. \binom{8}{6} + 35. \binom{8}{7} & \text{if } m = 8 \text{ and } j = 1, \\ \binom{8}{4} + 10. \binom{8}{5} + 45. \binom{8}{6} + 70. \binom{8}{7} & \text{if } m = 8 \text{ and } j = 2, \\ 21. \binom{8}{6} + 42. \binom{8}{7} & \text{if } m = 8 \text{ and } j = 3, \\ 20. \binom{8}{6} + 84. \binom{8}{7} & \text{if } m = 8 \text{ and } j = 4, \\ 20. \binom{9}{4} + 75. \binom{9}{5} + 90. \binom{9}{6} + 35. \binom{9}{7} & \text{if } m = 9 \text{ and } j = 1, \\ \binom{9}{4} + 10. \binom{9}{5} + 45. \binom{9}{6} + 70. \binom{9}{7} + 34. \binom{9}{8} & \text{if } m = 9 \text{ and } j = 2, \\ 21. \binom{9}{6} + 42. \binom{9}{7} + 21. \binom{9}{8} & \text{if } m = 9 \text{ and } j = 3, \\ 20. \binom{9}{6} + 84. \binom{9}{7} + 120. \binom{9}{8} & \text{if } m = 9 \text{ and } j = 4, \\ 28. \binom{9}{8} & \text{if } m = 9 \text{ and } j = 5, \\ 20. \binom{10}{4} + 75. \binom{10}{5} + 90. \binom{10}{6} + 35. \binom{10}{7} & \text{if } m = 10 \text{ and } j = 1, \\ \binom{10}{4} + 10. \binom{10}{5} + 45. \binom{10}{6} + 70. \binom{10}{7} + 34. \binom{10}{8} & \text{if } m = 10 \text{ and } j = 2, \\ 21. \binom{10}{6} + 42. \binom{10}{7} + 21. \binom{10}{8} & \text{if } m = 10 \text{ and } j = 3, \\ 20. \binom{10}{6} + 84. \binom{10}{7} + 120. \binom{10}{8} + 56. \binom{10}{9} & \text{if } m = 10 \text{ and } j = 4, \\ 28. \binom{10}{8} + 63. \binom{10}{9} & \text{if } m = 10 \text{ and } j = 5, \\ 20. \binom{11}{4} + 75. \binom{11}{5} + 90. \binom{11}{6} + 35. \binom{11}{7} & \text{if } m = 11 \text{ and } j = 1, \\ \binom{11}{4} + 10. \binom{11}{5} + 45. \binom{11}{6} + 70. \binom{11}{7} + 34. \binom{11}{8} & \text{if } m = 11 \text{ and } j = 2, \\ 21. \binom{11}{6} + 42. \binom{11}{7} + 21. \binom{11}{8} & \text{if } m = 11 \text{ and } j = 3, \\ 20. \binom{11}{6} + 84. \binom{11}{7} + 120. \binom{11}{8} + 56. \binom{11}{9} & \text{if } m = 11 \text{ and } j = 4, \\ 28. \binom{11}{8} + 63. \binom{11}{9} + 35. \binom{11}{10} & \text{if } m = 11 \text{ and } j = 5, \\ 10. \binom{11}{10} & \text{if } m = 11 \text{ and } j = 6, \end{array} \right.$$

$$\dim(Q_{12}^{\otimes m})^{\text{Param}_{(j)}^0} = \begin{cases} 20 \cdot \binom{12}{4} + 75 \cdot \binom{12}{5} + 90 \cdot \binom{12}{6} + 35 \cdot \binom{12}{7} & \text{if } m = 12 \text{ and } j = 1, \\ \binom{12}{4} + 10 \cdot \binom{12}{5} + 45 \cdot \binom{12}{6} + 70 \cdot \binom{12}{7} + 34 \cdot \binom{12}{8} & \text{if } m = 12 \text{ and } j = 2, \\ 21 \cdot \binom{12}{6} + 42 \cdot \binom{12}{7} + 21 \cdot \binom{12}{8} & \text{if } m = 12 \text{ and } j = 3, \\ 20 \cdot \binom{12}{6} + 84 \cdot \binom{12}{7} + 120 \cdot \binom{12}{8} + 56 \cdot \binom{12}{9} & \text{if } m = 12 \text{ and } j = 4, \\ 28 \cdot \binom{12}{8} + 63 \cdot \binom{12}{9} + 35 \cdot \binom{12}{10} & \text{if } m = 12 \text{ and } j = 5, \\ 10 \cdot \binom{12}{10} + 10 \cdot \binom{12}{11} & \text{if } m = 12 \text{ and } j = 6, \\ 20 \cdot \binom{m}{4} + 75 \cdot \binom{m}{5} + 90 \cdot \binom{m}{6} + 35 \cdot \binom{m}{7} & \text{if } m \geq 13 \text{ and } j = 1, \\ \binom{m}{4} + 10 \cdot \binom{m}{5} + 45 \cdot \binom{m}{6} + 70 \cdot \binom{m}{7} + 34 \cdot \binom{m}{8} & \text{if } m \geq 13 \text{ and } j = 2, \\ 21 \cdot \binom{m}{6} + 42 \cdot \binom{m}{7} + 21 \cdot \binom{m}{8} & \text{if } m \geq 13 \text{ and } j = 3, \\ 20 \cdot \binom{m}{6} + 84 \cdot \binom{m}{7} + 120 \cdot \binom{m}{8} + 56 \cdot \binom{m}{9} & \text{if } m \geq 13 \text{ and } j = 4, \\ 28 \cdot \binom{m}{8} + 63 \cdot \binom{m}{9} + 35 \cdot \binom{m}{10} & \text{if } m \geq 13 \text{ and } j = 5, \\ 10 \cdot \binom{m}{10} + 10 \cdot \binom{m}{11} & \text{if } m \geq 13 \text{ and } j = 6, \\ \binom{m}{12} & \text{if } m \geq 13 \text{ and } j = 7, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim(Q_{12}^{\otimes m})^{\text{Param}_{(j)}^{>0}} = \begin{cases} 75 & \text{if } m = 5 \text{ and } j = 1 \text{ (see also [PS17])}, \\ 10 & \text{if } m = 5 \text{ and } j = 2 \text{ (see also [PS17])}, \\ 90 & \text{if } m = 6 \text{ and } j = 1 \text{ (see also [Tin22])}, \\ 45 & \text{if } m = 6 \text{ and } j = 2 \text{ (see also [Tin22])}, \\ 21 & \text{if } m = 6 \text{ and } j = 3, \\ 20 & \text{if } m = 6 \text{ and } j = 4, \\ 35 & \text{if } m = 7 \text{ and } j = 1, \\ 70 & \text{if } m = 7 \text{ and } j = 2, \\ 42 & \text{if } m = 7 \text{ and } j = 3, \\ 84 & \text{if } m = 7 \text{ and } j = 4, \\ 34 & \text{if } m = 8 \text{ and } j = 2 \text{ (see also [Phu23a, Phu22b])}, \\ 21 & \text{if } m = 8 \text{ and } j = 3 \text{ (see also [Phu23a, Phu22b])}, \\ 120 & \text{if } m = 8 \text{ and } j = 4 \text{ (see also [Phu23a, Phu22b])}, \\ 28 & \text{if } m = 8 \text{ and } j = 5 \text{ (see also [Phu23a, Phu22b])}, \\ 56 & \text{if } m = 9 \text{ and } j = 4, \\ 63 & \text{if } m = 9 \text{ and } j = 5, \\ 35 & \text{if } m = 10 \text{ and } j = 5, \\ 10 & \text{if } m = 10 \text{ and } j = 6, \\ 10 & \text{if } m = 11 \text{ and } j = 6, \\ 1 & \text{if } m = 12 \text{ and } j = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Now, as was presented at the end of Sect. 2, we only need to prove the theorem for $m \geq 9$. Notice that for each $m \geq 13$, since $Q_{12}^{\otimes m} = (Q_{12}^{\otimes m})^0$, the $G(m)$ -coinvariant $(\mathbb{F}_2 \otimes_{G(m)} P_{\mathcal{A}}(\mathcal{P}_m)^*)_{12}$ vanishes. For $9 \leq m \leq 12$, due to Corollaries 4.3.1 and 4.3.2 and the fact that Kameko's squaring operation $(\widetilde{Sq_*^0})_{(m,12)} : Q_{12}^{\otimes m} \longrightarrow Q_{(12-m)/2}^{\otimes m}$ is an epimorphism for $m = 10, 12$, we must have the following isomorphisms:

$$Q_{12}^{\otimes m} \cong \begin{cases} \bigoplus_{1 \leq j \leq 3} (Q_{12}^{\otimes m})^{\text{Param}_j^0} \bigoplus \left(\bigoplus_{4 \leq j \leq 5} (Q_{12}^{\otimes m})^{\text{Param}_j} \right) & \text{if } m = 9, \\ \bigoplus_{1 \leq j \leq 5} (Q_{12}^{\otimes m})^{\text{Param}_j^0} \bigoplus (Q_{12}^{\otimes m})^{\text{Param}_{(6)}} & \text{if } m = 11, \\ \text{Ker}((\widetilde{Sq_*^0})_{(m,12)}) \bigoplus Q_{(12-m)/2}^{\otimes m}, & m = 10, 12, \\ \bigoplus_{1 \leq j \leq 4} (Q_{12}^{\otimes m})^{\text{Param}_j^0} \bigoplus \left(\bigoplus_{5 \leq j \leq 6} (Q_{12}^{\otimes m})^{\text{Param}_j} \right) \bigoplus \langle \{[x_i]_{(1,0)} : 1 \leq i \leq 10\} \rangle & \text{if } m = 10, \\ \bigoplus_{1 \leq j \leq 6} (Q_{12}^{\otimes m})^{\text{Param}_j^0} \bigoplus \langle [x_1 x_2 \dots x_{12}]_{\text{Param}_{(7)}} \rangle & \text{if } m = 12. \end{cases}$$

It must be noted that Mothebe et al. showed in [MKR16, Theorem 2] that $\dim Q_{12}^{\otimes 9} = 38392$. This result, however, is incorrect. Indeed, by the above isomorphism and Corollary 4.3.2, we derive $\dim Q_{12}^{\otimes 9} = \sum_{1 \leq j \leq 3} \dim(Q_{12}^{\otimes 9})^{\text{Param}_j^0} + \sum_{4 \leq j \leq 5} \dim(Q_{12}^{\otimes 9})^{\text{Param}_j} = 38402$. Then, for $9 \leq m \leq 11$, the $G(m)$ -invariants space $(Q_1^{\otimes m})^{G(m)}$ and $((Q_{12}^{\otimes m})^{\text{Param}_j^0})^{G(m)}$ ($1 \leq j \leq 5$) are trivial. So, based on the above isomorphism, the following estimates can be derived:

$$\dim(Q_{12}^{\otimes m})^{G(m)} \leq \begin{cases} \sum_{4 \leq j \leq 5} \dim((Q_{12}^{\otimes m})^{\text{Param}_j})^{G(m)} & \text{if } m = 9, \\ \sum_{5 \leq j \leq 6} \dim((Q_{12}^{\otimes m})^{\text{Param}_j})^{G(m)} & \text{if } m = 10, \\ \dim((Q_{12}^{\otimes m})^{\text{Param}_{(6)}})^{G(m)} & \text{if } m = 11. \end{cases}$$

By combining the homomorphisms $\mathfrak{S}_t : \mathcal{P}_m \longrightarrow \mathcal{P}_m$, $1 \leq t \leq 11$, $9 \leq m \leq 11$ (cf. Notation 4.1(iv)) with Corollary 4.3.2 and our previous results [Sum15, PS15], we state that all spaces of $G(m)$ -invariants $((Q_{12}^{\otimes m})^{\text{Param}_j})^{G(m)}$ are trivial. For instance, with $m = 11$, by Corollary 4.3.2, one has an isomorphism $(Q_{12}^{\otimes 11})^{\text{Param}_{(6)}} \cong (Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^0} \bigoplus (Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^{>0}}$, where $\dim(Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^0} = 110$ and $\dim(Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^{>0}} = 10$.

It is straightforward to see that an allowable basis for $(Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^{>0}}$ consists of all classes represented by the following monomials:

$$\begin{aligned} a_1 &= x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}^2, & a_2 &= x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}^2 x_{11}, \\ a_3 &= x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9^2 x_{10} x_{11}, & a_4 &= x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8^2 x_9 x_{10} x_{11}, \\ a_5 &= x_1 x_2 x_3 x_4 x_5 x_6 x_7^2 x_8 x_9 x_{10} x_{11}, & a_6 &= x_1 x_2 x_3 x_4 x_5 x_6^2 x_7 x_8 x_9 x_{10} x_{11}, \\ a_7 &= x_1 x_2 x_3 x_4 x_5^2 x_6 x_7 x_8 x_9 x_{10} x_{11}, & a_8 &= x_1 x_2 x_3 x_4^2 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}, \\ a_9 &= x_1 x_2 x_3^2 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}, & a_{10} &= x_1 x_2^2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}. \end{aligned}$$

Under the action of the symmetric group Σ_{11} , we get $((Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^{>0}})^{\Sigma_{11}} = 0$. Indeed suppose that $[f]_{\text{Param}_{(6)}} \in ((Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^{>0}})^{\Sigma_{11}}$. Then, we have $f \sim_{\text{Param}_{(6)}} \sum_{1 \leq j \leq 10} \gamma_j a_j$ with $\gamma_j \in \mathbb{F}_2$, $j = 1, 2, \dots, 10$.

Let us consider the homomorphisms $\mathfrak{S}_t : \mathcal{P}_{11} \longrightarrow \mathcal{P}_{11}$, $1 \leq t \leq 10$. An easy calculation shows that

$$\begin{aligned} \mathfrak{S}_1(f) &\sim_{\text{Param}_{(6)}} \sum_{1 \leq j \leq 9} (\gamma_j + \gamma_{10}) a_j + \gamma_{10} a_{10}, & \mathfrak{S}_2(f) &\sim_{\text{Param}_{(6)}} \sum_{1 \leq j \leq 8} \gamma_j a_j + \gamma_9 a_{10} + \gamma_{10} a_9, \\ \mathfrak{S}_3(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 8, 9} \gamma_j a_j + \gamma_8 a_9 + \gamma_9 a_8, & \mathfrak{S}_4(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 7, 8} \gamma_j a_j + \gamma_7 a_8 + \gamma_8 a_7, \\ \mathfrak{S}_5(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 6, 7} \gamma_j a_j + \gamma_6 a_7 + \gamma_7 a_6, & \mathfrak{S}_6(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 5, 6} \gamma_j a_j + \gamma_5 a_6 + \gamma_6 a_5, \end{aligned}$$

$$\begin{aligned}\mathfrak{S}_7(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 4, 5} \gamma_j a_j + \gamma_4 a_5 + \gamma_5 a_4, & \mathfrak{S}_8(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 3, 4} \gamma_j a_j + \gamma_3 a_4 + \gamma_4 a_3, \\ \mathfrak{S}_9(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 2, 3} \gamma_j a_j + \gamma_2 a_3 + \gamma_3 a_2, & \mathfrak{S}_{10}(f) &\sim_{\text{Param}_{(6)}} \sum_{j \neq 1, 2} \gamma_j a_j + \gamma_1 a_2 + \gamma_2 a_1.\end{aligned}$$

By these equalities and the relations $\mathfrak{S}_t(f) + f \sim_{\text{Param}_{(6)}} 0$, $1 \leq t \leq 10$ we get $\gamma_j = 0$, for every j . Thus, $((Q_{12}^{\otimes 11})^{\text{Param}_{(6)}})^{G(11)} = ((Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^0})^{G(11)}$. Using the maps $\mathfrak{S}_t : \mathcal{P}_{11} \longrightarrow \mathcal{P}_{11}$, $1 \leq t \leq 11$, we can easily deduce that $((Q_{12}^{\otimes 11})^{\text{Param}_{(6)}^0})^{G(11)} = 0$, and so is, $((Q_{12}^{\otimes 11})^{\text{Param}_{(6)}})^{G(11)}$. Thus, by the above inequality, $(Q_{12}^{\otimes m})^{G(m)}$ is trivial for $9 \leq m \leq 11$. Similarly, since

$$Q_{12}^{\otimes 12} \cong \bigoplus_{1 \leq j \leq 6} (Q_{12}^{\otimes 12})^{\text{Param}_{(j)}^0} \bigoplus ([x_1 x_2 \dots x_{12}]_{\text{Param}_{(7)}}),$$

by Corollary 4.3.2 and the homomorphisms $\mathfrak{S}_t : \mathcal{P}_{12} \longrightarrow \mathcal{P}_{12}$ ($1 \leq t \leq 12$), it may be claimed that $\dim(Q_{12}^{\otimes 12})^{G(12)} = 0$. (However, computing this $G(12)$ -invariant is also quite complicated.) Then, by dualizing, $(\mathbb{F}_2 \otimes_{G(m)} P_{\mathcal{A}}(\mathcal{P}_m)^*)_{12} = 0$ for $9 \leq m \leq 12$. Now, the theorem follows from the fact that by Tangora [Tan70], Lin [Lin08], Chen [Che12], Bruner and Rognes [Bru97, BR22], the cohomology groups $\text{Ext}_{\mathcal{A}}^{m, m+12}(\mathbb{F}_2, \mathbb{F}_2)$ are trivial for $m \geq 1$.

FINAL REMARK

Thanks to the statement of Corollary 4.3.2, it follows that

Corollary 4.3.3. *Conjecture 3.2.1 holds for all $m \geq 1$ and degree 12.*

The previous findings in [MKR16] and [Phu22b] allow us to easily verify the following technical assertion.

Corollary 4.3.4. *Let us consider the parameter vectors:*

$$\begin{aligned}\text{Param}_{(1, m)} &:= (m, 0), & \text{Param}_{(2, m)} &:= (m-1, 1), & \text{Param}_{(3, m)} &:= (m-2, 2), \\ \text{Param}_{(4, m)} &:= (m, 1), & \text{Param}_{(5, m)} &:= (m-1, 2).\end{aligned}$$

Then the dimensions of the cohit modules $(Q_{m+j}^{\otimes m})^{\text{Param}_{(k, m)}}$, $0 \leq j \leq 3$, $1 \leq k \leq 5$, are determined as follows:

- $\dim(Q_m^{\otimes m})^{\text{Param}_{(1, m)}} = \dim(Q_m^{\otimes m})^{\text{Param}_{(1, m)}^0} = 1$, for any $m > 0$;
- $\dim(Q_{m+1}^{\otimes m})^{\text{Param}_{(2, m)}} = \dim(Q_{m+1}^{\otimes m})^{\text{Param}_{(2, m)}^0} + \dim(Q_{m+1}^{\otimes m})^{\text{Param}_{(2, m)}^{>0}} = m^2 - 1$, for arbitrary $m > 1$, wherein $\dim(Q_{m+1}^{\otimes m})^{\text{Param}_{(2, m)}^0} = m(m-1)$, and $\dim(Q_{m+1}^{\otimes m})^{\text{Param}_{(2, m)}^{>0}} = m-1$;
- $\dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(3, m)}} = \dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(3, m)}^0} + \dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(3, m)}^{>0}}$
 $= \left(\binom{m}{2} + m \right) \left(\binom{m-1}{2} - 1 \right)$ for all $m > 3$, wherein
 $\dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(3, m)}^0} = \left[\binom{m}{2} + m - 1 \right] \left[\binom{m-1}{2} - 1 \right]$, and $\dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(3, m)}^{>0}} = \binom{m-1}{2} - 1$;
- $\dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(4, m)}} = \dim(Q_{m+2}^{\otimes m})^{\text{Param}_{(4, m)}^0} = m$, for every $m > 3$;
- $\dim(Q_{m+3}^{\otimes m})^{\text{Param}_{(5, m)}} = \dim(Q_{m+3}^{\otimes m})^{\text{Param}_{(5, m)}^0} + \dim(Q_{m+3}^{\otimes m})^{\text{Param}_{(5, m)}^{>0}} = 3 \binom{m}{m-3} + m(m-2)$ for
all $m > 3$, wherein $\dim(Q_{m+3}^{\otimes m})^{\text{Param}_{(5, m)}^0} = 3 \binom{m}{m-3}$, and $\dim(Q_{m+3}^{\otimes m})^{\text{Param}_{(5, m)}^{>0}} = m(m-2)$.

It should be noted that $\text{Param}_{(1, 12)} = \text{Param}_{(7)}$, $\text{Param}_{(2, 11)} = \text{Param}_{(6)}$, $\text{Param}_{(3, 10)} = \text{Param}_{(5)}$, $\text{Param}_{(4, 10)} = \text{Param}_{(6)}$, and $\text{Param}_{(5, 9)} = \text{Param}_{(5)}$, wherein the parameter vectors $\text{Param}_{(j)}$, $5 \leq j \leq 7$, are described as in Corollary 4.3.2.

As a result, Conjecture 3.2.1 is confirmed for parameter vectors $\text{Param}_{(j, m)}$ with $1 \leq j \leq 5$.

5. Appendix

In this section, our first objective is to study Conjecture 3.2.1 for any $m \geq 1$ and parameter vectors of degree ≤ 8 . Afterwards, as previously mentioned at the end of Section 2, we will provide a clear description of the set $(\mathcal{C}_{n_1}^{\otimes 5})^{\widetilde{\text{Param}}^{>0}}$. Finally, we present a comprehensive list of allowable monomials in $\mathcal{C}_{n_0}^{\otimes 5}$, along with an enumeration of the strictly nonallowable monomials in $(\mathcal{P}_5^{>0})_{n_1}$.

5.1. On Conjecture 3.2.1 for any $m \geq 1$ and parameter vectors of degrees ≤ 8

Conjecture 3.2.1 has already been shown to hold for parameter vectors of degrees ≤ 8 and $1 \leq m \leq 5$ in previous works, such as [Pet87, Kam90, Sum15, Tin17]. Nevertheless, we will provide a thorough verification for the sake of completeness. Therefore, it suffices to consider the cases where $m \geq 6$. It is well to note that in this subsection, the Kameko homomorphism $(\widetilde{Sq}_*^0)_{(m,m)} : Q_m^{\otimes m} \longrightarrow Q_0^{\otimes m} = \mathbb{F}_2$ is an epimorphism for every m , and so,

$$Q_m^{\otimes m} \cong \text{Ker}(\widetilde{Sq}_*^0)_{(m,m)} \bigoplus \langle [x_1 x_2 \dots x_m]_{(m,0)} \rangle.$$

For $n = 1$, and $x \in \mathcal{C}_1^{\otimes m}$, one has that $\text{Param}(x) := \text{Param} = (1, 0)$ and $\dim(Q_1^{\otimes m})^{\text{Param}} = \dim Q_1^{\otimes m} = m$ for any $m \geq 1$. Further, $(Q_1^{\otimes m})^{\text{Param}} = \langle \{[x_i]_{\text{Param}} = [x_i] : 1 \leq i \leq m\} \rangle$.

For $n = 2$, and $x \in \mathcal{C}_2^{\otimes m}$, then $\text{Param}(x) := \text{Param} = (2, 0)$. So, from a result in [MKR16], we deduce that $\dim(Q_2^{\otimes m})^{\text{Param}} = \dim Q_2^{\otimes m} = \binom{m}{2}$ for any $m \geq 2$. (Note that since $\beta(2) = 2 > 1$, by Theorem 2.1, $Q_2^{\otimes 1} = 0$.) Further, $(Q_2^{\otimes m})^{\text{Param}} = \langle \{[x_i x_j]_{\text{Param}} = [x_i x_j] : 1 \leq i < j \leq m\} \rangle$.

For $n = 3$, and $x \in \mathcal{C}_3^{\otimes m}$, as we will see, either $\text{Param}(x) := \text{Param}_{(1)} = (1, 1)$ or $\text{Param}(x) := \text{Param}_{(2)} = (3, 0)$ for any $m \geq 1$. By Peterson [Pet87], one gets the following: For $m = 1$, we have $\dim(Q_3^{\otimes 1})^{\text{Param}_{(1)}} = 1$ and $\dim(Q_3^{\otimes 1})^{\text{Param}_{(2)}} = 0$. For $m = 2$, we have $\dim(Q_3^{\otimes 2})^{\text{Param}_{(1)}} = 3$ and $\dim(Q_3^{\otimes 2})^{\text{Param}_{(2)}} = 0$. For $m = 3$, we have an isomorphism $Q_3^{\otimes 3} \cong \text{Ker}(\widetilde{Sq}_*^0)_{(3,3)} \bigoplus \langle [x_1 x_2 x_3]_{\text{Param}_{(2)}} \rangle$, where $\text{Ker}(\widetilde{Sq}_*^0)_{(3,3)} = (Q_3^{\otimes 3})^{\text{Param}_{(1)}}$ and $\langle [x_1 x_2 x_3]_{\text{Param}_{(2)}} \rangle = (Q_3^{\otimes 3})^{\text{Param}_{(2)}}$. So, by Kameko [Kam90], $\dim(Q_3^{\otimes 3})^{\text{Param}_{(1)}} = 6$. For $m \geq 4$, as $Q_3^{\otimes m} = (Q_3^{\otimes m})^0$, we get

$$\dim(Q_3^{\otimes m})^{\text{Param}_{(1)}} = \binom{m}{1} + \binom{m}{2} \text{ and } \dim(Q_3^{\otimes m})^{\text{Param}_{(2)}} = \binom{m}{3}.$$

Further,

$$\begin{aligned} (Q_3^{\otimes m})^{\text{Param}_{(1)}} &= \langle \{[x_i x_j^2]_{\text{Param}_{(1)}} : 1 \leq i \leq j \leq m, 1 \leq k \leq m\} \rangle, \text{ and} \\ (Q_3^{\otimes m})^{\text{Param}_{(2)}} &= \langle \{[x_i x_j x_t]_{\text{Param}_{(2)}} : 1 \leq i < j < t \leq m\} \rangle. \end{aligned}$$

For $n = 4$, and $x \in \mathcal{C}_4^{\otimes m}$, by Theorem 2.1(I), $Q_4^{\otimes 1} = 0$. For each $m \geq 2$, it is straightforward to see that either $\text{Param}(x) := \text{Param}_{(1)} = (2, 1)$ or $\text{Param}(x) := \text{Param}_{(2)} = (4, 0)$. For $m = 2$, by Theorem 2.1(II), $Q_4^{\otimes 2} \cong Q_1^{\otimes 2}$, and so, $\dim(Q_4^{\otimes 2})^{\text{Param}_{(1)}} = 2$ and $\dim(Q_4^{\otimes 2})^{\text{Param}_{(2)}} = 0$. For $m = 3$, by Kameko [Kam90], $\dim(Q_4^{\otimes 3})^{\text{Param}_{(1)}} = 2$ and $\dim(Q_4^{\otimes 3})^{\text{Param}_{(2)}} = 6$. For $m = 4$, one has an isomorphism $Q_4^{\otimes 4} \cong \text{Ker}(\widetilde{Sq}_*^0)_{(4,4)} \bigoplus \langle [x_1 x_2 x_3 x_4]_{\text{Param}_{(2)}} \rangle$, where $\text{Ker}(\widetilde{Sq}_*^0)_{(4,4)} = (Q_4^{\otimes 4})^{\text{Param}_{(1)}}$ and $\langle [x_1 x_2 x_3 x_4]_{\text{Param}_{(2)}} \rangle = (Q_4^{\otimes 4})^{\text{Param}_{(2)}}$. So, by Sum [Sum15], $\dim(Q_4^{\otimes 4})^{\text{Param}_{(1)}} = 16$. For $m \geq 5$, as $Q_4^{\otimes m} = (Q_4^{\otimes m})^0$, we get

$$\dim(Q_4^{\otimes m})^{\text{Param}_{(1)}} = 2 \cdot \binom{m}{2} + 2 \cdot \binom{m}{3} \text{ and } \dim(Q_4^{\otimes m})^{\text{Param}_{(2)}} = \binom{m}{4}.$$

Further,

$$\begin{aligned} (Q_4^{\otimes m})^{\text{Param}_{(1)}} &= \langle \{[x_i x_j^2 x_k]_{\text{Param}_{(1)}} : i < j, i \neq k, 1 \leq i, j, k \leq m\} \rangle, \text{ and} \\ (Q_4^{\otimes m})^{\text{Param}_{(2)}} &= \langle \{[x_i x_j x_k x_\ell]_{\text{Param}_{(2)}} : 1 \leq i < j < k < \ell \leq m\} \rangle. \end{aligned}$$

For $n = 5$, and $x \in \mathcal{C}_5^{\otimes m}$, by Theorem 2.1(I), $Q_5^{\otimes 1} = 0 = Q_5^{\otimes 2}$. For each $m \geq 3$, direct calculations point out that either $\text{Param}(x) := \text{Param}_{(1)} = (3, 1)$ or $\text{Param}(x) := \text{Param}_{(2)} = (5, 0)$. For $m = 3$, by Theorem 2.1(II), $Q_5^{\otimes 3} \cong Q_1^{\otimes 3}$, and so, by Kameko [Kam90], $\dim(Q_5^{\otimes 3})^{\text{Param}_{(1)}} = 3$ and $\dim(Q_5^{\otimes 3})^{\text{Param}_{(2)}} = 0$. For $m = 4$, by Sum [Sum15], $\dim(Q_5^{\otimes 4})^{\text{Param}_{(1)}} = 15$ and $\dim(Q_5^{\otimes 4})^{\text{Param}_{(2)}} = 0$. For $m = 5$, one has an isomorphism $Q_5^{\otimes 5} \cong \text{Ker}(\widetilde{Sq}_*^0)_{(5,5)} \bigoplus \langle [x_1 x_2 x_3 x_4 x_5]_{\text{Param}_{(2)}} \rangle$, where $\text{Ker}(\widetilde{Sq}_*^0)_{(5,5)} = (Q_5^{\otimes 5})^{\text{Param}_{(1)}}$ and $\langle [x_1 x_2 x_3 x_4 x_5]_{\text{Param}_{(2)}} \rangle = (Q_5^{\otimes 5})^{\text{Param}_{(2)}}$. So, by Sum [Sum14], $\dim(Q_5^{\otimes 5})^{\text{Param}_{(1)}} = 45$. For $m \geq 6$, as $Q_5^{\otimes m} = (Q_5^{\otimes m})^0$, we get

$$\dim(Q_5^{\otimes m})^{\text{Param}_{(1)}} = 3 \cdot \binom{m}{3} + 3 \cdot \binom{m}{4} \text{ and } \dim(Q_5^{\otimes m})^{\text{Param}_{(2)}} = \binom{m}{5}.$$

Further,

$$(Q_5^{\otimes m})^{\text{Param}_{(1)}} = \langle \{ [x_i x_j^2 x_k x_\ell]_{\text{Param}_{(1)}}, [x_i x_j x_k^2 x_\ell]_{\text{Param}_{(1)}}, [x_i x_j x_k x_\ell^2]_{\text{Param}_{(1)}}, [x_u x_v x_w^3]_{\text{Param}_{(1)}} : 1 \leq i < j < k < \ell \leq m, 1 \leq u < v < w \leq m \} \rangle, \text{ and}$$

$$(Q_5^{\otimes m})^{\text{Param}_{(2)}} = \langle \{ [x_i x_j x_k x_\ell x_t]_{\text{Param}_{(2)}} : 1 \leq i < j < k < \ell < t \leq m \} \rangle.$$

For $n = 6$, and $x \in \mathcal{C}_6^{\otimes m}$, by Theorem 2.1(I), $Q_6^{\otimes 1} = 0$. For each $m \geq 2$, it is not difficult to check that either $\text{Param}(x) := \text{Param}_{(1)} = (2, 2)$ or $\text{Param}(x) := \text{Param}_{(2)} = (4, 1)$ or $\text{Param}(x) := \text{Param}_{(3)} = (6, 0)$. For $m = 2$, according to Theorem 2.1(II), we have an isomorphism $Q_6^{\otimes 2} \cong Q_2^{\otimes 2}$, and so, due to Peterson [Pet87], $\dim(Q_6^{\otimes 2})^{\text{Param}_{(1)}} = 1$ and $\dim(Q_6^{\otimes 2})^{\text{Param}_{(j)}} = 0$ for $j = 2, 3$. For $m = 3$, by Kameko [Kam90], $\dim(Q_6^{\otimes 3})^{\text{Param}_{(1)}} = 6$ and $\dim(Q_6^{\otimes 3})^{\text{Param}_{(j)}} = 0$ for $j = 2, 3$. For $m = 4$, due to Sum [Sum15], $\dim(Q_6^{\otimes 4})^{\text{Param}_{(1)}} = 20$, $\dim(Q_6^{\otimes 4})^{\text{Param}_{(2)}} = 4$ and $\dim(Q_6^{\otimes 4})^{\text{Param}_{(3)}} = 0$. For $m = 5$, by our previous work [Phu16], $\dim(Q_6^{\otimes 5})^{\text{Param}_{(1)}} = 50$, $\dim(Q_6^{\otimes 5})^{\text{Param}_{(2)}} = 24$ and $\dim(Q_6^{\otimes 5})^{\text{Param}_{(3)}} = 0$. For $m = 6$, one has an isomorphism $Q_6^{\otimes 6} \cong \text{Ker}(\widetilde{Sq}_*^0)_{(6,6)} \bigoplus \langle [x_1 x_2 x_3 x_4 x_5 x_6]_{\text{Param}_{(3)}} \rangle$, where $\text{Ker}(\widetilde{Sq}_*^0)_{(6,6)} = (Q_6^{\otimes 6})^{\text{Param}_{(1)}} \bigoplus (Q_6^{\otimes 6})^{\text{Param}_{(2)}}$ and $\langle [x_1 x_2 x_3 x_4 x_5 x_6]_{\text{Param}_{(3)}} \rangle = (Q_6^{\otimes 6})^{\text{Param}_{(3)}}$. So, a simple computation shows: $\dim(Q_6^{\otimes 6})^{\text{Param}_{(1)}} = \binom{6}{2} + 3 \cdot \binom{6}{3} + 2 \cdot \binom{6}{4} = 105$ and $\dim(Q_6^{\otimes 6})^{\text{Param}_{(2)}} = 4 \cdot \binom{6}{4} + 4 \cdot \binom{6}{5} = 84$. For $m \geq 7$, as $Q_6^{\otimes m} = (Q_6^{\otimes m})^0$, we get

$$\dim(Q_6^{\otimes m})^{\text{Param}_{(1)}} = \binom{m}{2} + 3 \cdot \binom{m}{3} + 2 \cdot \binom{m}{4}, \quad \dim(Q_6^{\otimes m})^{\text{Param}_{(2)}} = 4 \cdot \binom{m}{4} + 4 \cdot \binom{m}{5}, \text{ and}$$

$$\dim(Q_6^{\otimes m})^{\text{Param}_{(3)}} = \binom{m}{6}.$$

Further,

$$(Q_6^{\otimes m})^{\text{Param}_{(1)}} = \langle \{ [x_i x_j x_k^2 x_\ell^2]_{\text{Param}_{(1)}}, [x_i x_j^2 x_k x_\ell^2]_{\text{Param}_{(1)}}, [x_p x_q^2 x_r^3]_{\text{Param}_{(1)}}, [x_s^3 x_t^3]_{\text{Param}_{(1)}} : 1 \leq i < j < k < \ell \leq m, p < q, p \neq r, q \neq r, 1 \leq p, q, r \leq m, 1 \leq s < t \leq m \} \rangle,$$

$$(Q_6^{\otimes m})^{\text{Param}_{(2)}} = \langle \{ [x_i x_j x_k x_\ell x_p^2]_{\text{Param}_{(2)}}, [x_i x_j x_k x_\ell^2 x_p]_{\text{Param}_{(2)}}, [x_i x_j x_k^2 x_\ell x_p]_{\text{Param}_{(2)}}, [x_i x_j^2 x_k x_\ell x_p]_{\text{Param}_{(2)}}, [x_q x_r x_s x_t^3]_{\text{Param}_{(2)}} : 1 \leq i < j < k < \ell < p \leq m, 1 \leq q < r < s < t \leq m \} \rangle,$$

$$(Q_6^{\otimes m})^{\text{Param}_{(3)}} = \langle \{ [x_i x_j x_k x_\ell x_p x_q]_{\text{Param}_{(3)}} : 1 \leq i < j < k < \ell < p < q \leq m \} \rangle.$$

For $n = 7$, and $x \in \mathcal{C}_7^{\otimes m}$, by routine calculations, one obtains that the parameter vector $\text{Param}(x)$ is one of the following sequences:

$$\text{Param}_{(1)} := (1, 1, 1), \quad \text{Param}_{(2)} := (1, 3), \quad \text{Param}_{(3)} := (3, 2), \quad \text{Param}_{(4)} := (5, 1), \quad \text{Param}_{(5)} := (7, 0).$$

In particular,

$$\text{Param}(x) \in \begin{cases} \{\text{Param}_{(1)}\} & \text{if } m = 1, 2 \text{ (see also [Pet87])}, \\ \{\text{Param}_{(1)}, \text{Param}_{(3)}\} & \text{if } m = 3 \text{ (see also [Kam90])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 3\} & \text{if } m = 4 \text{ (see also [Sum15])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 4\} & \text{if } m = 5 \text{ (see also [Tin17])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 4\} & \text{if } m = 6, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 5\} & \text{if } m \geq 7. \end{cases}$$

For $m = 1, 2$, by Peterson [Pet87], $(Q_7^{\otimes m})^{\text{Param}(1)}$ is 1-dimensional if $m = 1$ and is 3-dimensional if $m = 2$. For $m = 3$, by Kameko [Kam90], $\dim(Q_7^{\otimes 3})^{\text{Param}(1)} = 7$ and $\dim(Q_7^{\otimes 3})^{\text{Param}(3)} = 3$. For $m = 4$, by Sum [Sum15], $\dim(Q_7^{\otimes 4})^{\text{Param}(1)} = 14$, $\dim(Q_7^{\otimes 4})^{\text{Param}(2)} = 1$ and $\dim(Q_7^{\otimes 4})^{\text{Param}(3)} = 20$. For $m = 5$, by Tin [Tin17], $\dim(Q_7^{\otimes 5})^{\text{Param}(1)} = 25$, $\dim(Q_7^{\otimes 5})^{\text{Param}(2)} = 5$, $\dim(Q_7^{\otimes 5})^{\text{Param}(3)} = 75$ and $\dim(Q_7^{\otimes 5})^{\text{Param}(4)} = 5$. For $m = 6$, by direct calculations, we get $\dim(Q_7^{\otimes 6})^{\text{Param}(1)} = \binom{6}{1} + \binom{6}{2} + \binom{6}{3} = 41$, $\dim(Q_7^{\otimes 6})^{\text{Param}(2)} = \binom{6}{4} = 15$, $\dim(Q_7^{\otimes 6})^{\text{Param}(3)} = 3 \cdot \binom{6}{3} + 8 \cdot \binom{6}{4} + 5 \cdot \binom{6}{5} = 210$ and $\dim(Q_7^{\otimes 6})^{\text{Param}(4)} = 5 \cdot \binom{6}{5} + 5 \cdot \binom{6}{6} = 35$. For $m = 7$, one has an isomorphism $Q_7^{\otimes 7} \cong \text{Ker}(\widetilde{Sq}_*^0)_{(7,7)} \oplus \langle [x_1 x_2 x_3 x_4 x_5 x_6 x_7]_{\text{Param}(5)} \rangle$, where $\text{Ker}(\widetilde{Sq}_*^0)_{(7,7)} = \bigoplus_{1 \leq j \leq 4} (Q_7^{\otimes 7})^{\text{Param}(j)}$ and $\langle [x_1 x_2 x_3 x_4 x_5 x_6 x_7]_{\text{Param}(5)} \rangle = (Q_7^{\otimes 7})^{\text{Param}(5)}$. So, a simple computation shows: $\dim(Q_7^{\otimes 7})^{\text{Param}(1)} = \binom{7}{1} + \binom{7}{2} + \binom{7}{3} = 63$, $\dim(Q_7^{\otimes 7})^{\text{Param}(2)} = \binom{7}{4} = 35$, $\dim(Q_7^{\otimes 7})^{\text{Param}(3)} = 3 \cdot \binom{7}{3} + 8 \cdot \binom{7}{4} + 5 \cdot \binom{7}{5} = 490$ and $\dim(Q_7^{\otimes 7})^{\text{Param}(4)} = 5 \cdot \binom{7}{5} + 5 \cdot \binom{7}{6} = 140$.

For $m \geq 8$, as $Q_7^{\otimes m} = (Q_7^{\otimes m})^0$, we get

$$\begin{aligned} \dim(Q_7^{\otimes m})^{\text{Param}(1)} &= \binom{m}{1} + \binom{m}{2} + \binom{m}{3}, \quad \dim(Q_7^{\otimes m})^{\text{Param}(2)} = \binom{m}{4}, \\ \dim(Q_7^{\otimes m})^{\text{Param}(3)} &= 3 \cdot \binom{m}{3} + 8 \cdot \binom{m}{4} + 5 \cdot \binom{m}{5}, \quad \dim(Q_7^{\otimes m})^{\text{Param}(4)} = 5 \cdot \binom{m}{5} + 5 \cdot \binom{m}{6}, \\ \dim(Q_7^{\otimes m})^{\text{Param}(5)} &= \binom{m}{7}. \end{aligned}$$

For $n = 8$, and $x \in \mathcal{C}_8^{\otimes m}$, by Theorem 2.1(I), $Q_8^{\otimes 1} = 0$. For each $m \geq 2$, elementary computations in these cases show that the parameter vector $\text{Param}(x)$ is one of the following sequences:

$$\text{Param}_{(1)} := (2, 1, 1), \quad \text{Param}_{(2)} := (2, 3), \quad \text{Param}_{(3)} := (4, 2), \quad \text{Param}_{(4)} := (6, 1), \quad \text{Param}_{(5)} := (8, 0).$$

In particular,

$$\text{Param}(x) \in \begin{cases} \{\text{Param}_{(1)}\} & \text{if } m = 2, 3 \text{ (see also [Pet87, Kam90])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 3\} & \text{if } m = 4, 5 \text{ (see also [Sum15, Tin17])}, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 4\} & \text{if } m = 6, 7, \\ \{\text{Param}_{(j)} : 1 \leq j \leq 5\} & \text{if } m \geq 8. \end{cases}$$

For $m = 2$, due to Peterson [Pet87], $\dim(Q_8^{\otimes 2})^{\text{Param}(1)} = 3$. For $m = 3$, according to Kameko [Kam90], $\dim(Q_8^{\otimes 3})^{\text{Param}(1)} = 15$. For $m = 4$, by Sum [Sum15], $\dim(Q_8^{\otimes 4})^{\text{Param}(1)} = 45$, $\dim(Q_8^{\otimes 4})^{\text{Param}(2)} = 4$ and $\dim(Q_8^{\otimes 4})^{\text{Param}(3)} = 6$. For $m = 5$, due to Tin [Tin17], $\dim(Q_8^{\otimes 5})^{\text{Param}(1)} = 105$, $\dim(Q_8^{\otimes 5})^{\text{Param}(2)} = 24$ and $\dim(Q_8^{\otimes 5})^{\text{Param}(3)} = 45$. For $m = 6$, by our previous work [Phu22d], $\dim(Q_8^{\otimes 6})^{\text{Param}(1)} = 210$, $\dim(Q_8^{\otimes 6})^{\text{Param}(2)} = 84$ and $\dim(Q_8^{\otimes 6})^{\text{Param}(3)} = 189$. For the other inclusion, we observe that $Q_8^{\otimes 6} \cong \text{Ker}((\widetilde{Sq}_*^0)_{(6,8)}) \oplus Q_1^{\otimes 6}$, where $\text{Ker}((\widetilde{Sq}_*^0)_{(6,8)}) \cong \bigoplus_{1 \leq i \leq 3} (Q_8^{\otimes 6})^{\text{Param}(i)}$. So, $(Q_8^{\otimes 6})^{\text{Param}(4)} \cong$

$\varphi(Q_1^{\otimes 6}) = \langle [x_1 \dots x_6 u^2]_{\text{Param}(4)} : u \in \mathcal{C}_1^{\otimes 6} \rangle$ where $\varphi : Q_1^{\otimes 6} \longrightarrow Q_8^{\otimes 6}$, $[u] \longmapsto [x_1 x_2 \dots x_6 u^2]$ and $\mathcal{C}_1^{\otimes 6} = \{x_i : 1 \leq i \leq 6\}$. We see therefore that $\dim(Q_8^{\otimes 6})^{\text{Param}(4)} = 6$. For $m = 7$, by direct calculations, we get:

$$\begin{aligned} \dim(Q_8^{\otimes 7})^{\text{Param}(1)} &= 3 \cdot \binom{7}{2} + 6 \cdot \binom{7}{3} + 3 \cdot \binom{7}{4} = 378, \quad \dim(Q_8^{\otimes 7})^{\text{Param}(2)} = 4 \cdot \binom{7}{4} + 4 \cdot \binom{7}{5} = 224, \\ \dim(Q_8^{\otimes 7})^{\text{Param}(3)} &= 6 \cdot \binom{7}{4} + 15 \cdot \binom{7}{5} + 9 \cdot \binom{7}{6} = 588, \quad \dim(Q_8^{\otimes 7})^{\text{Param}(4)} = 6 \cdot \binom{7}{6} + 6 \cdot \binom{7}{7} = 48. \end{aligned}$$

For $m = 8$, one has that $Q_8^{\otimes 8} \cong \text{Ker}(\widetilde{Sq_*^0})_{(8,8)} \bigoplus \langle [x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8]_{\text{Param}_{(5)}} \rangle$, where

$$\text{Ker}(\widetilde{Sq_*^0})_{(8,8)} = \bigoplus_{1 \leq j \leq 4} (Q_8^{\otimes 8})^{\text{Param}_{(j)}} \text{ and } \langle [x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8]_{\text{Param}_{(5)}} \rangle = (Q_8^{\otimes 8})^{\text{Param}_{(5)}}.$$

So, by direct calculations, we obtain:

$$\begin{aligned} \dim(Q_8^{\otimes 8})^{\text{Param}_{(1)}} &= 3 \cdot \binom{8}{2} + 6 \cdot \binom{8}{3} + 3 \cdot \binom{8}{4} = 630, & \dim(Q_8^{\otimes 8})^{\text{Param}_{(2)}} &= 4 \cdot \binom{8}{4} + 4 \cdot \binom{8}{5} = 504, \\ \dim(Q_8^{\otimes 8})^{\text{Param}_{(3)}} &= 6 \cdot \binom{8}{4} + 15 \cdot \binom{8}{5} + 9 \cdot \binom{8}{6} = 1512, & \dim(Q_8^{\otimes 8})^{\text{Param}_{(4)}} &= 6 \cdot \binom{8}{6} + 6 \cdot \binom{8}{7} = 216. \end{aligned}$$

For $m \geq 9$, as $Q_8^{\otimes m} = (Q_8^{\otimes m})^0$, we get

$$\begin{aligned} \dim(Q_8^{\otimes m})^{\text{Param}_{(1)}} &= 3 \cdot \binom{m}{2} + 6 \cdot \binom{m}{3} + 3 \cdot \binom{m}{4}, \\ \dim(Q_8^{\otimes m})^{\text{Param}_{(2)}} &= 4 \cdot \binom{m}{4} + 4 \cdot \binom{m}{5}, \\ \dim(Q_8^{\otimes m})^{\text{Param}_{(3)}} &= 6 \cdot \binom{m}{4} + 15 \cdot \binom{m}{5} + 9 \cdot \binom{m}{6}, \\ \dim(Q_8^{\otimes m})^{\text{Param}_{(4)}} &= 6 \cdot \binom{m}{6} + 6 \cdot \binom{m}{7}, \\ \dim(Q_8^{\otimes m})^{\text{Param}_{(5)}} &= \binom{m}{8}. \end{aligned}$$

Thus, as a consequence of the above calculations, one derives

Corollary 5.1.1. *Conjecture 3.2.1 holds for any $m \geq 1$ and parameter vectors of degrees $n \leq 8$.*

Moreover, by amalgamating our previous findings in [Phu23a, Phu22b] with Corollaries 4.3.3 and 5.1.1, we declare that Conjecture 3.2.1 holds true for all $m \geq 1$ and parameter vectors of degrees not exceeding 12.

5.2. Some allowable monomials in $\mathcal{C}_{n_0}^{\otimes 5}$

Let us recall that $Q_{n_0}^{\otimes 5} \cong \bigoplus_{1 \leq i \leq 6} (Q_{n_0}^{\otimes 5})^{\overline{\text{Param}_{[i]}}}$, where the weights $\overline{\text{Param}_{[i]}}$ are determined as in Proposition 4.1.6. Moreover, by a simple computation, we get the following:

$$\begin{aligned} (\mathcal{C}_{n_0}^{\otimes 5})^{\overline{\text{Param}_{[1]}}} &= \widetilde{\Phi}_*((\mathcal{C}_{n_0}^{\otimes 4})^{\overline{\text{Param}_{[1]}}}) \bigcup \left(\bigcup_{1 \leq l \leq 5} \mathcal{C}_{(l, \overline{\text{Param}_{[1]}})} \right) \bigcup \{x_1 x_2^2 x_3^4 x_4^9 x_5^2, x_1^3 x_2^4 x_3 x_4^2 x_5^8\} \\ &= \{X_j \mid 1 \leq j \leq 300\}, \\ (\mathcal{C}_{n_0}^{\otimes 5})^{\overline{\text{Param}_{[4]}}} &= \widetilde{\Phi}_*((\mathcal{C}_{n_0}^{\otimes 4})^{\overline{\text{Param}_{[4]}}}) \bigcup \left(\bigcup_{1 \leq l \leq 5} \mathcal{C}_{(l, \overline{\text{Param}_{[4]}})} \right) \\ &= \{Y_j \mid 1 \leq j \leq 110\}. \end{aligned}$$

Allowable monomials in $(\mathcal{C}_{n_0}^{\otimes 5})^{\text{Param}_{[1]}}$:

$$\begin{aligned}
X_1 &= x_4^3 x_5^{15}, & X_2 &= x_4^{15} x_5^3, & X_3 &= x_3^3 x_5^{15}, & X_4 &= x_3^3 x_4^{15}, \\
X_5 &= x_3^{15} x_5^3, & X_6 &= x_3^{15} x_4^3, & X_7 &= x_2^3 x_5^{15}, & X_8 &= x_2^3 x_4^{15}, \\
X_9 &= x_2^3 x_3^{15}, & X_{10} &= x_2^{15} x_5^3, & X_{11} &= x_2^{15} x_4^3, & X_{12} &= x_2^{15} x_3^3, \\
X_{13} &= x_1^3 x_5^{15}, & X_{14} &= x_1^3 x_4^{15}, & X_{15} &= x_1^3 x_3^{15}, & X_{16} &= x_1^3 x_2^{15}, \\
X_{17} &= x_1^{15} x_5^3, & X_{18} &= x_1^{15} x_4^3, & X_{19} &= x_1^{15} x_3^3, & X_{20} &= x_1^{15} x_2^3, \\
X_{21} &= x_4^7 x_5^{11}, & X_{22} &= x_3^7 x_5^{11}, & X_{23} &= x_3^7 x_4^{11}, & X_{24} &= x_2^7 x_5^{11}, \\
X_{25} &= x_2^7 x_4^{11}, & X_{26} &= x_2^7 x_3^{11}, & X_{27} &= x_1^7 x_5^{11}, & X_{28} &= x_1^7 x_4^{11}, \\
X_{29} &= x_1^7 x_3^{11}, & X_{30} &= x_1^7 x_2^{11}, & X_{31} &= x_3 x_4^2 x_5^{15}, & X_{32} &= x_3 x_4^{15} x_5^2, \\
X_{33} &= x_3^{15} x_4 x_5^2, & X_{34} &= x_2 x_4^2 x_5^{15}, & X_{35} &= x_2 x_4^{15} x_5^2, & X_{36} &= x_2 x_3^2 x_5^{15}, \\
X_{37} &= x_2 x_3^2 x_4^{15}, & X_{38} &= x_2 x_3^{15} x_5^2, & X_{39} &= x_2 x_3^{15} x_4^2, & X_{40} &= x_2^{15} x_4 x_5^2, \\
X_{41} &= x_2^{15} x_3 x_4^2, & X_{42} &= x_2^{15} x_3 x_4^2, & X_{43} &= x_1 x_4^2 x_5^{15}, & X_{44} &= x_1 x_4^{15} x_5^2, \\
X_{45} &= x_1 x_3^2 x_5^{15}, & X_{46} &= x_1 x_3^2 x_4^{15}, & X_{47} &= x_1 x_3^{15} x_5^2, & X_{48} &= x_1 x_3^{15} x_4^2, \\
X_{49} &= x_1 x_2^{15} x_5^2, & X_{50} &= x_1 x_2^{15} x_4^2, & X_{51} &= x_1 x_2^{15} x_3^2, & X_{52} &= x_1 x_2^2 x_5^{15}, \\
X_{53} &= x_1 x_2^2 x_4^{15}, & X_{54} &= x_1 x_2^2 x_3^{15}, & X_{55} &= x_1^{15} x_4 x_5^2, & X_{56} &= x_1^{15} x_3 x_5^2, \\
X_{57} &= x_1^{15} x_3 x_4^2, & X_{58} &= x_1^{15} x_2 x_5^2, & X_{59} &= x_1^{15} x_2 x_4^2, & X_{60} &= x_1^{15} x_2 x_3^2, \\
X_{61} &= x_3 x_4^3 x_5^{14}, & X_{62} &= x_3 x_4^{14} x_5^3, & X_{63} &= x_3^3 x_4 x_5^{14}, & X_{64} &= x_2 x_4^3 x_5^{14}, \\
X_{65} &= x_2 x_4^{14} x_5^3, & X_{66} &= x_2 x_3^3 x_5^{14}, & X_{67} &= x_2 x_3^3 x_4^{14}, & X_{68} &= x_2 x_3^{14} x_5^3, \\
X_{69} &= x_2 x_3^{14} x_4^3, & X_{70} &= x_2^3 x_4 x_5^{14}, & X_{71} &= x_2^3 x_3 x_5^{14}, & X_{72} &= x_2^3 x_3 x_4^{14}, \\
X_{73} &= x_1 x_4^3 x_5^{14}, & X_{74} &= x_1 x_4^{14} x_5^3, & X_{75} &= x_1 x_3^3 x_5^{14}, & X_{76} &= x_1 x_3^3 x_4^{14}, \\
X_{77} &= x_1 x_3^3 x_5^{14}, & X_{78} &= x_1 x_3^3 x_4^{14}, & X_{79} &= x_1 x_2^3 x_5^{14}, & X_{80} &= x_1 x_2^3 x_4^{14}, \\
X_{81} &= x_1 x_2^3 x_3^{14}, & X_{82} &= x_1 x_2^{14} x_3^3, & X_{83} &= x_1 x_2^{14} x_4^3, & X_{84} &= x_1 x_2^{14} x_3^3, \\
X_{85} &= x_1^3 x_4 x_5^{14}, & X_{86} &= x_1^3 x_3 x_5^{14}, & X_{87} &= x_1^3 x_3 x_4^{14}, & X_{88} &= x_1^3 x_2 x_5^{14}, \\
X_{89} &= x_1^3 x_2 x_4^{14}, & X_{90} &= x_1^3 x_2 x_3^{14}, & X_{91} &= x_3 x_4^6 x_5^{11}, & X_{92} &= x_2 x_4^6 x_5^{11}, \\
X_{93} &= x_2 x_3^6 x_5^{11}, & X_{94} &= x_2 x_3^6 x_4^{11}, & X_{95} &= x_1 x_4^6 x_5^{11}, & X_{96} &= x_1 x_3^6 x_5^{11}, \\
X_{97} &= x_1 x_3^6 x_4^{11}, & X_{98} &= x_1 x_2^6 x_5^{11}, & X_{99} &= x_1 x_2^6 x_4^{11}, & X_{100} &= x_1 x_2^6 x_3^{11}, \\
X_{101} &= x_3 x_4^7 x_5^{10}, & X_{102} &= x_3 x_4^7 x_5^{10}, & X_{103} &= x_2 x_4^7 x_5^{10}, & X_{104} &= x_2 x_3^7 x_5^{10}, \\
X_{105} &= x_2 x_3^7 x_4^{10}, & X_{106} &= x_2^7 x_4 x_5^{10}, & X_{107} &= x_2^7 x_3 x_5^{10}, & X_{108} &= x_2^7 x_3 x_4^{10}, \\
X_{109} &= x_1 x_4^7 x_5^{10}, & X_{110} &= x_1 x_3^7 x_5^{10}, & X_{111} &= x_1 x_3^7 x_4^{10}, & X_{112} &= x_1 x_2^7 x_5^{10}, \\
X_{113} &= x_1 x_2^7 x_4^{10}, & X_{114} &= x_1 x_2^7 x_3^{10}, & X_{115} &= x_1^7 x_4 x_5^{10}, & X_{116} &= x_1^7 x_3 x_5^{10}, \\
X_{117} &= x_1^7 x_3 x_4^{10}, & X_{118} &= x_1^7 x_2 x_5^{10}, & X_{119} &= x_1^7 x_2 x_4^{10}, & X_{120} &= x_1^7 x_2 x_3^{10}, \\
X_{121} &= x_3^3 x_4^{13} x_5^2, & X_{122} &= x_2^3 x_4^{13} x_5^2, & X_{123} &= x_2^3 x_3^{13} x_5^2, & X_{124} &= x_2^3 x_3^{13} x_4^2, \\
X_{125} &= x_1^3 x_4^{13} x_5^2, & X_{126} &= x_1^3 x_3^{13} x_5^2, & X_{127} &= x_1^3 x_3^{13} x_4^2, & X_{128} &= x_1^3 x_2^{13} x_5^2, \\
X_{129} &= x_1^3 x_2^{13} x_4^2, & X_{130} &= x_1^3 x_2^{13} x_3^2, & X_{131} &= x_3^3 x_4^3 x_5^{12}, & X_{132} &= x_3^3 x_4^3 x_5^{12}, \\
X_{133} &= x_2^3 x_3^3 x_5^{12}, & X_{134} &= x_2^3 x_3^3 x_4^{12}, & X_{135} &= x_1^3 x_4^3 x_5^{12}, & X_{136} &= x_1^3 x_3^3 x_5^{12}, \\
X_{137} &= x_1^3 x_3^3 x_4^{12}, & X_{138} &= x_1^3 x_2^3 x_5^{12}, & X_{139} &= x_1^3 x_2^3 x_4^{12}, & X_{140} &= x_1^3 x_2^3 x_3^{12}, \\
X_{141} &= x_3^3 x_4^5 x_5^{10}, & X_{142} &= x_2^3 x_4^5 x_5^{10}, & X_{143} &= x_2^3 x_3^5 x_5^{10}, & X_{144} &= x_2^3 x_3^5 x_4^{10}, \\
X_{145} &= x_1^3 x_4^5 x_5^{10}, & X_{146} &= x_1^3 x_3^5 x_5^{10}, & X_{147} &= x_1^3 x_3^5 x_4^{10}, & X_{148} &= x_1^3 x_2^5 x_5^{10}, \\
X_{149} &= x_1^3 x_2^5 x_4^{10}, & X_{150} &= x_1^3 x_2^5 x_3^{10}, & X_{151} &= x_2 x_3 x_4^2 x_5^{14}, & X_{152} &= x_2 x_3 x_4^{14} x_5^2, \\
X_{153} &= x_2 x_3^2 x_4 x_5^{14}, & X_{154} &= x_2 x_3^{14} x_4 x_5^2, & X_{155} &= x_1 x_3 x_4^2 x_5^{14}, & X_{156} &= x_1 x_3 x_4^{14} x_5^2, \\
X_{157} &= x_1 x_3^2 x_4 x_5^{14}, & X_{158} &= x_1 x_3^{14} x_4 x_5^2, & X_{159} &= x_1 x_2 x_4^2 x_5^{14}, & X_{160} &= x_1 x_2 x_4^{14} x_5^2, \\
X_{161} &= x_1 x_2^2 x_4 x_5^{14}, & X_{162} &= x_1 x_2^{14} x_4 x_5^2, & X_{163} &= x_1 x_2 x_3^2 x_5^{14}, & X_{164} &= x_1 x_2 x_3^{14} x_5^2, \\
X_{165} &= x_1 x_2^2 x_3 x_5^{14}, & X_{166} &= x_1 x_2^{14} x_3 x_5^2, & X_{167} &= x_1 x_2 x_3^2 x_4^{14}, & X_{168} &= x_1 x_2 x_3^{14} x_4^2, \\
X_{169} &= x_1 x_2^2 x_3 x_4^{14}, & X_{170} &= x_1 x_2^{14} x_3 x_4^2, & X_{171} &= x_2 x_3^2 x_4^2 x_5^{13}, & X_{172} &= x_1 x_3^2 x_4^2 x_5^{13}, \\
X_{173} &= x_1 x_2^2 x_3^2 x_5^{13}, & X_{174} &= x_1 x_2^2 x_3^2 x_4^{13}, & X_{175} &= x_1 x_2^2 x_3^2 x_4^{13}, & X_{176} &= x_2 x_3^2 x_4^2 x_5^{11}, \\
X_{177} &= x_1 x_2^2 x_3^2 x_4^{11}, & X_{178} &= x_1 x_2^2 x_3^2 x_4^{11}, & X_{179} &= x_1 x_2^2 x_3^2 x_5^{11}, & X_{180} &= x_1 x_2^2 x_3^2 x_4^{11}, \\
X_{181} &= x_2 x_3^2 x_4^2 x_5^8, & X_{182} &= x_2 x_3^2 x_4^2 x_5^8, & X_{183} &= x_2^7 x_3 x_4^2 x_5^8, & X_{184} &= x_1 x_3^2 x_4^2 x_5^8, \\
X_{185} &= x_1 x_3^2 x_4^2 x_5^8, & X_{186} &= x_1 x_2^2 x_4^2 x_5^8, & X_{187} &= x_1 x_2^2 x_4^2 x_5^8, & X_{188} &= x_1 x_2^2 x_3^2 x_5^8, \\
X_{189} &= x_1 x_2^2 x_3^2 x_4^8, & X_{190} &= x_1 x_2^2 x_3^2 x_4^8, & X_{191} &= x_1 x_2^2 x_3^2 x_4^8, & X_{192} &= x_1 x_3^2 x_4^2 x_5^8, \\
X_{193} &= x_1^7 x_2 x_4^2 x_5^8, & X_{194} &= x_1^7 x_2 x_3^2 x_5^8, & X_{195} &= x_1^7 x_2 x_3^2 x_4^8, & X_{196} &= x_2 x_3^2 x_4^2 x_5^{10}, \\
X_{197} &= x_2 x_3^6 x_4 x_5^{10}, & X_{198} &= x_1 x_3^6 x_4 x_5^{10}, & X_{199} &= x_1 x_3^6 x_4 x_5^{10}, & X_{200} &= x_1 x_2 x_4^6 x_5^{10},
\end{aligned}$$

$$\begin{aligned}
X_{201} &= x_1 x_2^6 x_4 x_5^{10}, & X_{202} &= x_1 x_2 x_3^6 x_5^{10}, & X_{203} &= x_1 x_2^6 x_3 x_5^{10}, & X_{204} &= x_1 x_2 x_3^6 x_4^{10}, \\
X_{205} &= x_1 x_2^6 x_3 x_4^{10}, & X_{206} &= x_2 x_3^2 x_4^3 x_5^{12}, & X_{207} &= x_2 x_3^2 x_4^3 x_5^{12}, & X_{208} &= x_2 x_3^3 x_4^2 x_5^{12}, \\
X_{209} &= x_2 x_3^3 x_4^2 x_5^{12}, & X_{210} &= x_2^3 x_3 x_4^2 x_5^{12}, & X_{211} &= x_2^3 x_3 x_4^2 x_5^{12}, & X_{212} &= x_1 x_3^2 x_4^3 x_5^{12}, \\
X_{213} &= x_1 x_3^2 x_4^3 x_5^{12}, & X_{214} &= x_1 x_3^3 x_4^2 x_5^{12}, & X_{215} &= x_1 x_3^3 x_4^2 x_5^{12}, & X_{216} &= x_1 x_3^3 x_4^2 x_5^{12}, \\
X_{217} &= x_1 x_2^3 x_3^2 x_5^{12}, & X_{218} &= x_1 x_2^3 x_3^2 x_4^{12}, & X_{219} &= x_1 x_2^2 x_3^3 x_5^{12}, & X_{220} &= x_1 x_2^2 x_3^3 x_5^{12}, \\
X_{221} &= x_1 x_2^3 x_4^2 x_5^{12}, & X_{222} &= x_1 x_2^2 x_3^3 x_5^{12}, & X_{223} &= x_1 x_2^2 x_3^3 x_4^{12}, & X_{224} &= x_1 x_2^2 x_3^3 x_5^{12}, \\
X_{225} &= x_1 x_2^3 x_4^2 x_5^{12}, & X_{226} &= x_1 x_2^2 x_3^3 x_4^{12}, & X_{227} &= x_1 x_2^3 x_3^2 x_4^{12}, & X_{228} &= x_1^3 x_3 x_4^2 x_5^{12}, \\
X_{229} &= x_1^3 x_2 x_4^2 x_5^{12}, & X_{230} &= x_1^3 x_2 x_3^2 x_5^{12}, & X_{231} &= x_1^3 x_2 x_3^2 x_4^{12}, & X_{232} &= x_1^3 x_3 x_4^2 x_5^{12}, \\
X_{233} &= x_1^3 x_2 x_4^2 x_5^{12}, & X_{234} &= x_1^3 x_2 x_3^2 x_5^{12}, & X_{235} &= x_1^3 x_2 x_3^2 x_4^{12}, & X_{236} &= x_2 x_3^3 x_4^2 x_5^{12}, \\
X_{237} &= x_1 x_3^2 x_4^5 x_5^{10}, & X_{238} &= x_1 x_2^2 x_4^5 x_5^{10}, & X_{239} &= x_1 x_2^2 x_3^5 x_5^{10}, & X_{240} &= x_1 x_2^2 x_3^5 x_4^{10}, \\
X_{241} &= x_2 x_3^3 x_4^4 x_5^{10}, & X_{242} &= x_2^3 x_3 x_4^4 x_5^{10}, & X_{243} &= x_1 x_3^3 x_4^4 x_5^{10}, & X_{244} &= x_1 x_2^3 x_4^4 x_5^{10}, \\
X_{245} &= x_1 x_2^3 x_4^4 x_5^{10}, & X_{246} &= x_1 x_2^3 x_4^4 x_5^{10}, & X_{247} &= x_1^3 x_3 x_4^4 x_5^{10}, & X_{248} &= x_1^3 x_2 x_4^4 x_5^{10}, \\
X_{249} &= x_1^3 x_2 x_3^4 x_5^{10}, & X_{250} &= x_1^3 x_2 x_3^4 x_4^{10}, & X_{251} &= x_2 x_3^3 x_4^6 x_5^8, & X_{252} &= x_2^3 x_3 x_4^6 x_5^8, \\
X_{253} &= x_1 x_3^3 x_4^6 x_5^8, & X_{254} &= x_1 x_2^3 x_4^6 x_5^8, & X_{255} &= x_1 x_2^3 x_3^6 x_5^8, & X_{256} &= x_1 x_2^3 x_3^6 x_4^8, \\
X_{257} &= x_1^3 x_3 x_4^6 x_5^8, & X_{258} &= x_1^3 x_2 x_4^6 x_5^8, & X_{259} &= x_1^3 x_2 x_3^6 x_5^8, & X_{260} &= x_1^3 x_2 x_3^6 x_4^8, \\
X_{261} &= x_2^3 x_5^2 x_4^8 x_5^8, & X_{262} &= x_2^3 x_5^2 x_4^8 x_5^8, & X_{263} &= x_1^3 x_5^2 x_4^8 x_5^8, & X_{264} &= x_1^3 x_5^2 x_4^8 x_5^8, \\
X_{265} &= x_1^3 x_5^2 x_4^8 x_5^8, & X_{266} &= x_1^3 x_5^2 x_4^8 x_5^8, & X_{267} &= x_1^3 x_5^2 x_4^8 x_5^8, & X_{268} &= x_1^3 x_5^2 x_4^8 x_5^8, \\
X_{269} &= x_1^3 x_5^2 x_4^8 x_5^8, & X_{270} &= x_1^3 x_5^2 x_4^8 x_5^8, & X_{271} &= x_2^3 x_3^4 x_4^8 x_5^8, & X_{272} &= x_1^3 x_3^4 x_4^8 x_5^8, \\
X_{273} &= x_1^3 x_3^4 x_4^8 x_5^8, & X_{274} &= x_1^3 x_3^4 x_4^8 x_5^8, & X_{275} &= x_1^3 x_3^4 x_4^8 x_5^8, & X_{276} &= x_1 x_2^2 x_3^4 x_5^{12}, \\
X_{277} &= x_1 x_2^2 x_3^4 x_5^{12}, & X_{278} &= x_1 x_2^2 x_3^4 x_5^{12}, & X_{279} &= x_1 x_2^2 x_3^4 x_5^{12}, & X_{280} &= x_1 x_2^2 x_3^4 x_5^{12}, \\
X_{281} &= x_1 x_2^2 x_3^4 x_5^{10}, & X_{282} &= x_1 x_2^2 x_3^4 x_5^{10}, & X_{283} &= x_1 x_2^2 x_3^4 x_5^{10}, & X_{284} &= x_1 x_2^2 x_3^4 x_5^{10}, \\
X_{285} &= x_1 x_2^6 x_3 x_4^2 x_5^8, & X_{286} &= x_1 x_2^6 x_3 x_4^2 x_5^8, & X_{287} &= x_1 x_2^2 x_3^6 x_4^8 x_5^8, & X_{288} &= x_1 x_2^2 x_3^6 x_4^8 x_5^8, \\
X_{289} &= x_1 x_2^2 x_3^6 x_4^8 x_5^8, & X_{290} &= x_1 x_2^2 x_3^6 x_4^8 x_5^8, & X_{291} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{292} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, \\
X_{293} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{294} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{295} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{296} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, \\
X_{297} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{298} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{299} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8, & X_{300} &= x_1 x_2^3 x_3^2 x_4^8 x_5^8.
\end{aligned}$$

Allowable monomials in $(\mathcal{C}_{n_0}^{\otimes 5})^{\text{Param}_{[4]}}$:

$$\begin{aligned}
Y_1 &= x_2 x_3 x_4 x_5^{15}, & Y_2 &= x_2 x_3 x_4^{15} x_5, & Y_3 &= x_2 x_3^{15} x_4 x_5, & Y_4 &= x_2^{15} x_3 x_4 x_5, \\
Y_5 &= x_1 x_3 x_4 x_5^{15}, & Y_6 &= x_1 x_3 x_4^{15} x_5, & Y_7 &= x_1 x_3^{15} x_4 x_5, & Y_8 &= x_1 x_2 x_4 x_5^{15}, \\
Y_9 &= x_1 x_2 x_4^{15} x_5, & Y_{10} &= x_1 x_2 x_3 x_5^{15}, & Y_{11} &= x_1 x_2 x_3^{15} x_5, & Y_{12} &= x_1 x_2 x_3 x_4^{15}, \\
Y_{13} &= x_1 x_2 x_3^{15} x_4, & Y_{14} &= x_1 x_2^{15} x_4 x_5, & Y_{15} &= x_1 x_2^{15} x_3 x_5, & Y_{16} &= x_1 x_2^{15} x_3 x_4, \\
Y_{17} &= x_1^{15} x_3 x_4 x_5, & Y_{18} &= x_1^{15} x_2 x_4 x_5, & Y_{19} &= x_1^{15} x_2 x_3 x_5, & Y_{20} &= x_1^{15} x_2 x_3 x_4, \\
Y_{21} &= x_2 x_3 x_4^3 x_5^{13}, & Y_{22} &= x_2 x_3^3 x_4 x_5^{13}, & Y_{23} &= x_2 x_3^3 x_4^{13} x_5, & Y_{24} &= x_2^3 x_3 x_4 x_5^{13}, \\
Y_{25} &= x_2^3 x_3 x_4^{13} x_5, & Y_{26} &= x_2^3 x_3^{13} x_4 x_5, & Y_{27} &= x_1 x_3 x_4^3 x_5^{13}, & Y_{28} &= x_1 x_3^3 x_4 x_5^{13}, \\
Y_{29} &= x_1 x_3^3 x_4^{13} x_5, & Y_{30} &= x_1 x_2 x_4^3 x_5^{13}, & Y_{31} &= x_1 x_2 x_3^3 x_5^{13}, & Y_{32} &= x_1 x_2 x_3^3 x_4^{13}, \\
Y_{33} &= x_1 x_2^3 x_4 x_5^{13}, & Y_{34} &= x_1 x_2^3 x_4^{13} x_5, & Y_{35} &= x_1 x_2^3 x_3 x_5^{13}, & Y_{36} &= x_1 x_2^3 x_3^{13} x_5, \\
Y_{37} &= x_1 x_2^3 x_3 x_4^{13}, & Y_{38} &= x_1 x_2^3 x_3^{13} x_4, & Y_{39} &= x_1^3 x_3 x_4 x_5^{13}, & Y_{40} &= x_1^3 x_3 x_4^{13} x_5, \\
Y_{41} &= x_1^3 x_3^{13} x_4 x_5, & Y_{42} &= x_1^3 x_2 x_4 x_5^{13}, & Y_{43} &= x_1^3 x_2 x_4^{13} x_5, & Y_{44} &= x_1^3 x_2 x_3 x_5^{13}, \\
Y_{45} &= x_1^3 x_2 x_3^{13} x_5, & Y_{46} &= x_1^3 x_2 x_3 x_4^{13}, & Y_{47} &= x_1^3 x_2 x_3^{13} x_4, & Y_{48} &= x_1^3 x_2^{13} x_4 x_5, \\
Y_{49} &= x_1^3 x_2^{13} x_3 x_5, & Y_{50} &= x_1^3 x_2^{13} x_3 x_4, & Y_{51} &= x_2 x_3^3 x_4^5 x_5^9, & Y_{52} &= x_2^3 x_3 x_4^5 x_5^9, \\
Y_{53} &= x_2^3 x_3^5 x_4 x_5^9, & Y_{54} &= x_2^3 x_3^5 x_4^9 x_5, & Y_{55} &= x_1 x_3^3 x_4^5 x_5^9, & Y_{56} &= x_1 x_3^3 x_4^5 x_5^9, \\
Y_{57} &= x_1 x_2^3 x_3^5 x_5^9, & Y_{58} &= x_1 x_2^3 x_3^5 x_4^9, & Y_{59} &= x_1^3 x_3 x_4^5 x_5^9, & Y_{60} &= x_1^3 x_3^5 x_4 x_5^9, \\
Y_{61} &= x_1^3 x_3^5 x_4^9 x_5, & Y_{62} &= x_1^3 x_2 x_4^5 x_5^9, & Y_{63} &= x_1^3 x_2 x_3^5 x_5^9, & Y_{64} &= x_1^3 x_2 x_3^5 x_4^9, \\
Y_{65} &= x_1^3 x_2^5 x_4 x_5^9, & Y_{66} &= x_1^3 x_2^5 x_4^9 x_5, & Y_{67} &= x_1^3 x_2^5 x_3 x_5^9, & Y_{68} &= x_1^3 x_2^5 x_3 x_4^9, \\
Y_{69} &= x_1^3 x_2^5 x_3^9 x_5, & Y_{70} &= x_1^3 x_2^5 x_3^9 x_4, & Y_{71} &= x_1 x_2 x_3^2 x_4^5 x_5^9, & Y_{72} &= x_1 x_2 x_3 x_4^3 x_5^{12}, \\
Y_{73} &= x_1 x_2 x_3^3 x_4 x_5^{12}, & Y_{74} &= x_1 x_2 x_3^3 x_4^{12} x_5, & Y_{75} &= x_1 x_2 x_3^3 x_4^9 x_5^9, & Y_{76} &= x_1 x_2 x_3^3 x_4^5 x_8, \\
Y_{77} &= x_1 x_2^3 x_3 x_4^9 x_5, & Y_{78} &= x_1 x_2^3 x_3 x_4^5 x_8, & Y_{79} &= x_1 x_2^3 x_3^5 x_4 x_8, & Y_{80} &= x_1 x_2^3 x_3^5 x_4^8 x_5, \\
Y_{81} &= x_1 x_2^{14} x_3 x_4 x_5, & Y_{82} &= x_1 x_2^2 x_3 x_4 x_5^{13}, & Y_{83} &= x_1 x_2^2 x_3 x_4^{13} x_5, & Y_{84} &= x_1 x_2^2 x_3 x_4^5 x_9, \\
Y_{85} &= x_1 x_2^2 x_3^{13} x_4 x_5, & Y_{86} &= x_1 x_2^2 x_3^5 x_4 x_5^9, & Y_{87} &= x_1 x_2^3 x_4 x_4^9 x_5, & Y_{88} &= x_1 x_2^2 x_3^5 x_4^9 x_5, \\
Y_{89} &= x_1 x_2^3 x_3^4 x_4^9 x_5, & Y_{90} &= x_1^3 x_2 x_3 x_4^4 x_5^9, & Y_{91} &= x_1^3 x_2 x_3 x_4^5 x_8, & Y_{92} &= x_1^3 x_2 x_3^5 x_4^8 x_5, \\
Y_{93} &= x_1^3 x_2 x_3^5 x_4^8 x_5, & Y_{94} &= x_1^3 x_2 x_3^4 x_4^9 x_5, & Y_{95} &= x_1^3 x_2 x_3^4 x_4^9 x_5, & Y_{96} &= x_1 x_2^3 x_3 x_4 x_5^{12}, \\
Y_{97} &= x_1 x_2^3 x_3 x_4^{12} x_5, & Y_{98} &= x_1 x_2^3 x_3^{12} x_4 x_5, & Y_{99} &= x_1^3 x_2 x_3 x_4 x_5^{12}, & Y_{100} &= x_1^3 x_2 x_3 x_4^{12} x_5, \\
Y_{101} &= x_1^3 x_2^5 x_3 x_4 x_5^{14}, & Y_{102} &= x_1^3 x_2^5 x_3 x_4^8 x_5, & Y_{103} &= x_1^3 x_2 x_3^{12} x_4 x_5, & Y_{104} &= x_1^3 x_2^5 x_3^8 x_4 x_5, \\
Y_{105} &= x_1 x_2 x_3 x_4 x_5^{14}, & Y_{106} &= x_1 x_2 x_3 x_4^{14} x_5, & Y_{107} &= x_1 x_2 x_3^{14} x_4 x_5, & Y_{108} &= x_1 x_2 x_3 x_4^2 x_5^{13}, \\
Y_{109} &= x_1 x_2 x_3^2 x_4 x_5^{13}, & Y_{110} &= x_1 x_2 x_3^2 x_4^{13} x_5.
\end{aligned}$$

5.3. Allowable monomials in $(\mathcal{C}_{n_1}^{\otimes 5})^{\text{Param}^0}$

$$\begin{aligned}
X_1 &= x_1 x_2 x_3^2 x_4^6 x_5^{31}, & X_2 &= x_1 x_2 x_3^2 x_4^7 x_5^{30}, & X_3 &= x_1 x_2 x_3^2 x_4^{14} x_5^{23}, & X_4 &= x_1 x_2 x_3^2 x_4^{15} x_5^{22}, \\
X_5 &= x_1 x_2 x_3^2 x_4^{30} x_5^7, & X_6 &= x_1 x_2 x_3^2 x_4^{31} x_5^6, & X_7 &= x_1 x_2 x_3^2 x_4^6 x_5^{30}, & X_8 &= x_1 x_2 x_3^3 x_4^{14} x_5^{22}, \\
X_9 &= x_1 x_2 x_3^3 x_4^{30} x_5^6, & X_{10} &= x_1 x_2 x_3^6 x_4^2 x_5^{31}, & X_{11} &= x_1 x_2 x_3^6 x_4^3 x_5^{30}, & X_{12} &= x_1 x_2 x_3^6 x_4^6 x_5^{27}, \\
X_{13} &= x_1 x_2 x_3^6 x_4^7 x_5^{26}, & X_{14} &= x_1 x_2 x_3^6 x_4^{10} x_5^{23}, & X_{15} &= x_1 x_2 x_3^6 x_4^{11} x_5^{22}, & X_{16} &= x_1 x_2 x_3^6 x_4^{14} x_5^{19}, \\
X_{17} &= x_1 x_2 x_3^6 x_4^{15} x_5^{18}, & X_{18} &= x_1 x_2 x_3^6 x_4^{26} x_5^7, & X_{19} &= x_1 x_2 x_3^6 x_4^{27} x_5^6, & X_{20} &= x_1 x_2 x_3^6 x_4^{30} x_5^5, \\
X_{21} &= x_1 x_2 x_3^6 x_4^{31} x_5^2, & X_{22} &= x_1 x_2 x_3^7 x_4^2 x_5^{30}, & X_{23} &= x_1 x_2 x_3^7 x_4^6 x_5^{26}, & X_{24} &= x_1 x_2 x_3^7 x_4^{10} x_5^{22}, \\
X_{25} &= x_1 x_2 x_3^7 x_4^{14} x_5^{18}, & X_{26} &= x_1 x_2 x_3^7 x_4^{26} x_5^6, & X_{27} &= x_1 x_2 x_3^7 x_4^{30} x_5^2, & X_{28} &= x_1 x_2 x_3^{14} x_4^2 x_5^{23}, \\
X_{29} &= x_1 x_2 x_3^{14} x_4^3 x_5^{22}, & X_{30} &= x_1 x_2 x_3^{14} x_4^6 x_5^{19}, & X_{31} &= x_1 x_2 x_3^{14} x_4^7 x_5^{18}, & X_{32} &= x_1 x_2 x_3^{14} x_4^{18} x_5^7, \\
X_{33} &= x_1 x_2 x_3^{14} x_4^{19} x_5^6, & X_{34} &= x_1 x_2 x_3^{14} x_4^{22} x_5^3, & X_{35} &= x_1 x_2 x_3^{14} x_4^{23} x_5^2, & X_{36} &= x_1 x_2 x_3^{15} x_4^2 x_5^{22}, \\
X_{37} &= x_1 x_2 x_3^{15} x_4^6 x_5^{18}, & X_{38} &= x_1 x_2 x_3^{15} x_4^{18} x_5^6, & X_{39} &= x_1 x_2 x_3^{15} x_4^{22} x_5^2, & X_{40} &= x_1 x_2 x_3^{30} x_4^2 x_5^7, \\
X_{41} &= x_1 x_2 x_3^{30} x_4^3 x_5^6, & X_{42} &= x_1 x_2 x_3^{30} x_4^6 x_5^3, & X_{43} &= x_1 x_2 x_3^{30} x_4^7 x_5^2, & X_{44} &= x_1 x_2 x_3^{31} x_4^2 x_5^6, \\
X_{45} &= x_1 x_2 x_3^{31} x_4^6 x_5^2, & X_{46} &= x_1 x_2^2 x_3 x_4^{15} x_5^{22}, & X_{47} &= x_1 x_2^2 x_3 x_4^{30} x_5^7, & X_{48} &= x_1 x_2^2 x_3 x_4^{14} x_5^{23}, \\
X_{49} &= x_1 x_2^2 x_3 x_4^{15} x_5^{22}, & X_{50} &= x_1 x_2^2 x_3 x_4^{30} x_5^7, & X_{51} &= x_1 x_2^2 x_3 x_4^{31} x_5^6, & X_{52} &= x_1 x_2^2 x_3 x_4^{15} x_5^{22}, \\
X_{53} &= x_1 x_2^2 x_3 x_4^{30} x_5^6, & X_{54} &= x_1 x_2^2 x_3 x_4^{31} x_5^5, & X_{55} &= x_1 x_2^2 x_3 x_4^{12} x_5^{23}, & X_{56} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, \\
X_{57} &= x_1 x_2^2 x_3 x_4^{15} x_5^{20}, & X_{58} &= x_1 x_2^2 x_3 x_4^{28} x_5^7, & X_{59} &= x_1 x_2^2 x_3 x_4^{29} x_5^6, & X_{60} &= x_1 x_2^2 x_3 x_4^{31} x_5^4, \\
X_{61} &= x_1 x_2^2 x_3 x_4^{31} x_5^3, & X_{62} &= x_1 x_2^2 x_3 x_4^{27} x_5^{27}, & X_{63} &= x_1 x_2^2 x_3 x_4^{11} x_5^{23}, & X_{64} &= x_1 x_2^2 x_3 x_4^{15} x_5^{19}, \\
X_{65} &= x_1 x_2^2 x_3 x_4^{27} x_5^7, & X_{66} &= x_1 x_2^2 x_3 x_4^{31} x_5^3, & X_{67} &= x_1 x_2^2 x_3 x_4^{25} x_5^{31}, & X_{68} &= x_1 x_2^2 x_3 x_4^{53} x_5^{30}, \\
X_{69} &= x_1 x_2^2 x_3 x_4^{25} x_5^{26}, & X_{70} &= x_1 x_2^2 x_3 x_4^{25} x_5^{26}, & X_{71} &= x_1 x_2^2 x_3 x_4^{10} x_5^{23}, & X_{72} &= x_1 x_2^2 x_3 x_4^{11} x_5^{22}, \\
X_{73} &= x_1 x_2^2 x_3 x_4^{14} x_5^{19}, & X_{74} &= x_1 x_2^2 x_3 x_4^{15} x_5^{18}, & X_{75} &= x_1 x_2^2 x_3 x_4^{25} x_5^{26}, & X_{76} &= x_1 x_2^2 x_3 x_4^{26} x_5^7, \\
X_{77} &= x_1 x_2^2 x_3 x_4^{30} x_5^3, & X_{78} &= x_1 x_2^2 x_3 x_4^{31} x_5^2, & X_{79} &= x_1 x_2^2 x_3 x_4^{30} x_5^2, & X_{80} &= x_1 x_2^2 x_3 x_4^{33} x_5^{28}, \\
X_{81} &= x_1 x_2^2 x_3 x_4^{27} x_5^{27}, & X_{82} &= x_1 x_2^2 x_3 x_4^{25} x_5^{26}, & X_{83} &= x_1 x_2^2 x_3 x_4^{27} x_5^{24}, & X_{84} &= x_1 x_2^2 x_3 x_4^{28} x_5^{23}, \\
X_{85} &= x_1 x_2^2 x_3 x_4^{27} x_5^{22}, & X_{86} &= x_1 x_2^2 x_3 x_4^{11} x_5^{20}, & X_{87} &= x_1 x_2^2 x_3 x_4^{12} x_5^{19}, & X_{88} &= x_1 x_2^2 x_3 x_4^{13} x_5^{18}, \\
X_{89} &= x_1 x_2^2 x_3 x_4^{15} x_5^{16}, & X_{90} &= x_1 x_2^2 x_3 x_4^{24} x_5^7, & X_{91} &= x_1 x_2^2 x_3 x_4^{25} x_5^6, & X_{92} &= x_1 x_2^2 x_3 x_4^{27} x_5^5, \\
X_{93} &= x_1 x_2^2 x_3 x_4^{28} x_5^3, & X_{94} &= x_1 x_2^2 x_3 x_4^{29} x_5^2, & X_{95} &= x_1 x_2^2 x_3 x_4^{12} x_5^{23}, & X_{96} &= x_1 x_2^2 x_3 x_4^{12} x_5^{19}, \\
X_{97} &= x_1 x_2^2 x_3 x_4^{12} x_5^{19}, & X_{98} &= x_1 x_2^2 x_3 x_4^{12} x_5^{23}, & X_{99} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, & X_{100} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, \\
X_{101} &= x_1 x_2^2 x_3 x_4^{13} x_5^{19}, & X_{102} &= x_1 x_2^2 x_3 x_4^{13} x_5^{18}, & X_{103} &= x_1 x_2^2 x_3 x_4^{13} x_5^{17}, & X_{104} &= x_1 x_2^2 x_3 x_4^{13} x_5^{16}, \\
X_{105} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, & X_{106} &= x_1 x_2^2 x_3 x_4^{13} x_5^{23}, & X_{107} &= x_1 x_2^2 x_3 x_4^{15} x_5^{22}, & X_{108} &= x_1 x_2^2 x_3 x_4^{15} x_5^{20}, \\
X_{109} &= x_1 x_2^2 x_3 x_4^{15} x_5^{19}, & X_{110} &= x_1 x_2^2 x_3 x_4^{15} x_5^{18}, & X_{111} &= x_1 x_2^2 x_3 x_4^{15} x_5^{16}, & X_{112} &= x_1 x_2^2 x_3 x_4^{15} x_5^{16}, \\
X_{113} &= x_1 x_2^2 x_3 x_4^{15} x_5^{17}, & X_{114} &= x_1 x_2^2 x_3 x_4^{15} x_5^{19}, & X_{115} &= x_1 x_2^2 x_3 x_4^{15} x_5^{20}, & X_{116} &= x_1 x_2^2 x_3 x_4^{15} x_5^{21}, \\
X_{117} &= x_1 x_2^2 x_3 x_4^{28} x_5^3, & X_{118} &= x_1 x_2^2 x_3 x_4^{28} x_5^3, & X_{119} &= x_1 x_2^2 x_3 x_4^{29} x_5^2, & X_{120} &= x_1 x_2^2 x_3 x_4^{29} x_5^6, \\
X_{121} &= x_1 x_2^2 x_3 x_4^{29} x_5^3, & X_{122} &= x_1 x_2^2 x_3 x_4^{29} x_5^2, & X_{123} &= x_1 x_2^2 x_3 x_4^{31} x_5^6, & X_{124} &= x_1 x_2^2 x_3 x_4^{31} x_5^4, \\
X_{125} &= x_1 x_2^2 x_3 x_4^{31} x_5^3, & X_{126} &= x_1 x_2^2 x_3 x_4^{31} x_5^5, & X_{127} &= x_1 x_2^2 x_3 x_4^{31} x_5^6, & X_{128} &= x_1 x_2^2 x_3 x_4^{31} x_5^{14}, \\
X_{129} &= x_1 x_2^2 x_3 x_4^{30} x_5^6, & X_{130} &= x_1 x_2^2 x_3 x_4^{32} x_5^{31}, & X_{131} &= x_1 x_2^2 x_3 x_4^{32} x_5^{30}, & X_{132} &= x_1 x_2^2 x_3 x_4^{32} x_5^{28}, \\
X_{133} &= x_1 x_2^2 x_3 x_4^{12} x_5^{23}, & X_{134} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, & X_{135} &= x_1 x_2^2 x_3 x_4^{15} x_5^{20}, & X_{136} &= x_1 x_2^2 x_3 x_4^{15} x_5^{28}, \\
X_{137} &= x_1 x_2^2 x_3 x_4^{29} x_5^6, & X_{138} &= x_1 x_2^2 x_3 x_4^{31} x_5^4, & X_{139} &= x_1 x_2^2 x_3 x_4^{31} x_5^6, & X_{140} &= x_1 x_2^2 x_3 x_4^{31} x_5^{28}, \\
X_{141} &= x_1 x_2^2 x_3 x_4^{12} x_5^{22}, & X_{142} &= x_1 x_2^2 x_3 x_4^{14} x_5^{20}, & X_{143} &= x_1 x_2^2 x_3 x_4^{14} x_5^{28}, & X_{144} &= x_1 x_2^2 x_3 x_4^{14} x_5^{30}, \\
X_{145} &= x_1 x_2^2 x_3 x_4^{14} x_5^{31}, & X_{146} &= x_1 x_2^2 x_3 x_4^{14} x_5^{30}, & X_{147} &= x_1 x_2^2 x_3 x_4^{14} x_5^{27}, & X_{148} &= x_1 x_2^2 x_3 x_4^{14} x_5^{26}, \\
X_{149} &= x_1 x_2^2 x_3 x_4^{10} x_5^{23}, & X_{150} &= x_1 x_2^2 x_3 x_4^{11} x_5^{22}, & X_{151} &= x_1 x_2^2 x_3 x_4^{14} x_5^{19}, & X_{152} &= x_1 x_2^2 x_3 x_4^{14} x_5^{18}, \\
X_{153} &= x_1 x_2^2 x_3 x_4^{26} x_5^7, & X_{154} &= x_1 x_2^2 x_3 x_4^{27} x_5^6, & X_{155} &= x_1 x_2^2 x_3 x_4^{30} x_5^3, & X_{156} &= x_1 x_2^2 x_3 x_4^{31} x_5^2, \\
X_{157} &= x_1 x_2^2 x_3 x_4^{23} x_5^{30}, & X_{158} &= x_1 x_2^2 x_3 x_4^{26} x_5^{26}, & X_{159} &= x_1 x_2^2 x_3 x_4^{10} x_5^{22}, & X_{160} &= x_1 x_2^2 x_3 x_4^{14} x_5^{18}, \\
X_{161} &= x_1 x_2^2 x_3 x_4^{26} x_5^6, & X_{162} &= x_1 x_2^2 x_3 x_4^{30} x_5^2, & X_{163} &= x_1 x_2^2 x_3 x_4^{30} x_5^6, & X_{164} &= x_1 x_2^2 x_3 x_4^{30} x_5^{28}, \\
X_{165} &= x_1 x_2^2 x_3 x_4^{27} x_5^6, & X_{166} &= x_1 x_2^2 x_3 x_4^{26} x_5^{26}, & X_{167} &= x_1 x_2^2 x_3 x_4^{27} x_5^{24}, & X_{168} &= x_1 x_2^2 x_3 x_4^{28} x_5^{23}, \\
X_{169} &= x_1 x_2^2 x_3 x_4^{27} x_5^{22}, & X_{170} &= x_1 x_2^2 x_3 x_4^{11} x_5^{20}, & X_{171} &= x_1 x_2^2 x_3 x_4^{12} x_5^{19}, & X_{172} &= x_1 x_2^2 x_3 x_4^{12} x_5^{18}, \\
X_{173} &= x_1 x_2^2 x_3 x_4^{24} x_5^6, & X_{174} &= x_1 x_2^2 x_3 x_4^{25} x_5^6, & X_{175} &= x_1 x_2^2 x_3 x_4^{27} x_5^4, & X_{176} &= x_1 x_2^2 x_3 x_4^{28} x_5^3, \\
X_{177} &= x_1 x_2^2 x_3 x_4^{29} x_5^2, & X_{178} &= x_1 x_2^2 x_3 x_4^{28} x_5^2, & X_{179} &= x_1 x_2^2 x_3 x_4^{26} x_5^{26}, & X_{180} &= x_1 x_2^2 x_3 x_4^{26} x_5^{24}, \\
X_{181} &= x_1 x_2^2 x_3 x_4^{28} x_5^{22}, & X_{182} &= x_1 x_2^2 x_3 x_4^{10} x_5^{20}, & X_{183} &= x_1 x_2^2 x_3 x_4^{12} x_5^{18}, & X_{184} &= x_1 x_2^2 x_3 x_4^{12} x_5^{16}, \\
X_{185} &= x_1 x_2^2 x_3 x_4^{26} x_5^6, & X_{186} &= x_1 x_2^2 x_3 x_4^{28} x_5^2, & X_{187} &= x_1 x_2^2 x_3 x_4^{12} x_5^{23}, & X_{188} &= x_1 x_2^2 x_3 x_4^{12} x_5^{22}, \\
X_{189} &= x_1 x_2^2 x_3 x_4^{12} x_5^{19}, & X_{190} &= x_1 x_2^2 x_3 x_4^{12} x_5^{18}, & X_{191} &= x_1 x_2^2 x_3 x_4^{12} x_5^{17}, & X_{192} &= x_1 x_2^2 x_3 x_4^{12} x_5^{16}, \\
X_{193} &= x_1 x_2^2 x_3 x_4^{12} x_5^{22}, & X_{194} &= x_1 x_2^2 x_3 x_4^{12} x_5^{22}, & X_{195} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, & X_{196} &= x_1 x_2^2 x_3 x_4^{13} x_5^{18}, \\
X_{197} &= x_1 x_2^2 x_3 x_4^{13} x_5^{18}, & X_{198} &= x_1 x_2^2 x_3 x_4^{13} x_5^{22}, & X_{199} &= x_1 x_2^2 x_3 x_4^{14} x_5^{22}, & X_{200} &= x_1 x_2^2 x_3 x_4^{14} x_5^{23},
\end{aligned}$$

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$$\begin{aligned}
X_{857} &= x_1^3 x_2^{13} x_3^2 x_4^3 x_5^{20}, & X_{858} &= x_1^3 x_2^{29} x_3^2 x_4^3 x_5^4, & X_{859} &= x_1^3 x_2 x_3^4 x_4^7 x_5^{26}, & X_{860} &= x_1^3 x_2 x_3^6 x_4^7 x_5^{24}, \\
X_{861} &= x_1^3 x_2 x_3^{28} x_4^2 x_5^2, & X_{862} &= x_1^3 x_2 x_3^4 x_4^7 x_5^{24}, & X_{863} &= x_1^3 x_2 x_3^5 x_4^2 x_5^{24}, & X_{864} &= x_1^3 x_2 x_3^4 x_4^3 x_5^{24}, \\
X_{865} &= x_1^3 x_2 x_3^4 x_4^3 x_5^2, & X_{866} &= x_1^{15} x_2 x_3^6 x_4^3 x_5^{20}, & X_{867} &= x_1 x_2 x_3^6 x_4^3 x_5^3, & X_{868} &= x_1 x_2 x_3^{14} x_4^{22} x_5^3, \\
X_{869} &= x_1 x_2 x_3^{30} x_4^6 x_5^3, & X_{870} &= x_1 x_2 x_3^5 x_4^{30} x_5^3, & X_{871} &= x_1 x_2 x_3^{12} x_4^{23} x_5^3, & X_{872} &= x_1 x_2 x_3^{13} x_4^{22} x_5^3, \\
X_{873} &= x_1 x_2 x_3^{15} x_4^{20} x_5^3, & X_{874} &= x_1 x_2 x_3^{29} x_4^6 x_5^3, & X_{875} &= x_1 x_2 x_3^4 x_4^{30} x_5^3, & X_{876} &= x_1 x_2 x_3^6 x_4^{28} x_5^3, \\
X_{877} &= x_1 x_2 x_3^{12} x_4^{22} x_5^3, & X_{878} &= x_1 x_2 x_3^{14} x_4^{20} x_5^3, & X_{879} &= x_1 x_2 x_3^{28} x_4^6 x_5^3, & X_{880} &= x_1 x_2 x_3^{30} x_4^4 x_5^3, \\
X_{881} &= x_1 x_2 x_3^6 x_4^{30} x_5^3, & X_{882} &= x_1 x_2^{14} x_3 x_4^{22} x_5^3, & X_{883} &= x_1 x_2^{15} x_3^2 x_4^{20} x_5^3, & X_{884} &= x_1 x_2^{30} x_3 x_4^6 x_5^3, \\
X_{885} &= x_1 x_2 x_3^{28} x_4^7 x_5^3, & X_{886} &= x_1 x_2 x_3^6 x_4^{26} x_5^3, & X_{887} &= x_1 x_2 x_3^{30} x_4^2 x_5^7, & X_{888} &= x_1 x_2 x_3^2 x_4^{30} x_5^7, \\
X_{889} &= x_1 x_2 x_3^2 x_4^{27} x_5^7, & X_{890} &= x_1 x_2 x_3^5 x_4^{26} x_5^7, & X_{891} &= x_1 x_2 x_3^2 x_4^{24} x_5^7, & X_{892} &= x_1 x_2 x_3^{29} x_4^7 x_5^7, \\
X_{893} &= x_1 x_2 x_3^4 x_4^{26} x_5^7, & X_{894} &= x_1 x_2 x_3^6 x_4^{24} x_5^7, & X_{895} &= x_1 x_2 x_3^{28} x_4^2 x_5^7, & X_{896} &= x_1 x_2 x_3^6 x_4^{26} x_5^7, \\
X_{897} &= x_1 x_2 x_3^7 x_4^{24} x_5^7, & X_{898} &= x_1 x_2^{30} x_3 x_4^2 x_5^7, & X_{899} &= x_1 x_2 x_3^2 x_4^6 x_5^{31}, & X_{900} &= x_1 x_2 x_3^6 x_4^2 x_5^{31}, \\
X_{901} &= x_1 x_2 x_3^2 x_4^6 x_5^{31}, & X_{902} &= x_1 x_2 x_3^5 x_4^2 x_5^{31}, & X_{903} &= x_1 x_2 x_3^4 x_4^2 x_5^{31}, & X_{904} &= x_1 x_2 x_3^6 x_4^2 x_5^{31}, \\
X_{905} &= x_1^3 x_2 x_3^4 x_4^{30} x_5^3, & X_{906} &= x_1^3 x_2 x_3^6 x_4^{28} x_5^3, & X_{907} &= x_1^3 x_2 x_3^{12} x_4^{22} x_5^3, & X_{908} &= x_1^3 x_2 x_3^{14} x_4^{20} x_5^3, \\
X_{909} &= x_1^3 x_2 x_3^{28} x_4^6 x_5^3, & X_{910} &= x_1^3 x_2 x_3^{30} x_4^4 x_5^3, & X_{911} &= x_1^3 x_2 x_3^5 x_4^{28} x_5^3, & X_{912} &= x_1^3 x_2 x_3^6 x_4^{24} x_5^3, \\
X_{913} &= x_1^3 x_2 x_3^5 x_4^{10} x_5^3, & X_{914} &= x_1^3 x_2 x_3^5 x_4^{26} x_5^3, & X_{915} &= x_1^3 x_2 x_3^{13} x_4^2 x_5^3, & X_{916} &= x_1^3 x_2 x_3^{29} x_4^4 x_5^3, \\
X_{917} &= x_1^3 x_2 x_3^4 x_4^{26} x_5^7, & X_{918} &= x_1^3 x_2 x_3^6 x_4^{24} x_5^7, & X_{919} &= x_1^3 x_2 x_3^{28} x_4^2 x_5^7, & X_{920} &= x_1^3 x_2 x_3^4 x_4^{24} x_5^7, \\
X_{921} &= x_1^3 x_2 x_3^5 x_4^{24} x_5^7, & X_{922} &= x_1^3 x_2 x_3^4 x_4^2 x_5^{31}, & X_{923} &= x_1^3 x_2 x_3^4 x_4^2 x_5^{31}, & X_{924} &= x_1^7 x_2 x_3^2 x_4^{24} x_5^7, \\
X_{925} &= x_1^{15} x_2 x_3^2 x_4^{20} x_5^3.
\end{aligned}$$

5.4. Some strictly nonallowable monomials in $(\mathcal{P}_5^{\geq 0})_{n_1}$

$$\begin{aligned}
T_1 &= x_1 x_2^7 x_3^{24} x_4^3 x_5^6, & T_2 &= x_1 x_2^{14} x_3^{19} x_4^5 x_5^2, & T_3 &= x_1 x_2^{15} x_3^{18} x_4^5 x_5^2, & T_4 &= x_1 x_2^6 x_3^3 x_4^{15} x_5^{16}, \\
T_5 &= x_1 x_2^6 x_3^9 x_4^3 x_5^{22}, & T_6 &= x_1 x_2^6 x_3^9 x_4^3 x_5^{22}, & T_7 &= x_1 x_2^6 x_3^{11} x_4^{16} x_5^7, & T_8 &= x_1 x_2^6 x_3^{11} x_4^{19} x_5^4, \\
T_9 &= x_1 x_2^6 x_3^{15} x_4^3 x_5^{16}, & T_{10} &= x_1 x_2^6 x_3^{15} x_4^{16} x_5^3, & T_{11} &= x_1 x_2^6 x_3^{15} x_4^{17} x_5^2, & T_{12} &= x_1 x_2^6 x_3^{25} x_4^3 x_5^6, \\
T_{13} &= x_1 x_2^7 x_3^8 x_4^3 x_5^{22}, & T_{14} &= x_1 x_2^7 x_3^8 x_4^{19} x_5^6, & T_{15} &= x_1 x_2^7 x_3^9 x_4^{18} x_5^6, & T_{16} &= x_1 x_2^7 x_3^{10} x_4^{16} x_5^7, \\
T_{17} &= x_1 x_2^7 x_3^{10} x_4^{19} x_5^4, & T_{18} &= x_1 x_2^7 x_3^{14} x_4^3 x_5^{16}, & T_{19} &= x_1 x_2^7 x_3^{14} x_4^{17} x_5^2, & T_{20} &= x_1 x_2^7 x_3^{25} x_4^2 x_5^6, \\
T_{21} &= x_1 x_2^7 x_3^{25} x_4^6 x_5^2, & T_{22} &= x_1 x_2^{14} x_3^3 x_4^7 x_5^{16}, & T_{23} &= x_1 x_2^{14} x_3^3 x_4^{16} x_5^7, & T_{24} &= x_1 x_2^{14} x_3^3 x_4^{19} x_5^4, \\
T_{25} &= x_1 x_2^{14} x_3^7 x_4^{16} x_5^3, & T_{26} &= x_1 x_2^{14} x_3^7 x_4^{16} x_5^3, & T_{27} &= x_1 x_2^{14} x_3^7 x_4^{17} x_5^2, & T_{28} &= x_1 x_2^{14} x_3^{16} x_4^3 x_5^7, \\
T_{29} &= x_1 x_2^{14} x_3^{16} x_4^7 x_5^3, & T_{30} &= x_1 x_2^{14} x_3^{17} x_4^6 x_5^3, & T_{31} &= x_1 x_2^{14} x_3^{17} x_4^6 x_5^3, & T_{32} &= x_1 x_2^{14} x_3^{17} x_4^7 x_5^2, \\
T_{33} &= x_1 x_2^{14} x_3^{19} x_4^3 x_5^4, & T_{34} &= x_1 x_2^{15} x_3^6 x_4^3 x_5^{16}, & T_{35} &= x_1 x_2^{15} x_3^6 x_4^{16} x_5^3, & T_{36} &= x_1 x_2^{15} x_3^6 x_4^{17} x_5^2, \\
T_{37} &= x_1 x_2^{15} x_3^{16} x_4^3 x_5^6, & T_{38} &= x_1 x_2^{15} x_3^{16} x_4^6 x_5^3, & T_{39} &= x_1 x_2^{15} x_3^{16} x_4^7 x_5^2, & T_{40} &= x_1 x_2^{15} x_3^{17} x_4^2 x_5^6, \\
T_{41} &= x_1 x_2^{15} x_3^{17} x_4^6 x_5^2, & T_{42} &= x_1 x_2^{15} x_3^{18} x_4^4 x_5^3, & T_{43} &= x_1 x_2^{15} x_3^{19} x_4^2 x_5^4, & T_{44} &= x_1 x_2^{15} x_3^{19} x_4^4 x_5^2, \\
T_{45} &= x_1^3 x_2 x_3^{12} x_4^{19} x_5^6, & T_{46} &= x_1^3 x_2 x_3^{14} x_4^{15} x_5^{18}, & T_{47} &= x_1^3 x_2 x_3^4 x_4^9 x_5^{19}, & T_{48} &= x_1^3 x_2 x_3^{15} x_4^{18} x_5^6, \\
T_{49} &= x_1^3 x_2 x_3^{15} x_4^{17} x_5^{16}, & T_{50} &= x_1^3 x_2 x_3^{15} x_4^{15} x_5^{16}, & T_{51} &= x_1^3 x_2 x_3^{15} x_4^{16} x_5^2, & T_{52} &= x_1^3 x_2 x_3^{15} x_4^{16} x_5^2, \\
T_{53} &= x_1^3 x_2 x_3^{15} x_4^{19} x_5^4, & T_{54} &= x_1^3 x_2 x_3^{15} x_4^{16} x_5^4, & T_{55} &= x_1^3 x_2 x_3^{15} x_4^{16} x_5^4, & T_{56} &= x_1^3 x_2 x_3^{15} x_4^{16} x_5^4, \\
T_{57} &= x_1^3 x_2 x_3^{12} x_4^3 x_5^{22}, & T_{58} &= x_1^3 x_2 x_3^{12} x_4^{21} x_5^2, & T_{59} &= x_1^3 x_2 x_3^{12} x_4^{19} x_5^{25}, & T_{60} &= x_1^3 x_2 x_3^{13} x_4^{18} x_5^{25}, \\
T_{61} &= x_1^3 x_2 x_3^{13} x_4^{19} x_5^2, & T_{62} &= x_1^3 x_2 x_3^{15} x_4^4 x_5^{18}, & T_{63} &= x_1^3 x_2 x_3^{15} x_4^{17} x_5^2, & T_{64} &= x_1^3 x_2 x_3^{15} x_4^{16} x_5^2, \\
T_{65} &= x_1^3 x_2 x_3^{15} x_4^5 x_5^{16}, & T_{66} &= x_1^3 x_2 x_3^{15} x_4^5 x_5^{16}, & T_{67} &= x_1^3 x_2 x_3^{15} x_4^{17} x_5^2, & T_{68} &= x_1^3 x_2 x_3^{15} x_4^{20} x_5^2, \\
T_{69} &= x_1^3 x_2 x_3^{28} x_4^3 x_5^6, & T_{70} &= x_1^3 x_2 x_3^{28} x_4^3 x_5^6, & T_{71} &= x_1^3 x_2 x_3^{28} x_4^3 x_5^6, & T_{72} &= x_1^3 x_2 x_3^{28} x_4^3 x_5^6, \\
T_{73} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{74} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{75} &= x_1^7 x_2 x_3^{10} x_4^{16} x_5^7, & T_{76} &= x_1^7 x_2 x_3^{10} x_4^{19} x_5^4, \\
T_{77} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{78} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{79} &= x_1^7 x_2 x_3^{24} x_4^3 x_5^6, & T_{80} &= x_1^7 x_2 x_3^{25} x_4^2 x_5^6, \\
T_{81} &= x_1^7 x_2 x_3^{25} x_4^6 x_5^2, & T_{82} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{83} &= x_1^7 x_2 x_3^{11} x_4^{16} x_5^6, & T_{84} &= x_1^7 x_2 x_3^{11} x_4^{16} x_5^6, \\
T_{85} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{86} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{87} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, & T_{88} &= x_1^7 x_2 x_3^{14} x_4^3 x_5^{16}, \\
T_{89} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{90} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{91} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{92} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, \\
T_{93} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{94} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{95} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{96} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, \\
T_{97} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{98} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{99} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, & T_{100} &= x_1^7 x_2 x_3^{19} x_4^5 x_5^6, \\
T_{101} &= x_1^7 x_2 x_3^{24} x_4^3 x_5^6, & T_{102} &= x_1^7 x_2 x_3^{24} x_4^3 x_5^6, & T_{103} &= x_1^7 x_2 x_3^{24} x_4^3 x_5^6, & T_{104} &= x_1^7 x_2 x_3^{24} x_4^3 x_5^6, \\
T_{105} &= x_1^7 x_2 x_3^{25} x_4^3 x_5^6, & T_{106} &= x_1^7 x_2 x_3^{25} x_4^3 x_5^6, & T_{107} &= x_1^7 x_2 x_3^{25} x_4^3 x_5^6, & T_{108} &= x_1^7 x_2 x_3^{25} x_4^3 x_5^6, \\
T_{109} &= x_1^7 x_2 x_3^{25} x_4^3 x_5^6, & T_{110} &= x_1^7 x_2 x_3^{25} x_4^3 x_5^6, & T_{111} &= x_1^{15} x_2 x_3^6 x_4^{16} x_5^3, & T_{112} &= x_1^{15} x_2 x_3^6 x_4^{16} x_5^3,
\end{aligned}$$

$$\begin{aligned}
T_{113} &= x_1^{15} x_2 x_3^6 x_4^{17} x_5^2, & T_{114} &= x_1^{15} x_2 x_3^{16} x_4^3 x_5^6, & T_{115} &= x_1^{15} x_2 x_3^{16} x_4^6 x_5^3, & T_{116} &= x_1^{15} x_2 x_3^{16} x_4^7 x_5^2, \\
T_{117} &= x_1^{15} x_2 x_3^{17} x_4^6 x_5^2, & T_{118} &= x_1^{15} x_2 x_3^{17} x_4^6 x_5^2, & T_{119} &= x_1^{15} x_2 x_3^{18} x_4^4 x_5^3, & T_{120} &= x_1^{15} x_2 x_3^{18} x_4^5 x_5^2, \\
T_{121} &= x_1^{15} x_2 x_3^{19} x_4^2 x_5^2, & T_{122} &= x_1^{15} x_2 x_3^{19} x_4^2 x_5^2, & T_{123} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{124} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, \\
T_{125} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{126} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{127} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{128} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, \\
T_{129} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{130} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{131} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{132} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, \\
T_{133} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{134} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{135} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{136} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, \\
T_{137} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{138} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{139} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{140} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, \\
T_{141} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{142} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{143} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2, & T_{144} &= x_1^{15} x_2 x_3^{20} x_4^2 x_5^2.
\end{aligned}$$

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Future work

It is well-known that the polynomial algebra $\mathcal{P}_m = \mathbb{F}_2[x_1, \dots, x_m]$ admits the right action of the general linear group $G(m)$ by matrix substitution and the left action of the Steenrod algebra $\mathcal{A}$ at the prime 2. The matrix action preserves grading, whilst the algebra $\mathcal{A}$ is generated by elements Sq^k that raise the grading by k . The two actions commute, which implies that the algebra of invariants $\mathcal{P}_m^{\mathcal{G}}$, for any subgroup $\mathcal{G} \subseteq G(m)$, is an $\mathcal{A}$ -module. The hit problem is to find a minimal generating set for the $\mathcal{A}$ -module $\mathcal{P}_m^{\mathcal{G}}$ for varying m and $\mathcal{G}$. This topic pertains to the outcomes derived in this paper, wherein group $\mathcal{G}$ is trivial, and $m \geq 5$. Consider $\mathcal{G} = \Sigma_m$, the symmetric group on m elements, we have the symmetric polynomial algebra $\mathcal{P}_m^{\Sigma_m} = \mathbb{F}_2[\sigma_1, \dots, \sigma_m]$ on the elementary symmetric functions σ_j of degree j in m variables. We aim to study the symmetric hit problem for this algebra $\mathcal{P}_m^{\Sigma_m}$. The problem is known for the variables $m \leq 3$ (cf. [JW02, JW03, Jan11]). The intricacy of the problem magnifies considerably for higher variables, making the classification of elements into hit and non-hit classes an exceedingly challenging task, particularly in the context of four variables. Nonetheless, it is a compelling research subject worthy of exploration in the series of classical hit problems.

Let us recall that the group Σ_m acts on $\mathcal{P}_m$ by permuting the variables (i.e., for each $\theta \in \Sigma_m$, $\theta(x_1^{a_1} \dots x_m^{a_m}) = x_{\theta(1)}^{a_1} \dots x_{\theta(m)}^{a_m}$.) It is a known fact that the (left) action of the algebra $\mathcal{A}$ and the action of Σ_m on $\mathcal{P}_m$ commute. Therefore, an induced action of $\mathcal{A}$ on $\mathcal{B}_m := \mathcal{P}_m^{\Sigma_m}$, the graded $\mathcal{A}$ -submodule of the left $\mathcal{A}$ -module $\mathcal{P}_m$ containing elementary symmetric functions in m variables, can be established. This action has topological significance, as $\mathcal{B}_m$ can be identified with the (mod 2) cohomology of the Grassmannian of m -dimensional subspaces of $\mathbb{R}^\infty$ [WW18b]. In other point of view, the algebra $\mathcal{B}_m$ realizes the cohomology of the classifying space of the orthogonal group. It is by now a well known fact that the ideal in $\mathcal{P}_m$ generated by the m -th elementary symmetric function $\sigma_m = x_1 x_2 \dots x_m$ can be identified with the cohomology of the m -fold smash product of $\mathbb{R}P^\infty$ in positive dimensions. Thus, we may also write $\mathcal{B}_m = \mathbb{F}_2[\sigma_1, \dots, \sigma_m] = \bigoplus_{n \geq 0} (\mathcal{B}_m)_n$. For example,

$$\sigma_1(x_1, \dots, x_m) = \sum_{1 \leq j \leq m} x_j, \sigma_2(x_1, \dots, x_m) = \sum_{1 \leq i, j \leq m} x_i x_j, \text{ etc. The grading is inherited from } \mathcal{P}_m.$$

This forces σ_j to have grading j . A polynomial $g \in \mathcal{B}_m$ is hit, if it is in the image of the action $\overline{\mathcal{A}} \otimes_{\mathbb{F}_2} \mathcal{B}_m \longrightarrow \mathcal{B}_m$. In other words, in each positive degree n , the polynomial g can be represented as $g = \sum_{j \geq 1} Sq^j(g_j)$, $g_j \in (\mathcal{B}_m)_{n-j}$. In this case, we also say that g is *symmetrically hit* [Jan08]. The

quotient $\mathbb{F}_2$ -module of cohit elements in $\mathcal{B}_m$ is defined by $Q^{\otimes m}(\mathcal{B}_m) = \mathcal{B}_m / \overline{\mathcal{A}} \mathcal{B}_m \cong \mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{B}_m$, on which $\mathbb{F}_2$ concentrated in degree 0 and is given the trivial $\mathcal{A}$ -module structure through the augmentation $\mathcal{A} \longrightarrow \mathbb{F}_2$. Thus, in each positive degree n , one has the hit problem of determining

the dimension of the symmetric cohit space

$$Q_n^{\otimes m}(\mathcal{B}_m) = (\mathcal{B}_m)_n / (\mathcal{B}_m)_n \cap \overline{\mathcal{A}}\mathcal{B}_m.$$

(Here $(\mathcal{B}_m)_n \cap \overline{\mathcal{A}}\mathcal{B}_m$ is the submodule of symmetrically hit elements in $(\mathcal{B}_m)_n$.) Due to [JW02], all the monomials of degree n in $\mathcal{B}_m$ are hit if and only if $\beta(n) > m$. This means that $Q_n^{\otimes m}(\mathcal{B}_m) \neq 0$ if and only if $\beta(n) \leq m$. In particular, if $\beta(2n+m) = m$, then $Q_n^{\otimes m}(\mathcal{B}_m) \cong Q_{2n+m}^{\otimes m}(\mathcal{B}_m)$. These results are similar to Theorem 2.1 (see Sect.2). The expression for the action of Steenrod squares on elementary symmetric functions is known as the Wu formula. More precisely, for $1 \leq i \leq m$, let $\sigma_\ell \in \mathcal{B}_m$ be the ℓ -th elementary symmetric function, and let $0 \leq k \leq \ell$. Then

$$Sq^k(\sigma_\ell) = \sum_{0 \leq i \leq k} (-1)^i \binom{\ell - k + i - 1}{i} \sigma_{k-i} \sigma_{\ell+i}.$$

We refer the reader to [WW18b] for an explicit proof. According to [JW02], it is more convenient to work with the symmetrized monomial basis of $\mathcal{B}_m$ rather than with elementary symmetric function. As it is known, an element π in Σ_m acts on a monomial by permuting the variables. The notation $\pi_{(i,j)}$ is used for the element of Σ_m which switches x_i and x_j . The symmetrization of a monomial $f \in \mathcal{P}_m$ is the "smallest" symmetric polynomial $\sigma(f) \in \mathcal{B}_m$ containing f as a term. To be more precise, $\sigma(f) = \sum_{1 \leq j \leq t} f\pi_j$, where $\pi_1, \dots, \pi_t$ run through a set of left coset representatives

for the stabilizer of the monomial f in Σ_m . For instance, with $f = x_1x_2^2 \in \mathcal{P}_2$, then in $\mathcal{B}_2$, one has $\sigma(f) = f\pi_1 + f\pi_2 = f + f\pi_{(1,2)} = f + x_1^2x_2$, where π_1 is the identity and $\pi_2 = \pi_{(1,2)}$. In $\mathcal{B}_3$, $\sigma(f)$ has $|\Sigma_3| = 3! = 6$ terms, namely

$$\sigma(f) = f + f\pi_{(1,2)} + f\pi_{(2,3)} + f\pi_{(1,3)} + f\pi_{(1,2,3)} + f\pi_{(1,3,2)}.$$

In $\mathcal{B}_4$, $\sigma(f)$ has $|\Sigma_4| = 4! = 24$ terms, that is,

$$\begin{aligned} \sigma(f) = & f + f\pi_{(1,2)} + f\pi_{(1,3)} + f\pi_{(1,4)} + f\pi_{(2,3)} + f\pi_{(2,4)} + f\pi_{(3,4)} \\ & + f\pi_{(1,2)(3,4)} + f\pi_{(1,3)(2,4)} + f\pi_{(1,4)(2,3)} + f\pi_{(1,2,3)} + f\pi_{(1,3,2)} \\ & + f\pi_{(1,2,4)} + f\pi_{(1,4,2)} + f\pi_{(1,3,4)} + f\pi_{(1,4,3)} + f\pi_{(2,3,4)} + f\pi_{(2,4,3)} \\ & + f\pi_{(1,2,3,4)} + f\pi_{(1,2,4,3)} + f\pi_{(1,3,2,4)} + f\pi_{(1,3,4,2)} + f\pi_{(1,4,2,3)} + f\pi_{(1,4,3,2)}. \end{aligned}$$

The following open conjecture [Jan11] plays an important role in partitioning the elements of $\mathcal{P}_m$ into hit and non-hit classes: *If a monomial f is hit in $\mathcal{P}_m$, then $\sigma(f)$ is symmetrically hit in $\mathcal{B}_m$.* It should be mentioned that the converse of this conjecture is not generally true (see also [Jan08] for a counterexample.) The conjecture has been established by Janfada [Jan08] for $m \leq 3$. While the conjecture remains unproven or refuted in its general form, it can be established through examination of the following known specific cases. First, if $f \in (\mathcal{P}_m)_n$ is lower than a minimal spike of degree n , then by Theorem 3.2.5 (see Subsect.3.2), f is hit in $(\mathcal{P}_m)_n$, and so, by [JW02], $\sigma(f)$ is hit in $(\mathcal{B}_m)_n$. Next, if f is of the form $f = gh^{2^k}$, where every exponent of g is less than 2^k and $\deg(g) < (2^k - 1)\beta(\deg(h))$, then by Silverman [Sil98], f must be hit, and therefore, by [JW02], $\sigma(f)$ is symmetrically hit. Finally, according to Janfada's proof in [Jan11], if the exponents of f are distinct and f is hit, then $\sigma(f)$ is symmetrically hit.

As far as our knowledge extends, during his PhD program [JW02, JW03], Janfada, in collaboration with his supervisor Reginald Wood, studied the first symmetric hit problem for the number of variables not exceeding three. Their works were focused on the symmetric version of the Peterson conjecture and the Kameko isomorphism. Following [JW03], in the 1-variable case, we have that $\mathcal{B}_1 = \mathcal{P}_1$ and $\mathcal{P}_1$ is minimally generated by the spikes x^{2^s-1} . The situation for $m = 2$ is similar, and more precisely, that the symmetrized spikes $\sigma(x_1^{2^{t+s}-1}x_2^{2^t-1})$ for $t, s \geq 0$, form a minimal generating set for the $\mathcal{A}$ -module $\mathcal{B}_2$. When $m = 3$, the dimension of the symmetric cohit space $Q_n^{\otimes 3}(\mathcal{B}_3)$ is zero unless $n = 2^{u+t+s} + 2^{u+t} + 2^u - 3$, whenever $u, t, s \geq 0$. In this case, the dimension is independent of u when $s > 0$ and depends on t, s as follows:

$\dim Q_{2^{u+t+s}+2^{u+t}+2^u-3}^{\otimes 3}(\mathcal{B}_3)$	$u = 0$	$u \geq 0$			
	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s > 3$
$t = 0$	1	1	2	3	4
$t > 0$	1	1	2	2	2

Therefrom, as immediate consequences, *the best upper bounds for the dimension of $Q_n^{\otimes m}(\mathcal{B}_m)$ in cases $m = 1, 2, 3$ are respectively 1, 1, 4* [JW03]. The strategy used in those works is based on a splicing method in binary block notation (see also the monograph [WW18b] for more information). The results also reduce the study of symmetric hit problem for the cases $\beta(n) < m$. In 2014, Janfada [Jan14] exhibited the criteria for basis elements to be non-hit, and thus, the symmetric hit problem has been resolved for variables up to 3. Motivated entirely by the already described achievements, our objective, which we strive to accomplish in due course, is to address the symmetric hit problem for the four-variable case. The arduousness of this task lies in the derivation of basis elements in four variables, originating from generators in three variables, and subsequently proving their linear independence. It is inherently challenging to identify that a polynomial in $\mathcal{B}_m$ is not symmetrically hit. Despite this, initial calculations reveal that in most cases the minimal generating set of $Q_n^{\otimes 3}(\mathcal{B}_3)$ [JW03] provides most of the minimal generating set of $Q_n^{\otimes 4}(\mathcal{B}_4)$. The challenge arises in a limited number of cases, which will be clarified in our other works.

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