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Theorems in Topology

Open Mathematics Collaboration*†

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Abstract

This is an introductory collection of theorems in topology.

keywords: axioms, definitions, theorems, topology

The most updated version of this white paper is available at
https://osf.io/zm56w/download
https://zenodo.org/record/6205551

Introduction

1. We present some basic theorems in Topology using the fewest number of mathematical symbols, without losing information.

2. The idea is to provide a *global overview* of the *results* without worrying about their proofs.

3. This `white paper` is being *updated from time to time*.

*All *authors* with their *affiliations* appear at the end of this white paper.
†Corresponding author: mplobo@uft.edu.br | **Open Mathematics Collaboration**

Metalinguistic Symbols

4. A `metalinguistic` symbol is not part of the language.

5. The symbol $:=$ means that *what is on the left side is defined by the right side of it.*

6. Depending on the context, $\equiv$ and $\not\equiv$ are used to state a theorem.

7. For example, in the statement of a theorem, $\equiv$ can be read as *is, are, has, have* etc.

8. [1, 2]

Archimedean Property

9. `Theorem`

$$(\forall x \in F \; \exists n \in \mathbb{N} : n > x) \;\equiv\; (F \text{ has the } \texttt{Archimedean Property})$$

10. $F :=$ ordered field

11. (9) means that $\mathbb{N}$ is *unbounded* in F.

12. `Theorem`

$$\mathbb{Q}, \mathbb{R} \equiv \texttt{Archimedean Property}$$

13. $\mathbb{Q} :=$ set of rational numbers

14. $\mathbb{R} :=$ set of real numbers

15. [3]

Density Theorem

16. `Theorem`

$$(x, y \in \mathbb{R},\ x < y)\ \rightarrow\ (\exists q \in \mathbb{Q} : x < q < y)$$

17.

$$\mathbb{Q}\ \texttt{is dense in}\ \mathbb{R}$$

18. [3]

Open Interval

19. `Definition`

$$(a, b) = \{x \in X \mid a < x < b\}$$

20. (a, b) := open interval in X

21. X := set

22. [3]

Open Set

23. `Definition`

$$(\forall x \in X : x \in (a,b) \subseteq X) \;\equiv\; X := \texttt{open set in } \mathbb{R}$$

24. `Definition`

$$X := \texttt{open set} \text{ in } Y \text{ if}$$

(i) $X \subseteq Y$

(ii) $\forall x \in X : x \in (a,b) \subseteq X$

25. $X, Y :=$ sets

26. $(a,b) :=$ open interval in $\mathbb{R}$

27. [3]

Open Interval, Open in $\mathbb{R}$

28. `Theorem`

$$\forall (a,b) : (a,b) \equiv \texttt{open set} \text{ in } \mathbb{R}$$

29. `Theorem`

$$(\forall h.\ h \coloneqq \texttt{half-open interval} \text{ in } \mathbb{R}) \ \rightarrow \ (h \not\equiv \texttt{open set} \text{ in } \mathbb{R})$$

30. `Theorem`

$$(\forall c.\ c \coloneqq \texttt{closed interval} \text{ in } \mathbb{R}) \ \rightarrow \ (c \not\equiv \texttt{open set} \text{ in } \mathbb{R})$$

31. $(a,b) \coloneqq$ open interval in $\mathbb{R}$

32. [3]

Infinite Interval, Open in $\mathbb{R}$

33. `Theorem`

$$a \in \mathbb{R} \;\to\; (a, \infty) \equiv \texttt{open} \text{ in } \mathbb{R}$$

34. `Theorem`

$$b \in \mathbb{R} \;\to\; (-\infty, b) \equiv \texttt{open} \text{ in } \mathbb{R}$$

35. $(a, \infty), (-\infty, b) :=$ infinite open intervals in $\mathbb{R}$

36. [3]

Empty Set, Set of Real Numbers, Open in $\mathbb{R}$

37. `Theorem`

$$\varnothing, \mathbb{R} \equiv \texttt{open} \text{ in } \mathbb{R}$$

38. [3]

Open Subset of $\mathbb{R}$, Open in $\mathbb{R}$

39. `Theorem`

$$(X \equiv \texttt{open} \text{ in } \mathbb{R}) \;\leftrightarrow\; (\forall x \in X,\ \exists c > 0,\ c \in \mathbb{R} : (x - c, x + c) \subseteq X)$$

40. $X \subseteq \mathbb{R}$; $\quad (x - c, x + c) :=$ open interval in $\mathbb{R}$

41. [3]

Union of Open Sets in $\mathbb{R}$

42. `Theorem`

$$A \cup B \equiv \texttt{open} \text{ in } \mathbb{R}$$

43. $A, B :=$ open sets in $\mathbb{R}$

44. [3]

Union of a collection of sets

45. `Definition`

$$\bigcup X := \texttt{union} \text{ of all elements of } X$$

46. $\bigcup X := \{y \mid \exists Y \in X,\ y \in Y\}$

47. $X :=$ collection of sets

48. $Y :=$ set

49. [3]

Intersection of a collection of sets

50. `Definition`

$$\bigcap X := \texttt{intersection} \text{ of all elements of } X$$

51. $\bigcap X := \{y \mid \forall Y \in X,\ y \in Y\}$

52. $X :=$ collection of sets

53. $Y :=$ set

54. [3]

Union of Open Subsets of $\mathbb{R}$, Open in $\mathbb{R}$

55. `Theorem`

$$\bigcup X \equiv \texttt{open in } \mathbb{R}$$

56. $X :=$ set of open subsets of $\mathbb{R}$

57. $\bigcup X :=$ union of all elements of X

58. [3]

Upper/Lower Bound in $\mathbb{R}$

59. `Definition`

$$(\exists m \in \mathbb{R} : \forall s \in S,\ s \leq m) \ \equiv \ (m := \texttt{upper bound} \text{ of } S)$$

60. `Definition`

$$(\exists k \in \mathbb{R} : \forall s \in S,\ s \geq k) \ \equiv \ (k := \texttt{lower bound} \text{ of } S)$$

61. $\varnothing \neq S \subseteq \mathbb{R}$

62. [4]

Bounded Set

63. `Definition`

$$\texttt{bounded set} := \text{bounded above and below}$$

64. [3]

Bounded Open Intervals

65. `Theorem`

$$(\forall X \subseteq \mathbb{R},\ X \neq \varnothing,\ X \coloneqq \text{open set in } \mathbb{R}) \ \rightarrow\ (X = \bigcup B)$$

66. $B \coloneqq$ set of all `bounded open intervals` in X

67. $\bigcup B \coloneqq$ union of all elements of B

68. [3]

Pairwise Disjoint Open Intervals

69. `Theorem`

$$(\forall X \subseteq \mathbb{R},\ X \neq \varnothing,\ X \coloneqq \text{open set in } \mathbb{R}) \ \rightarrow\ (X = \bigcup_c P)$$

70. $\bigcup_c \coloneqq$ `countable union`

71. $P \coloneqq$ set of `pairwise disjoint open intervals`

72. [3]

Modal operators

73. `Definitions`

74. $\Box, \Diamond :=$ (unary) modal operators

75. $\Box :=$ *necessarily*

76. $\Diamond :=$ *possibly*

77. $\varphi :=$ formula

78. $\Diamond\varphi \equiv \neg \Box \neg\varphi$

79. [5]

Intersection of Open Sets in $\mathbb{R}$

80. `Theorem`

$$A \cap B \ \equiv \ \texttt{open} \text{ in } \mathbb{R}$$

81. `Theorem`

$$\bigcap_{i=1}^{n} A_i \equiv \texttt{open} \text{ in } \mathbb{R}$$

82. `Theorem`

$$\bigcap_{i=1}^{\infty} A_i \equiv \neg \Box \, \text{open in } \mathbb{R}$$

83. $A, B, A_i \coloneqq$ open sets in $\mathbb{R}$

84. $\bigcap A_i \coloneqq$ arbitrary intersection of A_i

85. $\neg\Box \coloneqq$ not necessarily

86. [3]

Closed Set in $\mathbb{R}$, Complement of a Set

87. `Definition`

$$(\mathbb{R}\backslash X \equiv \texttt{open} \text{ in } \mathbb{R}) \;\rightarrow\; (X \coloneqq \texttt{closed} \text{ in } \mathbb{R})$$

88. $X \subseteq \mathbb{R}$

89. $\mathbb{R}\backslash X \coloneqq \texttt{complement}$ of X in $\mathbb{R}$

90. [3]

Intersection of Closed Sets

91. `Theorem`

$$A \cap B \equiv \texttt{closed} \text{ in } \mathbb{R}$$

92. $A, B :=$ closed sets in $\mathbb{R}$

93. [3]

Accumulation Point of a Set in $\mathbb{R}$

94. `Definition`

$$(\mathbb{R} \ni x := \texttt{accumulation point} \text{ of } S \quad \leftrightarrow$$
$$\leftrightarrow \forall a, b \in \mathbb{R} \ (a < x < b \rightarrow \exists y \in S(a < y < b \wedge y \neq x)))$$
$$\equiv \textit{every open interval containing } x \textit{ contains at least one point of } S$$
$$\textit{different from } x$$

95. $S \subseteq \mathbb{R}$

96. $(\mathbb{R} \ni x) \equiv (x \in \mathbb{R})$

97. [3]

Closed in $\mathbb{R}$, Accumulation Points

98. `Theorem`

$$(C \equiv \texttt{closed}\text{ in } \mathbb{R}) \;\leftrightarrow\; (x \equiv \texttt{accumulation point}\text{ of } C \to x \in C)$$

99. $C \subseteq \mathbb{R}$

100. [3]

Closure of a Set in $\mathbb{R}$, Intersection

101. `Definition`

$$S \subseteq \mathbb{R} \;\to\; \overline{S} := \bigcap \{C \mid S \subseteq C,\; C \equiv \texttt{closed} \text{ in } \mathbb{R}\}$$

102. $\overline{S}$:= `closure` of S in $\mathbb{R}$

103. $\overline{S}$:= *intersection of all closed sets in* $\mathbb{R}$ *containing* S

104. `Theorem`

$$\overline{S} \equiv \texttt{closed} \text{ in } \mathbb{R}$$

105. [3]

Closure, Accumulation Point, Closed in $\mathbb{R}$

106. `Theorems`

107.

$$S \subseteq \overline{S}$$

108.

$$(C \equiv \texttt{closed} \text{ in } \mathbb{R},\ S \subseteq C) \ \rightarrow\ (\overline{S} \subseteq C)$$

109.

$$\overline{S} = S \cup \{x \in \mathbb{R} \mid x \equiv \texttt{accumulation point} \text{ of } S\}$$

110.

$$(S \equiv \texttt{closed} \text{ in } \mathbb{R}) \leftrightarrow\ (S = \overline{S})$$

111.

$$(x \in \overline{S})\ \leftrightarrow\ \forall X\ (x \in X \rightarrow \exists s \in X)$$

112. $s \in S \subseteq \mathbb{R}$

113. $\overline{S}$:= `closure` of S in $\mathbb{R}$

114. X := open interval

115. [3]

Open Disk in $\mathbb{C}$, Neighborhood

116. `Definition`

$$N_r(c) := \{z \in \mathbb{C} : |z - c| < r\}$$

117. $N_r(c) :=$ r-neighborhood of c (`open disk` in $\mathbb{C}$)

118. $c :=$ center of the open disk

119. $r :=$ radius of the open disk

120. [3]

Open in $\mathbb{C}$

121. `Definition`

$$(\forall z \in X \ \exists D : z \in D \subseteq X) \ \rightarrow \ (X := \texttt{open} \text{ in } \mathbb{C})$$

122. `Theorem`

$$(X \equiv \texttt{open} \text{ in } \mathbb{C}) \ \leftrightarrow \ (\forall w \in X \ \exists d > 0 : N_d(w) \subseteq X)$$

123. $w \in X \subseteq \mathbb{C}; \qquad d \in \mathbb{R}$

124. $D :=$ open disk

125. $N_d(w) :=$ open disk in $\mathbb{C}$ (d-neighborhood of w)

126. [3]

Closed in $\mathbb{C}$

127. `Definition`

$$(\mathbb{C}\backslash X \equiv \texttt{open} \text{ in } \mathbb{C}) \;\rightarrow\; (X \equiv \texttt{closed} \text{ in } \mathbb{C})$$

128. $X \subseteq \mathbb{C}$

129. $\mathbb{C}\backslash X :=$ complement of X in $\mathbb{C}$

130. [3]

Accumulation Point of a Set in $\mathbb{C}$

131. `Definition`

$$(\mathbb{C} \ni z \equiv \texttt{accumulation point} \text{ of } S \quad \leftrightarrow$$
$$\leftrightarrow \forall a \in \mathbb{C}\ \forall r \in \mathbb{R}^{+}\ (z \in N_r(a) \to \exists w \in S(w \in N_r(a) \land w \neq z)))$$
$$\equiv \textit{every open disk containing } z \textit{ contains at least one point of } S$$
$$\textit{different from } z$$

132. $S \subseteq \mathbb{C}$

133. $(\mathbb{C} \ni z) \equiv (z \in \mathbb{C})$

134. $N_r(a) :=$ open disk in $\mathbb{C}$ (r-neighborhood of a)

135. [3]

Closed in $\mathbb{C}$, Accumulation Points

136. `Theorem`

$$(C \equiv \texttt{closed in } \mathbb{C}) \;\leftrightarrow\; (\forall z \in \mathbb{C} : z \in C)$$

137. $C \subseteq \mathbb{C}$

138. $z :=$ accumulation point of C

139. [3]

Closure, Accumulation Point, Closed in $\mathbb{C}$

140. `Theorems`

141.

$$S \subseteq \overline{S}$$

142.

$$(C \equiv \texttt{closed} \text{ in } \mathbb{C},\ S \subseteq C)\ \rightarrow\ (\overline{S} \subseteq C)$$

143.

$$\overline{S} = S \cup \{z \in \mathbb{C} \mid z \equiv \texttt{accumulation point} \text{ of } S\}$$

144.

$$(S \equiv \texttt{closed} \text{ in } \mathbb{C}) \leftrightarrow\ (S = \overline{S})$$

145.

$$(z \in \overline{S})\ \leftrightarrow\ \forall D\ (z \in D \wedge \exists s \in D)$$

146. $s \in S \subseteq \mathbb{C}$

147. $\overline{S}$:= closure of S in $\mathbb{C}$

148. D := open disk in $\mathbb{C}$

149. [3]

Euclidean space

150. `Definition`

$$\mathbb{R}^n := \{(x_1, x_2, ..., x_n) \mid x_1, x_2, ..., x_n \in \mathbb{R}\}$$

151. $x, y \in \mathbb{R}^n; \quad k \in \mathbb{R}$

152. $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$

153. $kx = (kx_1, kx_2, ..., kx_n)$

154. $-x = (-x_1, -x_2, ..., -x_n)$

155. $x - y := x + (-y) = (x_1 - y_1, x_2 - y_2, ..., x_n - y_n)$

156. $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

157. $|x| :=$ norm (length/magnitude) of x

158. $\mathbb{R}^n := n$-`dimensional Euclidean space`

159. [3]

Unitary space

160. `Definition`

$$\mathbb{C}^n := \{(z_1, z_2, ..., z_n) \mid z_1, z_2, ..., z_n \in \mathbb{C}\}$$

161. $\mathbb{C}^n :=$ n-`dimensional unitary space`
(with the definitions of *sum*, *scalar multiple*, and *norm*)

162. [3]

Generalized Triangle Inequality

163. `Theorem`

$$|x + y| \leq |x| + |y|$$

164. $x, y \in \mathbb{R}^n$ (or $x, y \in \mathbb{C}^n$)

165. $|x| :=$ norm of x

166. [3]

Open Ball in $\mathbb{R}^n$, Neighborhood

167. `Definition`

$$B_d(a) \coloneqq \{y \in \mathbb{R}^n : |y - a| < d\}$$

168. $B_d(a) \coloneqq$ open ball in $\mathbb{R}^n$ (center a, radius d)
(d-neighborhood of $a \in R^n$)

169. [3]

Open in $\mathbb{R}^n$, Open Ball in $\mathbb{R}^n$

170. `Definition`

$$(\forall x \in X \ \exists B : x \in B, B \subseteq X) \ \rightarrow \ (X := \texttt{open} \text{ in } \mathbb{R}^n)$$

171. `Definition`

$$(\mathbb{R}^n \backslash X \equiv \texttt{open} \text{ in } \mathbb{R}^n) \ \rightarrow \ (X := \texttt{closed} \text{ in } \mathbb{R}^n)$$

172. $\mathbb{R}^n \backslash X :=$ complement of X in $\mathbb{R}^n$

173. $X \subseteq \mathbb{R}^n$; $\quad B :=$ open ball in $\mathbb{R}^n$

174. [3]

Accumulation Point of a Set in $\mathbb{R}^n$

175. `Definition`

$$\forall B \subseteq \mathbb{R}^n \ (x \in B,\ \exists s \in B,\ s \neq x) \ \rightarrow$$
$$\rightarrow \ \mathbb{R}^n \ni x := \texttt{accumulation point} \text{ of } S \text{ in } \mathbb{R}^n$$

176. $S \subseteq \mathbb{R}^n$

177. $B :=$ open ball in $\mathbb{R}^n$; $\quad s \in S$; $\quad (\mathbb{R}^n \ni x) \equiv (x \in \mathbb{R}^n)$

178. [3]

Closure of a Set in $\mathbb{R}^n$, Intersection

179. `Definition`

$$S \subseteq \mathbb{R}^n \;\; \rightarrow \;\; \overline{S} = \bigcap \{C \mid S \subseteq C \wedge C \equiv \texttt{closed} \text{ in } \mathbb{R}^n\}$$

180. $\overline{S}$:= `closure` of S in $\mathbb{R}^n$

181. $\overline{S}$:= *intersection of all closed sets in* $\mathbb{R}^n$ *containing* S

182. [3]

Open, Closed, Union, Intersection, Accumulation Point, Open Ball

183. `Theorems`

184.

$$(X \equiv \texttt{open in } \mathbb{R}^n) \;\leftrightarrow\; (\forall x \in \mathbb{R}^n \; \exists d > 0 : B_d(x) \subseteq X)$$

185.

$$\varnothing, \mathbb{R}^n \equiv \texttt{open} \text{ and } \texttt{closed} \text{ in } \mathbb{R}^n$$

186.

$$\bigcup A_i \equiv \texttt{open} \text{ in } \mathbb{R}^n$$

187.

$$\bigcap_{i=1}^{n} A_i \equiv \texttt{open} \text{ in } \mathbb{R}^n$$

188.

$$\bigcap K_i \equiv \texttt{closed} \text{ in } \mathbb{R}^n$$

189.

$$\bigcup_{i=1}^{n} K_i \equiv \texttt{closed} \text{ in } \mathbb{R}^n$$

190.

$$(C \equiv \texttt{closed} \text{ in } \mathbb{R}^n) \;\leftrightarrow\; (\forall y \in \mathbb{R}^n : y \in C)$$

191.

$$S \subseteq \mathbb{R}^n \;\rightarrow\; S \subseteq \overline{S}$$

192.

$$(C \equiv \texttt{closed} \text{ in } \mathbb{R}^n,\; S \subseteq C) \;\rightarrow\; (\overline{S} \subseteq C)$$

193.

$$(S \subseteq \mathbb{R}^n) \;\rightarrow\; (\overline{S} = S \cup \{x \in \mathbb{R}^n \mid x \equiv \texttt{accumulation point} \text{ of } S\})$$

194.

$$\left(S \equiv \texttt{closed} \text{ in } \mathbb{R}^n\right) \;\leftrightarrow\; \left(S = \overline{S}\right)$$

195.

$$\left(x \in \overline{S}\right) \;\leftrightarrow\; \forall B \; \left(x \in B, \exists s \in B\right)$$

196. $x \in \mathbb{R}^n$

197. $B_d(x)$:= open ball (center x, radius d)

198. A_i := open sets in $\mathbb{R}^n$; $i \in \mathbb{N} = \{1, 2, 3, ...\}$

199. $\bigcup A_i$:= union of all A_i

200. K_i := closed sets in $\mathbb{R}^n$

201. $\bigcap K_i$:= intersection of all K_i

202. $C \subseteq \mathbb{R}^n$

203. y := accumulation point of R^n

204. $\overline{S}$:= closure of S in $\mathbb{R}^n$

205. B := open ball

206. $s \in S$

207. [3]

Topology on S

208. `Definition`

$$\mathcal{T} := \texttt{topology on } S \text{ if}$$

(i) $\varnothing, S \in \mathcal{T}$

(ii) $X \subseteq \mathcal{T} \to \bigcup X \in \mathcal{T}$

(iii) $(Y \subseteq \mathcal{T} \ \wedge \ Y := \text{finite}) \ \to \ \bigcap Y \in \mathcal{T}$

209. (208.ii) means that $\mathcal{T}$ *is closed under taking arbitrary unions.*

210. (208.iii) means that $\mathcal{T}$ *is closed under taking finite intersections.*

211. $S :=$ set

212. $\mathcal{T}, X, Y :=$ sets of sets

213. $\mathcal{P}(S) :=$ power set of S (set of all subsets of S)

214. `Theorem`

$$\mathcal{T} \subseteq \mathcal{P}(S) \in \mathcal{P}(\mathcal{P}(S))$$

215. [3]

Trivial Topology

216. `Definition`

$$\mathcal{T} \coloneqq \{\varnothing, S\}$$

217. $\mathcal{T} \coloneqq$ `trivial topology` (or `indiscrete topology`) on S

218. $S \coloneqq$ set

219. [3]

Discrete Topology

220. `Definition`

$$\mathcal{P}(S)$$

221. $\mathcal{P}(S) \coloneqq$ `power set` $\coloneqq$ `discrete topology`

222. $S \coloneqq$ set

223. [3]

Topological Space

224. `Definition`

$$(S, \mathcal{T}) := \texttt{topological space}$$

225. $S :=$ set

226. $\mathcal{T} :=$ topology on S

227. [3]

Topology of a singleton

228. `Definition`

$$\mathcal{T} := \{\varnothing, \{a\}\}$$

229. $S = \{a\}$

230. $\mathcal{T}$:= unique topology on S

231. [3]

Topologies of a set with two elements

232. `Theorems`

$$\mathcal{T}_1 = \{\varnothing, \{a,b\}\}$$
$$\mathcal{T}_2 = \{\varnothing, \{a\}, \{a,b\}\}$$
$$\mathcal{T}_3 = \{\varnothing, \{b\}, \{a,b\}\}$$
$$\mathcal{T}_4 = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$$

233. $S = \{a, b\}$

234. $\mathcal{T}_i \coloneqq$ `topologies` on S

235. [3]

Coarser, finer, and incomparable topologies

236. `Definition`

$$\mathcal{T}_1 \subseteq \mathcal{T}_2$$

237. $\mathcal{T}_1$ is `coarser` than $\mathcal{T}_2$

238. $\mathcal{T}_2$ is `finer` than $\mathcal{T}_1$

239. `incomparable` := neither *finer* nor *coarser*

240. [3]

Standard topology on $\mathbb{R}$

241. `Definition`

$$\mathcal{T} := \{X \subseteq \mathbb{R} \mid \forall x \in X\ \exists a, b \in \mathbb{R}\ (x \in (a,b) \wedge (a,b) \subseteq X)\}$$

242. $\mathcal{T}$:= `standard topology` on $\mathbb{R}$

243. (a,b) := open interval

244. [3]

Standard topology on $\mathbb{C}$

245. `Definition`

$$\mathcal{T} = \{X \subseteq \mathbb{C} \mid \forall z \in X \ \exists a \in \mathbb{C} \ \exists r \in \mathbb{R}^{+}(z \in N_r(a) \wedge N_r(a) \subseteq X)\}$$

246. $N_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$:= open ball

247. $\mathcal{T}$:= `standard topology` on $\mathbb{C}$

248. [3]

Standard topology on $\mathbb{R}^n$

249. `Definition`

$$\mathcal{T} = \{X \subseteq \mathbb{R}^n \mid \forall x \in X \ \exists a \in \mathbb{R}^n \ \exists r \in \mathbb{R}^+ (x \in B_r(a) \wedge B_r(a) \subseteq X)\}$$

250. $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$:= open ball

251. $\mathcal{T}$:= `standard topology` on the Euclidean space $\mathbb{R}^n$

252. [3]

Topological Space, Neighborhood

253. `Definition`

$$(U \in \mathcal{T},\ x \in U) \ \equiv\ (U := \texttt{neighborhood} \text{ of } x)$$

254. $(S, \mathcal{T})$:= topological space

255. [3]

Accumulation Point of a Set

256. `Definition`

$$\begin{gathered}(S \ni x \equiv \texttt{accumulation point} \text{ of } A \leftrightarrow \\ \leftrightarrow \forall U \in \mathcal{T} \ (x \in U \to \exists y \in A(y \in U, y \neq x)))\end{gathered}$$

257. $(S, \mathcal{T}) :=$ topological space

258. $A \subseteq S$

259. $(S \ni x) \equiv (x \in S)$

260. [3]

Closed, Accumulation Points, Topological Space

261. `Theorem`

$$(C \equiv \texttt{closed}\text{ in } S) \;\leftrightarrow\; \forall x \in S\ (x \in C)$$

262. $(S, \mathcal{T}) :=$ topological space

263. $C \subseteq S$

264. $x :=$ accumulation points of C

265. [3]

Closure, Topological Space

266. `Definition`

$$A \subseteq S \;\rightarrow\; \overline{A} \coloneqq \bigcap\{C \mid A \subseteq C,\; C \text{ is } \texttt{closed} \text{ in } S\}$$

267. `Theorem`

$$\overline{A} \texttt{ is closed} \text{ in } S$$

268. $(S, \mathcal{T}) \coloneqq$ topological space

269. $\overline{A} \coloneqq$ closure of A in S

270. $\overline{A} \coloneqq$ intersection of all closed sets in S

271. [3]

Topological Space, Closed, Accumulation Point

272. `Theorems`

273.

$$A \subseteq \overline{A}$$

274.

$$(C \equiv \texttt{closed} \text{ in } S,\ A \subseteq C) \ \rightarrow \ (\overline{A} \subseteq C)$$

275.

$$\overline{A} = A \cup \{x \in S \mid x \equiv \texttt{accumulation point} \text{ of } A\}$$

276.

$$(A \equiv \texttt{closed} \text{ in } S) \ \leftrightarrow \ (A = \overline{A})$$

277.

$$(x \in \overline{A}) \ \leftrightarrow \ \forall O \subseteq S \ (x \in O, \exists y \in O, y \in A)$$

278. $(S, \mathcal{T}) :=$ topological space

279. $A \subseteq S$

280. $\overline{A} := \texttt{closure}$ of A in S

281. $O :=$ open set

282. [3]

Bases

283. `Definition`

$$(S, \mathcal{T}) := \text{topological space } \rightarrow$$
$$\rightarrow \ \mathcal{B} \subseteq \mathcal{T} : \forall x \in \mathcal{T} \left(x = \bigcup_i B_i \right)$$

284. *Every element of the topology can be written as a union of elements from the basis.*

285. $\mathcal{B}$:= `basis` for the topology $\mathcal{T}$

286. $B_i \in \mathcal{B}$

287. $\bigcup B_i$:= arbitrary union

288. $\mathcal{T}$:= `generated` by $\mathcal{B}$ (or $\mathcal{B}$ `generates` $\mathcal{T}$)

289. [3]

Cover, Intersection Containment Property

290. `Definition`

$$(\forall x \in S\ \exists A \in \mathcal{X}(x \in A))\ \rightarrow\ (\mathcal{X}\ \texttt{covers}\ S)$$

291. `Definition`

$$\mathcal{X} \text{ has the } \texttt{intersection containment property} \text{ on } S$$
$$\text{if}$$
$$\forall x \in S\ \forall A, B \in \mathcal{X}(x \in A \cap B \rightarrow \exists C \in \mathcal{X}(x \in C, C \subseteq A \cap B))$$

292. $S, A, B, C :=$ sets; $\mathcal{X} :=$ set of sets

293. [3]

Basis, Cover, Intersection Containment Property

294. `Theorem`

$$\mathcal{B} \equiv \texttt{basis} \text{ for a topology on } S$$
$$\Longleftrightarrow$$
$$\mathcal{B}\ \texttt{covers}\ S,$$
$$\mathcal{B} \text{ has the } \texttt{intersection containment property} \text{ on } S$$

295. $S \neq \varnothing$

296. $\mathcal{B} :=$ collection of subsets of S

297. [3]

Subbasis

298. `Theorem`

$$\mathcal{X} = \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\} \not\equiv \text{basis for a topology on } \mathbb{R}$$

299. `Theorem`

$$\begin{aligned} \mathcal{B} := \text{collection of all finite intersections of sets in } \mathcal{X} \ \rightarrow \\ \rightarrow \ \mathcal{B} \text{ forms a basis for } \mathbb{R} \text{ (because } \mathcal{X} \text{ covers } \mathbb{R}) \ \rightarrow \\ \rightarrow \ \mathcal{X} \equiv \texttt{subbasis} \text{ for the topology generated by } \mathcal{B} \end{aligned}$$

300. standard topology on $\mathbb{R}$ ~> generated by $\mathcal{B}$

301. [3]

Basis, Subbasis, Union

302. `Theorem`

$$\begin{aligned} \mathcal{X} := \text{set of sets } \rightarrow \\ \rightarrow (\mathcal{X} \equiv \neg\Box \text{ basis for } \mathcal{T}) \wedge \\ \wedge (\mathcal{X} \equiv \texttt{subbasis} \text{ for } \mathcal{T}_{\cup x}) \end{aligned}$$

303. $\neg\Box$:= not necessarily

304. $\mathcal{T}$:= topology on $\mathbb{R}$

305. $\mathcal{T}_{\cup x}$:= topology on $\bigcup \mathcal{X}$

306. $\bigcup \mathcal{X}$:= union of all elements of $\mathcal{X}$

307. [3]

Basis, Intersection, Subspace Topology

308. `Theorem`

$$(S, \mathcal{T}) \equiv \text{topological space with basis } \mathcal{B},\ A \subseteq S \ \rightarrow$$
$$\rightarrow \ \mathcal{B}_A = \{U \cap A \mid U \in \mathcal{B}\} \equiv \texttt{basis for a topology } \mathcal{T}_A \text{ on } A$$

309. $\mathcal{T}$:= `subspace topology relative to` A

310. [3]

Topological Space, Subset, Intersection

311. `Theorem`

$$(S, \mathcal{T}) \equiv \text{topological space},\ A \subseteq S \ \rightarrow\ \mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

312. $\mathcal{T}_A :=$ topology on A

313. [3]

Product Topology (basis, Cartesian product)

314. `Definition`

$$S_1 \times S_2 := \{(x, y) \mid x \in S_1,\ y \in S_2\}$$

315. `Theorem`

$$(\mathcal{T}_1 := \text{topology on } S_1,\ \mathcal{T}_2 := \text{topology on } S_2) \ \rightarrow$$
$$\rightarrow \ \mathcal{B} = \{U \times V \mid U \in \mathcal{T}_1,\ V \in \mathcal{T}_2\} \equiv \texttt{basis} \text{ for } \mathcal{T}$$

316. $\mathcal{T}$:= `product topology` on $S_1 \times S_2$

317. [3]

Product Topology, Basis, Cartesian Product

318. `Theorem`

$$(\mathcal{T}_1 \coloneqq \text{topology on } S_1 \text{ with basis } \mathcal{B}_1,$$
$$\mathcal{T}_2 \coloneqq \text{topology on } S_2 \text{ with basis } \mathcal{B}_2) \;\rightarrow$$
$$\rightarrow\; \mathcal{C} = \{U \times V \mid U \in \mathcal{B}_1,\; V \in \mathcal{B}_2\} \equiv \texttt{basis} \text{ for } \mathcal{T}$$

319. `Theorem`

$$\mathcal{X} = \{U \times S_2 \mid U \in \mathcal{B}_1\} \cup \{S_1 \times V \mid V \in \mathcal{B}_2\} \equiv \texttt{subbasis} \text{ for } \mathcal{T}$$

320. $\mathcal{T} \coloneqq$ `product topology` on $S_1 \times S_2$

321. [3]

Product Topology, Basis, Cartesian Product (n)

322. `Theorem`

$$\forall i \in \{1, 2, ..., n\} : \mathcal{T}_i \coloneqq \text{topology on } S_i \;\rightarrow$$
$$\rightarrow\; \mathcal{B} = \{U_1 \times U_2 \times ... \times U_n \mid U_1 \in \mathcal{T}_1,\; U_2 \in \mathcal{T}_2,\; ...,\; U_n \in \mathcal{T}_n\} \equiv \texttt{basis} \text{ for } \mathcal{T}$$

323. `Theorem`

$$\forall i \in \{1, 2, ..., n\} : \mathcal{B}_i \coloneqq \text{bases for } \mathcal{T}_i \;\rightarrow$$
$$\rightarrow\; \mathcal{B} = \{U_1 \times U_2 \times ... \times U_n \mid U_1 \in \mathcal{B}_1,\; U_2 \in \mathcal{B}_2,\; ...,\; U_n \in \mathcal{B}_n\} \equiv \texttt{basis} \text{ for } \mathcal{T}$$

324. $\mathcal{T} \coloneqq$ `product topology` on $S_1 \times S_2 \times ... \times S_n$

325. [3]

General Cartesian product

326. `Definition`

$$\prod_{k \in K} S_k := \{f : K \to S \mid \forall k \in K\ (f(k) \in S_k)\}$$

327. $K :=$ `index set`

328. $S_k :=$ set for each $k \in K :=$ union of all elements S_k

329. $S := \bigcup\{S_k \mid k \in K\}$

330. $\{S_k \mid k \in K\} :=$ indexed by k

331. $\prod S_k :=$ `general Cartesian product` of $\{S_k \mid k \in K\}$

332. [3]

Product Topology on a general Cartesian product, Box Topology

333. `Theorem`

$$\mathcal{B} = \left\{ \prod U_k \mid \forall k \in K : (U_k \in \mathcal{T}_k) \wedge (U_k = S_k) \right\} \equiv \texttt{basis} \text{ for } \mathcal{T}$$

334. `Theorem`

$$\mathcal{B}' = \left\{ \prod U_k \mid \forall k \in K' : (U_k \in \mathcal{T}_k) \wedge (U_k = S_k) \right\} \equiv \texttt{basis} \text{ for } \mathcal{T}'$$

335. $K :=$ *finite* index set

336. $K' :=$ *infinite* index set

337. $(S_k, \mathcal{T}_k) :=$ *topological space* for each $k \in K$

338. $\mathcal{T} :=$ `product topology` on $\prod S_k$

339. $\mathcal{T}' :=$ `box topology`

340. $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$,

$$\mathcal{T} \subset \mathcal{T}'.$$

341. [3]

Kolmogorov space (T_0-space)

342. `Definition`

$$\forall x, y \in S\ (x \neq y)\ \exists U \in \mathcal{T} : (x \in U \wedge y \notin U) \veebar (x \notin U \wedge y \in U) \rightarrow$$
$$\rightarrow (S, \mathcal{T}) \equiv \texttt{Kolmogorov space (T}_0\texttt{-space)}$$

343. $(S, \mathcal{T}) :=$ topological space

344. [3]

$\mathbf{T_1}$-space (Fréchet)

345. `Definition`

$$\forall x, y \in S \ (x \neq y) \ \exists U, V \in \mathcal{T} : (x \in U,\ y \notin U,\ x \notin V,\ y \in V)$$
$$\rightarrow \ (S, \mathcal{T}) \equiv \texttt{T}_1\texttt{-space (Fréchet or accessible space)}$$

346. $(S, \mathcal{T}) :=$ topological space

347. [3]

Cofinite Topology

348. `Definition`

$$\mathcal{T} \equiv \texttt{cofinite topology} \text{ on } S$$

349. $\mathcal{T} \coloneqq$ topology generated by the basis $\mathcal{B}$

350. $\mathcal{B} = \{X \subseteq S : S \backslash X \text{ is finite}\}$

351. [3]

Cofinite Topology on $\mathbb{R}$

352. `Definition`

$$(\mathcal{T} := \text{topology generated by the basis } \mathcal{B},$$
$$\mathcal{B} = \{X \subseteq \mathbb{R} : \mathbb{R}\backslash X \text{ is finite}\}) \to$$
$$\to \mathcal{T} \equiv \texttt{cofinite topology} \text{ on } \mathbb{R}$$

353. `Theorem:` $\mathcal{T}$ is *strictly coarser* than $\mathcal{T}_{st}$,

$$\mathcal{T} \subset \mathcal{T}_{st}.$$

354. `Theorem`

$$(\mathbb{R}, \mathcal{T}) \equiv \texttt{T}_1\texttt{-space}$$

355. $\mathcal{T}_{st}$:= standard topology

356. [3]

T_1-space, Closed

357. `Theorem`

$$((S,\mathcal{T}) \equiv \mathtt{T}_1\texttt{-space}) \leftrightarrow (\forall x \in S,\ \{x\} \equiv \mathtt{closed}\ \text{in}\ (S,\mathcal{T}))$$

358. $(S,\mathcal{T}) :=$ topological space

359. [3]

T_1-space, Accumulation Point

360. `Theorem`

$$x \equiv \texttt{accumulation point} \text{ of } A$$
$$\text{iff}$$
$$\forall \mathcal{O} \subseteq S \ (x \in \mathcal{O}) \ \exists_\infty a \in \mathcal{O}$$

361. $(S, \mathcal{T}) :=$ T_1-space

362. $A \subseteq S; \quad a \in A$

363. $\mathcal{O} :=$ open set

364. $\exists_\infty :=$ there are infinitely many

365. [3]

Multiple limits (multiple convergence)

366. `Theorem:` In $\texttt{T}_1$`-spaces`, *sequences* may *converge* to *multiple limits.*

367. [3]

Line with two origins

368.

$$(S, \mathcal{T}) \equiv \texttt{line with two origins (topological space)}$$

369. `Theorems`

370.

$$(S, \mathcal{T}) \equiv \texttt{T}_1\texttt{-space}$$

371.

$$\left(\frac{1}{n+1}\right) \to \{0, c\}$$

372.

$$\mathcal{B} \ \texttt{covers} \ S$$

373.

$$\mathcal{B} \text{ has the } \texttt{intersection containment property}$$

374. $S := \mathbb{R} \cup \{c\}$

375. $c \notin \mathbb{R}$

376. $\mathcal{B} := \{(a, b) \mid (a, b) \subseteq \mathbb{R}\} \cup \{(-a, 0) \cup \{c\} \cup (0, b)\}$

377. *There are two origins (0 and c).*

378. $\mathcal{B}$:= basis for a topology $\mathcal{T}$ on S

379. $s = \left(\frac{1}{n+1}\right)$:= sequence

380. s `converges` to `both` 0 and c.

381. (a, b) := open interval in $\mathbb{R}$

382. [3]

Cofinite Topology on $\mathbb{N}$

383. `Theorems`

384.

$$(\mathbb{N}, \mathcal{T}) \equiv \texttt{cofinite topology} \text{ on } \mathbb{N}$$

385.

$$(\mathbb{N}, \mathcal{T}) \equiv \texttt{T}_1\texttt{-space}$$

386.

$$\forall m \in \mathbb{N} : (n) \to m$$

387. $\mathcal{T}$:= topology generated by the basis $\mathcal{B}$

388. $\mathcal{B} = \{X \subseteq \mathbb{N} : \mathbb{N} \backslash X \text{ is finite}\}$

389. (n) := sequence

390. (n) *converges to every natural number in T_1-space.*

391. [3]

$\mathbf{T_2}$-space (Hausdorff)

392. `Definition`

$$\forall x, y \in S\ (x \neq y)\ \exists U, V \in \mathcal{T} : (x \in U,\ y \in V,\ U \cap V = \varnothing)$$
$$\rightarrow\ (S, \mathcal{T}) \equiv \mathtt{T_2\text{-}space\ (Hausdorff)}$$

393. $(S, \mathcal{T}) :=$ topological space

394. [3]

Unique Limit, Convergence, T_2-space

395. `Theorem`

$$((s_n) \equiv \text{convergent}) \;\rightarrow\; \left(\lim_{n\to\infty}(s_n) \equiv \texttt{unique}\right)$$

396. $(S, \mathcal{T})$:= T_2-space

397. (s_n) := sequence in T_2-space

398. [3]

T_3-space (Regular)

399. `Definition`

$$(S,\mathcal{T}) \equiv \mathrm{T}_1\text{-space},\ X \equiv \text{closed set},$$
$$\forall x \in S,\ X \subseteq S \backslash \{x\},\ \exists U,V \in \mathcal{T} : (x \in U,\ X \subseteq V,\ U \cap V = \varnothing) \rightarrow$$
$$\rightarrow\ (S,\mathcal{T}) \coloneqq \mathtt{T_3\text{-}space\ (Regular)}$$

400. $(S,\mathcal{T}) \coloneqq$ topological space

401. [3]

Clopen set

402. `Definition:` *Both* `open` and `closed`.

403. [3]

T_4-space (Normal)

404. `Definition`

$$(S,\mathcal{T}) \equiv \mathrm{T}_1\text{-space},\ \forall (X,Y): X \cap Y = \varnothing,\ X,Y \equiv \text{closed},\ X,Y \subseteq S, \\ \exists U,V \in \mathcal{T}: (X \subseteq U,\ Y \subseteq V,\ U \cap V = \varnothing) \rightarrow \\ \rightarrow\ (S,\mathcal{T}) \coloneqq \mathtt{T_4\text{-}space\ (Normal)}$$

405. $(S,\mathcal{T}) \coloneqq$ topological space

406. [3]

Separation and Countability Axioms

407. `Axioms`

408. `Separation Axioms` ≡ (definitions of) T_0, T_1, T_2, T_3, T_4.
(***they all "separate" points and/or closed sets from each other by open sets***)

409. `Countability Axioms`

    (*i*) Separable Spaces

    (*ii*) First-Countable Spaces

    (*iii*) Second-Countable Spaces

410. [3]

Separable Spaces

411. `Definition`

$$(\overline{A} = S) \rightarrow (A \coloneqq \texttt{dense})$$

412. `Definition`

$$(\exists A : A \equiv \text{dense, countable}) \rightarrow ((S, \mathcal{T}) \coloneqq \texttt{separable})$$

413. $A \subseteq S$

414. $\overline{A} \coloneqq$ closure of A in S

415. $(S, \mathcal{T}) \coloneqq$ topological space

416. [3]

First-Countable Spaces

417. `Definition`

$$\mathcal{B} := \texttt{countable basis} \text{ at } x \text{ if}$$

(i) $V \in \mathcal{B} \ \rightarrow \ x \in V$

(ii) $\forall U : x \in U, \ \exists V \in \mathcal{B} \ (V \subseteq U)$

418. $x \in S$

419. $U :=$ open set

420. $(S, \mathcal{T}) :=$ topological space

421. [3]

First-Countable, Closure, Sequence, Convergence

422. `Theorem`

$$((S, \mathcal{T}) \equiv \text{first-countable},\ x \in \overline{A}) \Rightarrow (\exists (s_n) : s_n \to x)$$

423. $\overline{A} :=$ closure of A in S

424. $(s_n) :=$ sequence

425. $(S, \mathcal{T}) :=$ topological space

426. [3]

Second-Countable Spaces

427. `Definition`

$$\exists \mathcal{B} \ \rightarrow \ (S, \mathcal{T}) := \texttt{second-countable}$$

428. `Theorem`

$$\forall \mathcal{T} \equiv \text{second-countable} \ \rightarrow \ \mathcal{T} \equiv \texttt{separable}$$

429. $\mathcal{B}$:= countable basis of $(S, \mathcal{T})$

430. $(S, \mathcal{T})$:= topological space

431. [3]

Metric (distance function)

432. `Definition`

$$d : S \times S \to \mathbb{R}$$

(a) $\forall x, y \in S : \; (d(x,y) = 0) \leftrightarrow (x = y)$

(b) $\forall x, y \in S : \; d(x,y) = d(y,x)$

(c) $\forall x, y, z \in S : \; d(x,z) \le d(x,y) + d(y,z)$

433. `Theorem:` $\forall x, y \in S : d(x,y) \ge 0$.

434. [3]

Metric space

435. `Definition`

$$(S, d) \coloneqq \texttt{metric space}$$

436. $S \coloneqq$ set

437. $d \coloneqq$ metric (distance function)

438. $d : S \times S \to \mathbb{R}$

439. [3]

Open ball

440. `Definition`

$$B_r(a) := \{x \in S \mid d(a,x) < r\}$$

441. $(S,d) :=$ metric space

442. $a \in S; \quad r \in \mathbb{R}^+$

443. $B_r(a) \equiv B_r(a;d) :=$ `open ball` (center a, radius r)

444. $\mathcal{B} = \{B_r(a) \mid a \in S,\ r \in \mathbb{R}^+\}$ covers S.

445. [3]

Metrizable Topological Space

446. `Definition`

$$(S, \mathcal{T}) \text{ is } \texttt{metrizable} \text{ if}$$
$$\exists d : \mathcal{T} \text{ is } \textit{generated} \text{ from the } \textit{open balls} \text{ in } (S, d)$$

447. $d :=$ metric; $\quad d : S \times S \to \mathbb{R}$

448. $(S, \mathcal{T}) :=$ topological space

449. d `induces` $\mathcal{T}$

450. $\mathcal{T} :=$ `metric topology` (on S) `induced` by d

451. ***Different metrics can induce the same topology.***

452. [3]

Euclidean Metric, Unitary Metric

453. `Definitions`

454.

$$d(x,y) = |x-y| := \texttt{Euclidean metric}$$

455.

$$k(z,w) = |z-w| := \texttt{unitary metric}$$

456. $n \in \mathbb{Z}^+ = \{1,2,3,...\}$

457. $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

458. $k : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

459. $(\mathbb{R}^n, d), (\mathbb{C}^n, k) :=$ metric spaces

460. $|x-y| :=$ Euclidean norm

461. $|z-w| :=$ unitary norm

462. [3]

Square Metric on $\mathbb{R}^n$

463. `Definition`

$$\rho(x, y) := \max\{|x_1 - y_1|, ..., |x_n - y_n|\} := \texttt{square metric} \text{ on } \mathbb{R}^n$$

464. $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

465. $n = 1 \ \rightarrow \ \rho :=$ Euclidean metric

466. ρ and d *induce* the *product topology* on $\mathbb{R}^n$.

467. [3]

Discrete Metric

468. `Definition`

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

469. $d : S \times S \to \mathbb{R}$

470. $d :=$ `discrete metric` on S

471. [3]

Standard Bounded Metric

472. Definition

$$(\exists M \in \mathbb{R}^+ : \forall x, y \in A,\ d(x,y) \leq M) \ \equiv\ (A := \texttt{bounded in } S)$$

473. Theorem

$$\overline{d} : S \times S \to \mathbb{R}; \quad \overline{d}(x,y) = \min\{1, d(x,y)\} \ \Rightarrow$$
$$\Rightarrow\ (S, \overline{d}) \equiv \texttt{metric space}$$

474. Theorem

$$\overline{d} \equiv \texttt{bounded}$$

475. Theorem

$$\overline{d}, d \texttt{ induce the same topology}$$

476. $(S, d) :=$ metric space

477. $A \subseteq S$

478. diam $A = \sup\{d(x,y) \mid x, y \in A\} :=$ `diameter` of A

479. $\overline{d} :=$ `standard bounded metric` *corresponding* to d

480. `boundedness` $\not\equiv$ *topological property*

481. [3]

Uniform Topology

482. `Definition`

$$\rho(x,y) := \sup\{\overline{d}(x_k, y_k) \mid k \in K\}$$

483. $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $\quad d(x,y) = |x - y|$; $\quad d \not\equiv$ bounded

484. $(S, d) \equiv$ metric space

485. $K :=$ index set

486. ${}^K S :=$ set of functions from K to S

487. $\rho : {}^K S \times {}^K S \to S$

488. $({}^K S, \rho) \equiv$ metric space

489. $\overline{d} :=$ standard bounded metric corresponding to d

490. $\rho :=$ `uniform metric` on ${}^K S$

491. `Theorem:` *ρ induces $\mathcal{T}_u$.*

492. $\mathcal{T}_u :=$ `uniform topology` on ${}^K S$

493. $\mathcal{T}_p :=$ *product topology* on ${}^K S$

494. `Theorem:` $\mathcal{T}_u$ is finer than $\mathcal{T}_p$,

$$\mathcal{T}_p \subseteq \mathcal{T}_u.$$

495. [3]

Convergence

496. `Definition`

$$(\forall U \subseteq S \ (s \in U) \ \exists K \in \mathbb{N} : n > K \rightarrow s_n \in U) \ \Rightarrow \ (s_n \rightarrow s)$$

497. $(S, \mathcal{T})$:= topological space

498. (s_n) := sequence

499. $\forall n \in \mathbb{N} : s_n \in S$

500. U := open set

501. $(s_n \rightarrow s)$:= s_n `converges` to $s \in S$

502. [3]

Metrizable Topological Space, Induced by a Metric, Convergence

503. `Theorem`

$$
\begin{aligned}
&s_n \to s \;\equiv \\
\equiv\; &\forall B_r(s) \subseteq S\ \exists K \in \mathbb{N} : n > K \;\Rightarrow\; s_n \in B_r(s) \;\equiv \\
\equiv\; &\forall r \in \mathbb{R}^+\ \exists K \in \mathbb{N} : n > K \;\Rightarrow\; d(s_n, s) < r
\end{aligned}
$$

504. $(S, \mathcal{T})$:= metrizable topological space

505. $\mathcal{T}$:= induced by the metric d

506. $s \in S$

507. (s_n) := sequence

508. $\forall n \in \mathbb{N}\ (s_n \in S)$

509. $B_r(s)$:= open ball (radius r, center s)

510. $(s_n \to s)$:= s_n `converges` to $s \in S$

511. [3]

Cauchy Sequence

512. `Definition`

$$\forall r \in \mathbb{R}^+ \ \exists K \in \mathbb{N} : m \geq n > K \ \Rightarrow \ d(s_m, s_n) < r$$

513. $(S, \mathcal{T}) :=$ metrizable topological space

514. $\mathcal{T} :=$ induced by the metric d

515. $(s_n) :=$ sequence in S; $\quad s_n \to s$

516. $(s_n) :=$ `Cauchy sequence`

517. $\neg\Box :=$ not necessarily

518. `Theorem:` $\forall (s'_n) \equiv$ `Cauchy sequence` $: \neg \Box \, (s'_n) \equiv$ `converges`.

519. [3]

Cauchy Sequence Bounded by a Rational Number

520. `Theorem`

$$\forall (x_n) : (x_n) \equiv \texttt{Cauchy sequence of rational numbers} \rightarrow$$
$$\rightarrow\ (x_n) \equiv \texttt{bounded by } q \in \mathbb{Q}$$

521. $(x_n) :=$ Cauchy sequence of rational numbers

522. [3]

Complete Metric Space

523. `Definition`

$$(\forall (s_n) : s_n \to s \in S) \ \Rightarrow \ (S, d) \coloneqq \texttt{complete metric space}$$

524. $(S, d) \coloneqq$ metric space

525. $(s_n) \coloneqq$ Cauchy sequence

526. [3]

Subsequence

527. `Definition`

$$(f \circ g : \mathbb{N} \to S) := \texttt{subsequence} \text{ of } f$$

528. $(s_n) :=$ sequence in S

529. $(s_n) \equiv (f : \mathbb{N} \to S,\ \forall n \in \mathbb{N} : f(n) = s_n)$

530. $g : \mathbb{N} \to \mathbb{N}$

531. $g :=$ strictly increasing function

532. [3]

Cauchy Sequence, Subsequence, Convergence

533. `Theorem`

$$(\exists s \in S : s_{n_k} \to s) \;\Rightarrow\; (s_n \to s)$$

534. $(S, d) :=$ metric space

535. $(s_n) :=$ Cauchy sequence in S

536. $(s_{n_k}) :=$ subsequence of (s_n)

537. [3]

Complete Metric Space, Cauchy Sequence, Subsequence, Convergence

538. `Theorem`

$$((S,d) \equiv \texttt{complete}) \;\Leftrightarrow\; (\forall (s_n) \text{ in } S,\ \exists (s_{n_k}):\ s_{n_k} \to s \in S)$$

539. $(S,d) :=$ metric space

540. $(s_n) :=$ Cauchy sequence in S

541. $(s_{n_k}) :=$ subsequence of (s_n)

542. [3]

Bounded Sequence

543. `Theorem`

$$\{s_n\} \equiv \text{bounded} \;\rightarrow\; (s_n) \equiv \texttt{bounded}$$
$$\equiv$$
$$(\exists M \in \mathbb{R}^+ : \forall n, m \in \mathbb{N},\; d(s_n, s_m) \leq M) \;\rightarrow\; ((s_n) \equiv \texttt{bounded} \text{ in } S)$$

544. (S, d) := metric space

545. (s_n) := sequence in S

546. [3]

Metric Space, Cauchy Sequence, Bounded

547. `Theorem`

$$(S,d) \equiv \text{metric space},\ (s_n) \equiv \text{Cauchy sequence in } S\ \Rightarrow$$
$$\Rightarrow\ (s_n) \equiv \texttt{bounded} \text{ in } S$$

548. [3]

Bolzano-Weierstrass Property

549. `Definition`

$$(\forall (s_n)\ \exists (s_{n_k})\ s_{n_k} \to s \in S) \Rightarrow$$
$$\Rightarrow (S,d) \text{ has the } \texttt{Bolzano-Weierstrass Property}$$

550. $(S,d) :=$ metric space

551. $(s_n) :=$ sequence *bounded* in S

552. $(s_{n_k}) :=$ subsequence of (s_n)

553. [3]

Euclidean Space, Unitary Space, Complete Metric Spaces $\mathbb{R}^n$, $\mathbb{C}^n$

554. `Theorem`

$$\forall n \in \mathbb{N} : (\mathbb{R}^n, d), (\mathbb{C}^n, k) \equiv \texttt{complete metric spaces}$$

555. $d :=$ Euclidean metric

556. $k :=$ unitary metric

557. [3]

Standard Bounded Metric on $\mathbb{R}$, Supremum, Complete Metric Space, Metric Induces the Product Topology on the Index Set

558. `Theorem`

$$S = {}^{\mathbb{N}}\mathbb{R}, \quad (\overline{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}) \equiv \text{standard bounded metric on } \mathbb{R},$$
$$d^* : {}^{\mathbb{N}}\mathbb{R} \times {}^{\mathbb{N}}\mathbb{R} \to \mathbb{R}, \quad d^*(x,y) = \sup\left\{\frac{\overline{d}(x_k, y_k)}{k+1} \mid k \in \mathbb{N}\right\} \;\Rightarrow$$
$$\Rightarrow \;(S, d^* \equiv \texttt{complete metric space}, \;\; d^* \text{ induces } \mathcal{T}_p)$$

559. $\mathcal{T}_p :=$ product topology on $S = {}^{\mathbb{N}}\mathbb{R}$

560. [3]

Covering

561. `Definition`

$$(\bigcup \mathcal{C} = S) \;\rightarrow\; (\mathcal{C} := \texttt{covering of } S)$$

$$\mathcal{C} \;\texttt{covers}\; S$$

562. `Definition`

$$(\mathcal{C} := \text{covering of } S) \;\wedge\; (\forall U \in \mathcal{C} : U \equiv \text{open set}) \;\rightarrow$$
$$\rightarrow\; \mathcal{C} := \texttt{open covering of } S$$

563. $(S, \mathcal{T})$:= topological space

564. $\mathcal{C}$:= collection of subsets of S

565. $\bigcup \mathcal{C}$:= union of all elements of $\mathcal{C}$

566. [3]

Compact, Subcover

567. `Definition`

$$(\forall \mathcal{C} : \exists_n \texttt{ subcover} \text{ of the covering}) \rightarrow (S, \mathcal{T}) := \texttt{compact}$$

568. $(S, \mathcal{T})$:= topological space

569. $\mathcal{C}$:= collection of subsets of S

570. $\mathcal{C}$:= open covering of S

571. $\exists_n$:= there is a finite number of

572. `subcover` of the covering := subcollection that covers S

573. [3]

Finite Intersection Property, Subcollection

574. `Definition`

    $(\forall \mathcal{D} \subseteq \mathcal{C} : \bigcap \mathcal{D} \neq \varnothing) \rightarrow (\mathcal{C}$ has the `finite intersection property`$)$

575. $(S, \mathcal{T})$:= topological space

576. $\mathcal{C}$:= collection of subsets of S

577. $\mathcal{D}$:= finite subcollection

578. $\bigcap \mathcal{C}$:= intersection of all elements of $\mathcal{C}$

579. [3]

Compact, Closed, Finite Intersection Property

580. `Theorem`

$$((S,\mathcal{T}) \equiv \texttt{compact}) \;\leftrightarrow\; (\forall \mathcal{C} : \bigcap \mathcal{C} \neq \varnothing)$$

581. $(S,\mathcal{T})$:= topological space

582. $\mathcal{C}$:= collection of *closed sets* in S with the *finite intersection property*

583. $\bigcap \mathcal{C}$:= intersection of all elements of $\mathcal{C}$

584. [3]

Nested sequence

585. `Definition`

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$$

586. $(S, \mathcal{T})$:= compact topological space

587. B_i := nonempty *closed sets* in S

588. `Theorem:` $\mathcal{C} = \{B_k \mid k \in \mathbb{N},\ j < k \to B_k \subseteq B_j\}$ has the *finite intersection property.*

589. $\mathcal{C}$:= `nested sequence` of nonempty *closed sets* in S

590. [3]

Compact, Finite Subcollection, Cover, Finite Intersection Property, Covering, Basis, Basic Open Sets, Basic Closed Sets

591. `Theorem`

$$(S,\mathcal{T}) \text{ is } \texttt{compact} \equiv$$
$$\equiv \forall \mathcal{C}_S : \mathcal{C}_S \text{ contains a } \texttt{finite subcollection} \text{ that } \texttt{covers } S \equiv$$
$$\equiv (\forall \mathcal{C}' : \mathcal{C}' \text{ has the } \texttt{finite intersection property}) \to \bigcup \mathcal{C}' \neq \varnothing$$

592. $(S,\mathcal{T})$:= topological space

593. $\mathcal{C}_S$:= covering of S by B_i

594. $\mathcal{B}$:= `basis` for the topology $\mathcal{T}$

595. $B_i \in \mathcal{B}$

596. B_i := `basic open sets` in S

597. $B_i' = S \setminus B_i$:= `basic closed sets` in S

598. $\mathcal{C}' = \{B_i'\}$

599. $\bigcup \mathcal{C}'$:= union of all elements of $\mathcal{C}'$

600. [3]

Tychonoff's Theorem

601. `Theorem`

$$(\forall k \in K : (S_k, \mathcal{T}_k) \equiv \text{compact topological space}) \rightarrow$$
$$\rightarrow \left(\prod_{k \in K} S_k \equiv \texttt{compact} \text{ in } \mathcal{T}_p\right)$$

602.

$$\prod_{k \in K} S_k = \{f : K \rightarrow S \mid \forall k \in K \ (f(k) \in S_k)\}$$

603. $K :=$ `index set`

604. $S_k :=$ set for each $k \in K$

605. $\{S_k \mid k \in K\} :=$ indexed by k

606. $S = \bigcup\{S_k \mid k \in K\}$

607. $\prod S_k :=$ general Cartesian product of $\{S_k \mid k \in K\}$

608. $\mathcal{T}_p :=$ product topology

609. [3]

Heine-Borel Theorem

610. `Theorem`

$$((A, \mathcal{T}_A) \equiv \texttt{compact}) \leftrightarrow (A \equiv \texttt{closed} \text{ and } \texttt{bounded} \text{ in } \mathbb{R})$$

611. $A \subseteq \mathbb{R}$

612. $(A, \mathcal{T}_A) \coloneqq$ topological space

613. [3]

Bounded, Supremum, Infimum, Closure

614. `Theorem`

$$(A \equiv \text{bounded}) \;\rightarrow\; (\sup A, \inf A \in \overline{A})$$

615. $A \neq \varnothing; \quad A \subseteq \mathbb{R}$

616. $\overline{A} \coloneqq$ closure of A in $\mathbb{R}$

617. [3]

Covering by Sets, Open Covering by Open Sets, Subspace

618. `Definition`

$$\mathcal{C} \coloneqq \texttt{covering of } A \text{ by } \texttt{sets} \text{ in } S$$

619. `Definition`

$$(\forall C \in \mathcal{C} : C \text{ open in } S) \rightarrow$$
$$\rightarrow (\mathcal{C} \coloneqq \texttt{open covering} \text{ of } A \text{ by } \texttt{open sets} \text{ in } S)$$

620. $(S, \mathcal{T}) \coloneqq$ topological space

621. $(A, \mathcal{T}_A) \coloneqq$ subspace of $(S, \mathcal{T})$

622. $\mathcal{C} = \{S_i \mid S_i \in S,\ A \subseteq \bigcup \mathcal{C}\}$

623. [3]

Compact, Subspace, Open Covering, Subcollection

624. `Theorem`

$$((A, \mathcal{T}_A) \equiv \texttt{compact}) \leftrightarrow (\forall \mathcal{C} : \mathcal{C}' \in \mathcal{C})$$

625. $(S, \mathcal{T})$:= topological space

626. $(A, \mathcal{T}_A)$:= subspace of $(S, \mathcal{T})$

627. $\mathcal{C}$:= open covering of A by open sets in S

628. $\mathcal{C}'$:= finite subcollection that covers A

629. [3]

T_2-space, Subspace, Compact, Closed

630. `Theorem`

$$((A, \mathcal{T}_A) \equiv \texttt{compact}) \;\rightarrow\; (A \equiv \texttt{closed} \text{ in } S)$$

631. $(S, \mathcal{T}) :=$ $\texttt{T}_2$ `space`

632. $(A, \mathcal{T}_A) :=$ `subspace` of $(S, \mathcal{T})$

633. [3]

Closed, Compact, Subspace

634. `Theorem`

$$(A \equiv \texttt{closed} \text{ in } S) \ \rightarrow \ (A, \mathcal{T}_A) \equiv \texttt{compact}$$

635. $(S, \mathcal{T}) :=$ `compact` topological space

636. $(A, \mathcal{T}_A) :=$ `subspace` of $(S, \mathcal{T})$

637. [3]

Generalized Heine-Borel Theorem

638. `Theorem`

$$((A, \mathcal{T}_A) \equiv \texttt{compact}) \leftrightarrow (A \equiv \texttt{closed} \text{ and } \texttt{bounded} \text{ in } \mathbb{R}^n)$$

639. $A \subseteq \mathbb{R}$

640. $(A, \mathcal{T}_A) :=$ topological space

641. [3]

Sequentially Compact

642. `Definition`

$$\forall n((s_n) \in S,\ \exists(s'_n) : s'_n \to s)\ \to\ (S,d) \equiv \texttt{sequentially compact}$$

643. (S,d) := metric space

644. (s_n) := sequence in S

645. (s'_n) := subsequence of (s_n)

646. [3]

Lebesgue number

647. `Definition`

$$(\forall A \subseteq S,\ \text{diam } A < \delta,\ \exists C \in \mathcal{C} : A \subseteq C) \rightarrow (\delta := \texttt{Lebesgue number} \text{ for } \mathcal{C})$$

648. $(S, \mathcal{T}) :=$ topological space

649. $\mathcal{C} :=$ open covering of S

650. $\delta \in \mathbb{R}^+$

651. diam $A :=$ diameter of A

652. [3]

Lebesgue's Covering Lemma

653. `Theorem`

$$(S, \mathcal{T}) \equiv \texttt{metrizable, sequentially compact} \rightarrow$$
$$\rightarrow \mathcal{C} \text{ has a } \texttt{Lebesgue number}$$

654. $(S, \mathcal{T}) :=$ topological space

655. $\mathcal{C} :=$ open covering of S

656. [3]

Totally bounded, r-net

657. `Definition`

$$(S = \bigcup\{B_r(x) \mid x \in A\}) \;\rightarrow\; (A \text{ has an } r\text{-}\texttt{net})$$

658. `Definition`

$$(\forall r \in \mathbb{R}^+ : (S,d) \text{ has an } r\text{-net}) \;\rightarrow\; (S,d) \equiv \texttt{totally bounded}$$

659. $(S,d) :=$ metric space

660. $r \in \mathbb{R}^+$

661. $A \in S$; $A :=$ finite

662. [3]

Sequentially Compact Metric Space, Totally Bounded

663. `Theorem`

$$(S, d) \equiv \texttt{sequentially compact} \rightarrow (S, d) \equiv \texttt{totally bounded}$$

664. $(S, d) :=$ metric space

665. [3]

Metrizable Space, Compact, Accumulation Point, Sequentially Compact, Infinite Subset

666. `Theorem`

$$\begin{aligned} &(S,\mathcal{T}) \text{ is } \texttt{compact} \equiv \\ &\equiv \forall S' : S' \text{ has an } \texttt{accumulation point} \equiv \\ &\equiv (S,\mathcal{T}) \text{ is } \texttt{sequentially compact} \end{aligned}$$

667. $(S,\mathcal{T}) :=$ metrizable space

668. $S' :=$ *infinite* subset of S

669. [3]

Locally Compact

670. `Definition`

$$\forall x \in S \; \exists (K, \mathcal{T}_K) : N(x) \subseteq K \;\; \rightarrow \;\; (S, \mathcal{T}) \coloneqq \texttt{locally compact}$$

671. $(S, \mathcal{T}) \coloneqq$ topological space

672. $(K, \mathcal{T}_K) \coloneqq$ compact subspace

673. $N(x) \coloneqq$-neighborhood of x

674. [3]

One-Point Compactification

675. `Definition`

$$(\overline{S}, \overline{\mathcal{T}}) := \texttt{one-point compactification} \text{ of } S$$

676. $(S, \mathcal{T}) :=$ locally compact T_2 space

677. $\overline{S} = S \cup \{\infty\}$

678. $\infty :=$ `point at infinity`

679. $\overline{\mathcal{T}} = \mathcal{T} \cup \{S \backslash K \mid (K, \mathcal{T}_K)\} := \texttt{compact subspace} \text{ of } (S, \mathcal{T})$

680. [3]

Locally Compact, T_2-space, One-Point Compactification

681. `Theorem`

$$(S,\mathcal{T}) \equiv \text{locally compact } \mathrm{T}_2\text{-space} \rightarrow$$
$$\rightarrow (\overline{\mathcal{S}},\overline{\mathcal{T}}) \equiv \texttt{compact } \mathtt{T}_2\texttt{-space}$$

682. $(\overline{\mathcal{S}},\overline{\mathcal{T}})$:= one-point compactification of $(S,\mathcal{T})$

683. [3]

Interior of a Set, Empty Interior, Nowhere Dense, Closure

684. `Definition`

$$A^\circ := \bigcup_i U_i := \texttt{interior of } A$$

685. `Definition`

$$(A = \varnothing) \;\to\; (A \text{ has } \texttt{empty interior})$$

686. `Definition`

$$(\overline{A})^\circ = \varnothing \;\to\; A := \texttt{nowhere dense} \text{ in } S$$

687. $(S, \mathcal{T}) :=$ topological space

688. $A \subseteq S$

689. $U_i \subseteq A$

690. $U_i :=$ open set

691. $\overline{A} :=$ closure of A

692. [3]

Nowhere Dense, Dense, Interior

693. `Theorem`

$$(A \equiv \texttt{nowhere dense in } S) \leftrightarrow (S \backslash A)^\circ \equiv \texttt{dense in } S$$

694. $(S, \mathcal{T})$:= topological space

695. $A \subseteq S$

696. $(S \backslash A)^\circ$:= interior of $(S \backslash A)$

697. [3]

Meagre, Nonmeagre, Comeagre, Countable Union, Nowhere Dense, First Category, Second Category

698. `Definition`

$$A = \bigcup_c A' \ \rightarrow$$
$$\rightarrow \ A := \texttt{meagre} \text{ (or a set of } \texttt{first category}\text{)}$$

699. `Definition`

$$(A \not\equiv \text{meagre}) \ \rightarrow \ A := \texttt{nonmeagre} \text{ (or a set of } \texttt{second category}\text{)}$$

700. `Definition`

$$(A \equiv \text{meagre}) \ \rightarrow \ S \backslash A := \texttt{comeagre}$$

701. $(S, \mathcal{T})$:= topological space

702. $A \subseteq S$

703. $\bigcup_c$:= countable union

704. A' := nowhere dense sets

705. [3]

Baire Category Theorem

706. `Theorem`

(*i*) $(\forall A : A \equiv \texttt{comeagre},\ A \subseteq S) \ \rightarrow \ A \equiv \texttt{dense}$ in S

(*ii*) $S \equiv \texttt{nonmeagre}$

707. $(S, \mathcal{T}) :=$ (locally compact T_2-space) $\vee$ (completely metrizable space)

708. [3]

Image, Inverse Image

709. `Definitions`

710. $f : X \to Y; \quad A \subseteq X; \quad B \subseteq Y$

711. $f[A] = \{f(x) \mid x \in A\} :=$ `image of` A `under` f

712. $f^{-1}[B] = \{x \in X \mid f(x) \in B\} :=$ `inverse image of` B `under` f

713. [3]

Continuous Functions

714. `Definition`

$$(\forall V \in \mathcal{U} : f^{-1}[V] \in \mathcal{T}) \;\rightarrow\; (f := \texttt{continuous})$$

715. `Theorem`

$$(\forall A : f^{-1}[A] := \texttt{open}) \;\rightarrow\; (f \equiv \texttt{continuous})$$

716. $(X,\mathcal{T}),(Y,\mathcal{U}) :=$ topological spaces

717. $f : X \rightarrow Y$

718. $A :=$ open set

719. *Continuity may depend on both, the function and the topologies.*

720. `Theorem`

$$(\forall V \in \mathcal{U},\; f(x) \in V,\; \exists U \in \mathcal{T},\; x \in U : f[U] \subseteq V) \;\rightarrow$$
$$\rightarrow\; (f \equiv \texttt{continuous} \text{ at } x \in X)$$

721. [3]

Continuous Function

722. `Theorem`

$$(f \equiv \texttt{continuous}) \leftrightarrow (\forall x \in A : f \equiv \texttt{continuous})$$

723. $(A, \mathcal{T}), (B, \mathcal{U}) :=$ topological spaces

724. $f : A \to B$

725. [3]

Continuity in Metrizable Spaces

726. `Theorem`

$$(f \equiv \texttt{continuous at } x \in A) \leftrightarrow$$
$$\leftrightarrow (\forall \epsilon > 0\ \exists \delta > 0 :\ d(x,y) < \delta \rightarrow \rho(f(x), f(y)) < \epsilon)$$

727. $(A, \mathcal{T}), (B, \mathcal{U}) :=$ metrizable topological spaces

728. $d, \rho :=$ metrics

729. $\mathcal{T}, \mathcal{U} :=$ induced by d and ρ, respectively

730. $f : A \rightarrow B$

731. [3]

Continuous Function, Metrizable Spaces, Open Ball, Sequence, Convergence

732. `Theorem`

$$\begin{aligned} f \text{ is } \texttt{continuous} \text{ at } x \ \equiv \\ \equiv\ \forall B_\epsilon(f(x);\rho)\ \exists B_\delta(x;d) : f[B_\delta(x;d)] \subseteq B_\epsilon(f(x);\rho)\ \equiv \\ \equiv\ (x_n \to x)\ \Rightarrow\ (f(x_n) \to f(x)) \end{aligned}$$

733. $(A,\mathcal{T}),(B,\mathcal{U}) :=$ metrizable topological spaces

734. $d,\rho :=$ metrics

735. $\mathcal{T},\mathcal{U} :=$ induced by d and ρ, respectively

736. $f : A \to B$

737. $x \in A$

738. $B_\epsilon(f(x);\rho) :=$ open ball in $\mathcal{U}$ (center $f(x)$, radius $\epsilon > 0$, metric ρ)

739. $B_\delta(x;d) :=$ open ball in $\mathcal{T}$ (center x, radius $\delta > 0$, metric d)

740. $(x_n) :=$ sequence

741. $(x_n \to x) := x_n$ converges to x

742. [3]

Uniformly Continuous Function

743. `Definition`

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall a, b \in A\ (d(a,b) < \delta\ \rightarrow\ \rho(f(a), f(b)) < \epsilon)\ \rightarrow$$
$$\rightarrow\ f := \texttt{uniformly continuous} \text{ on } A$$

744. $(A, \mathcal{T}), (B, \mathcal{U}) :=$ metrizable topological spaces

745. $d, \rho :=$ metrics

746. $\mathcal{T}, \mathcal{U} :=$ induced by d and ρ, respectively

747. $f : A \rightarrow B$

748. [3]

Homeomorphism

749. `Definition`

$$(f \equiv \texttt{bijection} : O \in \mathcal{T} \leftrightarrow f[O] \in \mathcal{U}) \;\rightarrow\; (f := \texttt{homeomorphism})$$

750. $(A, \mathcal{T}), (B, \mathcal{U}) :=$ topological spaces

751. $f : A \rightarrow B$

752. [3]

Homeomorphic (Topologically Equivalent) Spaces

753. `Definition`

$$\exists f_h \ \rightarrow \ (A, \mathcal{T}) \equiv_t (B, \mathcal{U})$$

754. $(A, \mathcal{T}), (B, \mathcal{U}) :=$ topological spaces

755. $(f_h : A \rightarrow B) :=$ homeomorphism

756. $\equiv_t$ $:=$ `topological equivalence`

757. [3]

Topological Property (Topological Invariant)

758. `Definition`

`topological invariant` := *property preserved under homeomorphisms*

$$(\mathcal{T} \text{ has } P) \wedge (\mathcal{U} \equiv_t \mathcal{T}) \;\rightarrow\; (\mathcal{U} \text{ has } P)$$

759. P := topological property

760. $(A,\mathcal{T}),(B,\mathcal{U})$:= topological spaces

761. $\equiv_t$:= topological equivalence

762. Examples of *topological invariants*: `compactness`, `being a` $\mathtt{T}_2$`-space`, `separation axioms` (T_0-T_4), `countability axioms`, `metrizability`.

763. [3]

T_2-space, Topological Invariance

764. `Theorem`

$$((A,\mathcal{T}) := \mathrm{T}_2\text{-space}) \land ((B,\mathcal{U}) \equiv_t (A,\mathcal{T})) \;\rightarrow\; (B,\mathcal{U}) \equiv \mathtt{T}_2\texttt{-space}$$

765. $(A,\mathcal{T}),(B,\mathcal{U})$:= topological spaces

766. $\equiv_t$:= topological equivalence

767. [3]

Tychonoff Space

768. `Definition`

$$(\mathcal{T} \equiv \mathrm{T}_1\text{-space}) \ \wedge$$
$$\wedge \ (\forall x \in S,\ \forall A \subseteq S \backslash \{x\},\ \exists f : S \to [0,1],\ f(x) = 0,\ f[A] = \{1\}) \to$$
$$\to \ \mathcal{T} := \texttt{Tychonoff space}$$

769. $(S, \mathcal{T})$:= topological space

770. A := closed set

771. f := continuous function

772. [3]

Completely Regular Space ($T_{3\frac{1}{2}}$-space)

773. `Definition`

$$(\forall x \in S,\ \forall A \subseteq S \backslash \{x\},\ \exists f : S \to [0,1],\ f(x) = 0,\ f[A] = \{1\}) \to$$
$$\to\ \mathcal{T} \coloneqq \texttt{completely regular space } (\texttt{T}_{3\frac{1}{2}}\texttt{-space})$$

774. $(S, \mathcal{T}) \coloneqq$ topological space

775. $A \coloneqq$ closed set

776. $f \coloneqq$ continuous function

777. [3]

Urysohn's Lemma

778. `Theorem`

$$((S,\mathcal{T}) \equiv \mathrm{T}_4\text{-space}) \ \wedge \ (A \cap B = \varnothing) \ \rightarrow$$
$$\rightarrow \ \exists f : S \rightarrow [0,1], \ f[A] = \{0\}, \ f[B] = \{1\}$$

779. $A, B :=$ closed subsets of S

780. $f :=$ continuous function

781. [3]

T_4-space, Tychonoff Space

782. `Theorem`

$$(\forall \mathcal{T} : \mathcal{T} \equiv \texttt{T}_4\texttt{-space}) \rightarrow (\mathcal{T} \equiv \texttt{Tychonoff space})$$

783. $\mathcal{T} :=$ topological space

784. [3]

Tietze Extension Theorem

785. `Theorem`

$$\begin{aligned} (S,\mathcal{T}) &\equiv \texttt{T}_4\texttt{-space}, \\ (C,\mathcal{T}_C) &\subseteq_s (S,\mathcal{T}), \\ C &\equiv \text{closed in } S, \\ a,b \in \mathbb{R} &: a < b, \\ f : C \to [a,b] &\text{ continuous} \\ &\Longrightarrow \\ f \texttt{ can be extended to } g : S \to [a,b] &\text{ continuous} \end{aligned}$$

786. $\subseteq_s$:= `subspace` relation

787. [3]

Disconnection, Disconnected, Connected

788. `Definitions`

789.

$$(U, V) := \texttt{disconnection} \text{ of } S$$

790.

$$\exists (U, V) \;\rightarrow\; (S, \mathcal{T}) := \texttt{disconnected}$$

791.

$$\nexists\, (U, V) \;\rightarrow\; (S, \mathcal{T}) := \texttt{connected}$$

792. $(S, \mathcal{T}) :=$ topological space

793. $U, V :=$ nonempty open sets

794. $U \cap V = \varnothing$

795. $U \cup V = S$

796. [3]

Connected, Clopen, Uniqueness

797. `Theorem`

$$(S, \mathcal{T}) \equiv \texttt{connected} \leftrightarrow \exists! \mathcal{C} : \mathcal{C} = \{S, \varnothing\}$$

798. $(S, \mathcal{T})$:= topological space

799. $\mathcal{C}$:= set of `clopen` sets in S

800. $\exists!$:= there is *exactly* one

801. [3]

Interval, Connected Subspace, Standard Topology

802. `Theorem`

$$I \subseteq \mathbb{R} \;\rightarrow\; (I, \mathcal{T}_I) \equiv \texttt{connected subspace} \text{ of } (\mathbb{R}, \mathcal{T})$$

803. I := interval

804. $\mathcal{T}$:= standard topology on $\mathbb{R}$

805. [3]

Connected Subspace, Standard Topology, Interval

806. `Theorem`

$$A \subseteq \mathbb{R}, \quad (A, \mathcal{T}_A) \equiv \texttt{connected subspace} \text{ of } (\mathbb{R}, \mathcal{T}) \ \rightarrow$$
$$\rightarrow \ A \equiv \texttt{interval}$$

807. $\mathcal{T}$:= standard topology on $\mathbb{R}$

808. [3]

Connected, Continuous, Image

809. `Theorem`

$$(A, \mathcal{T}) \equiv \texttt{connected}, \quad f : A \to B \text{ continuous } \to f[A] \equiv \texttt{connected}$$

810. $(A, \mathcal{T}), (B, \mathcal{U}) :=$ topological spaces

811. [3]

Intermediate Value Theorem

812. `Theorem`

$$(S,\mathcal{T}) \equiv \texttt{connected}, \quad f : S \to \mathbb{R} \text{ continuous } \to \\ \to f[S] \equiv \texttt{interval}$$

813. $(S,\mathcal{T}) :=$ topological space

814. [3]

Topology Generated by a Set, Connected Subspace

815. `Theorem`

$$\mathcal{A} = \{(A_k, \mathcal{T}_{A_k}) \mid k \in K\},$$
$$\bigcap\{A_k \mid k \in K\} \neq \varnothing,$$
$$A = \bigcup\{A_k \mid k \in K\},$$
$$\mathcal{T}^* \text{ is } \texttt{generated} \text{ by } \bigcup\{\mathcal{T}_{A_k} \mid k \in K\}$$
$$\Longrightarrow$$
$$(A, \mathcal{T}^*) \equiv \texttt{connected subspace} \text{ of } (S, \mathcal{T})$$

816. $(S, \mathcal{T})$:= topological space

817. $(A_k, \mathcal{T}_{A_k})$:= connected subspace of $(S, \mathcal{T})$

818. $\mathcal{T}^*$:= topology

819. [3]

Components

820. `Definitions`

821.

$$x \sim y \quad \leftrightarrow \quad \exists (A, \mathcal{T}_A) : x, y \in A$$

822.

$$\texttt{connected components} \text{ of } S := \textit{equivalence classes} \text{ of } \sim$$

823. $(S, \mathcal{T})$:= topological space

824. $\sim$:= equivalence relation on S

825. $(A, \mathcal{T}_A)$:= connected subspace of $(S, \mathcal{T})$

826. [3]

Connected Subspace, Clopen, Closed, Component

827. `Theorem`

(i) $\forall x \in S\ \exists! \mathcal{M} : x \in \mathcal{M}$

(ii) $(A, \mathcal{T}_A) \equiv$ `connected subspace` of $(S, \mathcal{T}) \;\rightarrow\; A \subseteq \mathcal{M}$

(iii) $(A, \mathcal{T}_A) \equiv$ `connected subspace` of $(S, \mathcal{T})$, $\;A \equiv$ `clopen` in $S \;\rightarrow$
$\rightarrow\; A \subseteq \mathcal{M}$

(iv) $\forall \mathcal{M} : \mathcal{M} \equiv$ `closed` in S

828. $(S, \mathcal{T}) :=$ topological space

829. $\mathcal{M} :=$ component of S

830. [3]

Totally Disconnected

831. `Definition`

$$\forall x, y \in S,\ x \neq y,\ \exists (U, V) : x \in U, y \in V \ \rightarrow$$
$$\rightarrow\ (S, \mathcal{T}) := \texttt{totally disconnected}$$

832. $(S, \mathcal{T}) :=$ topological space

833. $(U, V) :=$ disconnection of S

834. [3]

Locally Connected Space

835. `Definition`

$$\forall x \in S,\ \forall U : x \in U \ \rightarrow$$
$$\rightarrow\ \exists V,\ x \in V : (V, \mathcal{T}_V) \coloneqq \texttt{connected},\ V \subseteq U$$

836. $(S, \mathcal{T}) \coloneqq$ topological space

837. $U, V \coloneqq$ open sets in S

838. $\neg\Box \coloneqq$ not necessarily

839. `connected space` $\equiv \neg\Box$ `locally connected`

840. [3]

Function Space

841. `Definition`

$$\texttt{function space} := \text{vector space } + \text{ functions (elements)}$$

842. [3]

Example, Function Space, ${}^{\mathbb{R}}\mathbb{R}$

843. `Theorems`

844.
$${}^{\mathbb{R}}\mathbb{R} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$$

845.
$${}^{\mathbb{R}}\mathbb{R} \equiv \texttt{vector space over } \mathbb{R}$$

846.
$${}^{\mathbb{R}}\mathbb{R} \equiv \texttt{function space}$$

847. $f, g \in {}^{\mathbb{R}}\mathbb{R} \ \to \ f + g \in {}^{\mathbb{R}}\mathbb{R}, \ (f+g)(x) = f(x) + g(x)$

848. $f \in {}^{\mathbb{R}}\mathbb{R}, \ c \in \mathbb{R} \ \to \ cf \in {}^{\mathbb{R}}\mathbb{R}, \ (cf)(x) = c \cdot f(x)$

849. $\forall x \in \mathbb{R} : 0(x) = 0$

850. $\forall x \in \mathbb{R} : (-f)(x) = -f(x)$

851. $0 :=$ zero vector (function)

852. $-f :=$ additive inverse of $f \in {}^{\mathbb{R}}\mathbb{R}$

853. [3]

Normed Vector Space

854. `Definitions`

855.

$$(V, |\ \ |) \coloneqq \texttt{normed vector space (normed linear space)}$$

856. $(|\ \ | : V \to \mathbb{R}) \coloneqq$ `norm` (function)

$$\begin{aligned} &(i)\ \ \forall v \in V,\ \ |v| = 0 \leftrightarrow v = 0 \\ &(ii)\ \ \forall v, w \in V,\ \ |v + w| \leq |v| + |w| \\ &(iii)\ \ \forall v \in V,\ \ c \in \mathbb{F},\ \ |cv| = |c||v| \end{aligned}$$

857. $V \coloneqq$ `vector space` over a *field* $\mathbb{F}$

858. $v \in V \ \to\ |v| \coloneqq$ norm of v

859. $c \in \mathbb{F} \ \to\ |c| \coloneqq$ absolute value of v

860. [3]

Banach Space

861. `Definition`

$$(V, d) \equiv \texttt{complete metric space} \rightarrow V \coloneqq \texttt{Banach space}$$

862. $V \coloneqq$ normed vector space induced by d

863. $d \coloneqq$ metric

864. [3]

Metric Space, Continuous Function

865. Theorem

$$(S, d) \equiv \texttt{metric space} \rightarrow \mathcal{C}(S, \mathbb{R}) \leq \mathcal{B}(S, \mathbb{R})$$

866. $\mathcal{B}(S, \mathbb{R}) = \{f : S \to \mathbb{R} \mid f := \text{ bounded}\}$

867. $\mathcal{C}(S, \mathbb{R}) = \{f \in \mathcal{B}(S, \mathbb{R}) \mid f := \text{ continuous}\}$

868. $\leq$:= subspace relation

869. [3]

Closed Subspace of a Banach space

870. `Theorem`

```
closed subspace of a Banach space ≡ Banach space
```

871. $V :=$ Banach space induced by d

872. $d :=$ metric

873. $W :=$ subspace of V (closed in V)

874. [3]

Pointwise Convergence

875. `Definition`

$$\forall x \in S : f_n(x) \to f(x) \;\Rightarrow\; (f_n) := \texttt{converges pointwise} \text{ to } f$$

876. `Definition`

$$(x \in S,\ \epsilon > 0,\ \exists K \in \mathbb{N} : n > K \to |f_n(x) - f(x)| < \epsilon) \;\Rightarrow$$
$$\Rightarrow\; (f_n) := \texttt{converges pointwise} \text{ to } f$$

877. $(f_n) :=$ sequence in $\mathcal{C}(S, \mathbb{R})$

878. $\mathcal{B}(S, \mathbb{R}) = \{f : S \to \mathbb{R} \mid f := \text{ bounded}\}$

879. $\mathcal{C}(S, \mathbb{R}) = \{f \in \mathcal{B}(S, \mathbb{R}) \mid f := \text{ continuous}\}$

880. [3]

Uniform Convergence

881. `Definition`

$$(\epsilon > 0,\ \exists K \in \mathbb{N} : n > K \to \forall x \in S,\ |f_n(x) - f(x)| < \epsilon)\ \to$$
$$\to\ (f_n) \coloneqq \texttt{converges uniformly} \text{ to } f$$

882. $(f_n) \coloneqq$ sequence in $\mathcal{C}(S, \mathbb{R})$

883. $\mathcal{B}(S, \mathbb{R}) = \{f : S \to \mathbb{R} \mid f \coloneqq \text{ bounded}\}$

884. $\mathcal{C}(S, \mathbb{R}) = \{f \in \mathcal{B}(S, \mathbb{R}) \mid f \coloneqq \text{ continuous}\}$

885. [3]

Uniform Convergence, Metric Space

886. `Theorem`

$$(f_n) \equiv \texttt{converges uniformly} \text{ to } f \quad \Rightarrow \quad f \in \mathcal{C}(S, \mathbb{R})$$

887. $(S, d) :=$ metric space

888. $(f_n) :=$ sequence in $\mathcal{C}(S, \mathbb{R})$

889. $\mathcal{B}(S, \mathbb{R}) = \{f : S \to \mathbb{R} \mid f := \text{ bounded}\}$

890. $\mathcal{C}(S, \mathbb{R}) = \{f \in \mathcal{B}(S, \mathbb{R}) \mid f := \text{ continuous}\}$

891. [3]

Equicontinuous

892. `Definition`

$$(\epsilon > 0 \;\rightarrow\; (\delta > 0 : \forall x, y \in K,\; \forall f \in \mathcal{A},\; d(x,y) < \delta \rightarrow |f(x) - f(y)| < \epsilon)) \;\Rightarrow \\ \Rightarrow\; \mathcal{A} := \texttt{equicontinuous}$$

893. [3]

Arzela-Ascoli Theorem

894. `Theorem`

$$\mathcal{A} \equiv \texttt{compact} \;\leftrightarrow\; A \equiv (\texttt{bounded} \wedge \texttt{equicontinuous})$$

895. $(K, d) :=$ compact metric space

896. $\mathcal{A} \subseteq \mathcal{C}(K, \mathbb{R})$

897. $\mathcal{A} :=$ closed in K

898. [3]

Homotopy

899. `Definitions`

900.

$$F : X \times [0,1] \to Y \text{ continuous},$$
$$\forall x \in X : \; F(x,0) = f(x), \; F(x,1) = g(x) \; :=$$
$$:= \; \texttt{homotopy} \text{ from } f \text{ to } g$$

901.

$$F \; \texttt{deforms} \; f \text{ into } g$$

902.

$$\exists \; \texttt{homotopy} \text{ from } f \text{ to } g \;\to\; f \simeq g$$

903. $(X,\mathcal{T}),(Y,\mathcal{U})$:= topological spaces

904. $(f,x : X \to Y)$:= continuous

905. $\simeq$:= homotopic relation

906. [3]

The Pasting Theorem

907. `Theorem`

$$\forall x \in A \cup B,\ f(x) = g(x)\ \rightarrow\ h \equiv \text{continuous}$$

908. $(X,\mathcal{T}),(Y,\mathcal{U}) :=$ topological spaces

909. $A, B :\subseteq X;\quad A, B :=$ closed; $\quad X = A \cup B$

910. $f : A \to Y;\quad g : B \to Y;\quad f, g :=$ continuous

911. $h : X \to Y$

912. $h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$

913. [3]

Homotopy is an Equivalence Relation

914. `Theorem`

$$\simeq \text{ is an } \texttt{equivalence relation}$$

915. $\simeq :=$ homotopic relation

916. [3]

Path Connected

917. `Definitions`

918.

$$X = [0,1] \;\rightarrow\; f, g := \texttt{paths}$$

919.

$$a, b \in Y \;\rightarrow\; f' := \texttt{path} \text{ in } Y \text{ from } a \text{ to } b$$

920.

$$\forall a, b \in Y,\; \exists f : f \equiv \texttt{path} \text{ from } a \text{ to } b \;\rightarrow\; (Y, \mathcal{U}) := \texttt{path connected}$$

921. $(X, \mathcal{T}), (Y, \mathcal{U}) :=$ topological spaces

922. $f, g : X \rightarrow Y$

923. $f' : [0,1] \rightarrow Y$ continuous

924. $a :=$ `initial point`

925. $b :=$ `terminal point`

926. [3]

Convex

927. `Definition`

$$\forall t \in [0,1],\ x, y \in X \to (1-t)x + ty \in X \ \Rightarrow\ (X, \mathcal{T}) \coloneqq \texttt{convex}$$

928. `Theorem`

$$\forall (X, \mathcal{T}) \equiv \texttt{convex} \ \to\ (X, \mathcal{T}) \coloneqq \texttt{path connected}$$

929. $(X, \mathcal{T}) \coloneqq$ topological space

930. [3]

Path Homotopy

931. `Definitions`

932. $F :=$ `path homotopy` from f to g

$$(i)\ \ \forall s \in [0,1]:\ F(s,0) = f(s),\ F(s,1) = g(s)$$
$$(ii)\ \ \forall t \in [0,1]:\ F(0,t) = a,\ F(1,t) = b$$

933.

$$\exists\ \texttt{path homotopy} \text{ from } f \text{ to } g\ \rightarrow\ f \simeq_p g$$

934. $f, g : [0,1] \to Y;\quad f, g :=$ continuous functions

935. $a :=$ initial point of f and g

936. $b :=$ terminal point of f and g

937. $F : [0,1] \times [0,1] \to Y$

938. $\simeq_p\ :=$ path homotopic relation

939. [3]

Path Homotopy is an Equivalence Relation

940. `Theorem`

$$\simeq_p \text{ is an } \texttt{equivalence relation}$$

941. $\simeq_p$:= path homotopic relation

942. [3]

Product of Paths

943. `Definition`

$$(f \star g)(s) := \begin{cases} f(2s), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1), & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

944. $f :=$ path in Y from a to b

945. $g :=$ path in Y from b to c

946. $f \star g :=$ product of f and $g :=$ path in Y from a to c

947. [3]

Continuous, Paths, Path Homotopy, Composition of Functions

948. `Theorem`

$$\begin{aligned}(X, \mathcal{T}), (Y, \mathcal{U}) &\equiv \text{topological spaces},\\ h : X \to Y &\text{ continuous},\\ f, g : [0,1] \to X &\equiv \text{paths in } X\\ F : [0,1] \times [0,1] \to X &\equiv \text{path homotopy from } f \text{ to } g\\ &\Longrightarrow\\ h \circ f : [0,1] \times [0,1] \to Y &\equiv \text{path homotopy from } h \circ f \text{ to } h \circ g\end{aligned}$$

949. [3]

Continuous, Paths, Composition and Product of Functions

950. `Theorem`

$$\begin{aligned} (X,\mathcal{T}),(Y,\mathcal{U}) &\equiv \text{topological spaces},\\ h:X\to Y &\text{ continuous},\\ f,g:[0,1]\to X \equiv \text{paths in } X;\quad f(1)&=g(0)\\ &\Longrightarrow\\ h\circ(f\star g)&=(h\circ f)\star(h\circ g) \end{aligned}$$

951. [3]

Well-defined Operation on Path Homotopy Classes

952. `Theorem`

$$(f \star g)(s) = \begin{cases} f(2s), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1), & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

953. $(Y, \mathcal{T}) :=$ topological space

954. $f :=$ path in Y from a to b

955. $g :=$ path in Y from b to c

956. $f \star g :=$ product of f and $g :=$ path in Y from a to c

957. `Theorem`

$$[f] \mathbin{\bar{\star}} [g] = [f \star g]$$

   (i) `associativity:` $[f] \mathbin{\bar{\star}} ([g] \mathbin{\bar{\star}} [h]) = ([f] \mathbin{\bar{\star}} [g]) \mathbin{\bar{\star}} [h]$

   (ii) `left and right identities:` $y \in Y,\ c_y : [0,1] \to Y,\ c_y(s) = y$
   $(f : [0,1] \to Y \equiv \texttt{path}$ in Y from a to $b) \Rightarrow$
   $\Rightarrow\ ([c_a] \mathbin{\bar{\star}} [f] = [f],\ [f] \mathbin{\bar{\star}} [c_b] = [f])$

   (iii) `inverse:`
   $f : [0,1] \to Y \equiv$ path from a to b,
   $\overline{f} : [0,1] \to Y,\ \overline{f}(s) = f(1-s) \Rightarrow$
   $\Rightarrow\ \overline{f} \equiv$ path from b to a, $[f] \mathbin{\bar{\star}} [\overline{f}] = [c_a],\ [\overline{f}] \mathbin{\bar{\star}} [f] = [c_b]$

958. $[f], [g] :=$ homotopy classes

959. $\bar{\star} :=$ well-defined operation on $[f], [g]$

960. $\star$ induces $\bar{\star}$

961. $c_y :=$ constant path

962. [3]

Fundamental Group, Base Point, Loop

963. `Definition`

$$\texttt{loop} \text{ in } Y \text{ based at } a := \texttt{path} \text{ that begins and ends at } a$$

964. `Theorem`

$$(\pi_1(Y,a),\ \overline{\star}\ ) \equiv \texttt{fundamental group} \text{ of } Y \text{ relative to } a$$

965. $\pi_1(Y,a) = \{[f] \mid f := \text{loop based at } a\}$

966. $[f] :=$ path homotopy class of f

967. $a :=$ base point

968. [3]

Path Connected, Isomorphism

969. `Theorem`

$$(Y,\mathcal{U}) \equiv \texttt{path connected} \;\rightarrow\; \forall a,b \in Y : \pi_1(Y,a) \cong \pi_1(Y,b)$$

970. $(Y,\mathcal{U}) :=$ topological space

971. $\cong$:= isomorphic relation

972. [3]

Path Components, Equivalence Classes, Path Connected

973. `Definitions`

974.

$$a \sim b \;\leftrightarrow\; \exists \; \texttt{path} \text{ from } a \text{ to } b$$

975.

$$\texttt{path components} \coloneqq \texttt{equivalence classes} \text{ of } \sim$$

976. $(Y, \mathcal{U}) \coloneqq$ topological space

977. $\sim$ is an equivalence relation.

978. *Each equivalence class is path connected.*

979. [3]

Topological Equivalence, Isomorphism, Fundamental Groups

980. `Theorem`

$$(X, \mathcal{T}) \equiv_t (Y, \mathcal{U}) \;\rightarrow\; (G_X \cong G_Y)$$

981. $(X, \mathcal{T}), (Y, \mathcal{U})$:= topological spaces

982. $\equiv_t$:= topological equivalence

983. $\cong$:= isomorphic relation

984. G_X, G_Y := fundamental groups

985. [3]

The Fundamental Group of the Circle

986. `Theorem`

$$G_{\mathbb{S}^1} \cong (\mathbb{Z}, +)$$

987. $G_{\mathbb{S}^1}$:= fundamental group of $\mathbb{S}^1$

988. $\cong$:= isomorphic relation

989. [3]

Lifting

990. `Definition`

$$(h_f : X \to Z \text{ continuous},\ g \circ h_f = f) \ \coloneqq \ \texttt{lifting} \text{ of } f$$

$$f \ \texttt{lifts} \text{ to } h_f$$

991. $(X,\mathcal{T}),(Y,\mathcal{U}),(Z,\mathcal{V}) \coloneqq$ topological spaces

992. $f : X \to Y; \quad g : Z \to Y; \quad f, g \coloneqq$ continuous functions

993. [3]

Standard Covering Map of the Circle

994. `Definition`

$$g := \texttt{standard covering map} \text{ of } \mathbb{S}^1$$

995. $g : \mathbb{R} \to \mathbb{S}^1; \quad g(x) = (\cos 2\pi x, \sin 2\pi x)$

996. *g wraps around* the `unit circle` *infinitely many times.*

997. [3]

Standard Covering Map, Circle, Uniqueness, Lift

998. `Theorem`

$$d(0) = (x, y) \in \mathbb{S}^1, \quad z \in g^{-1}[\{x, y\}] \Rightarrow$$
$$\Rightarrow \exists! k_d : [0,1] \to \mathbb{R}, \; k_d(0) = z$$

999. $g : \mathbb{R} \to \mathbb{S}^1; \quad g(x) = (\cos 2\pi x, \sin 2\pi x)$

1000. $g :=$ standard covering map of $\mathbb{S}^1$

1001. $d : [0,1] \to \mathbb{S}^1 :=$ path

1002. $\exists! :=$ there is *exactly* one

1003. $k_d :=$ lift of d

1004. [3]

Retraction

1005. `Definition`

$$(f : X \to A) := \texttt{retraction} \text{ if } (\forall a \in A)\, f(a) = a$$

1006. $f :=$ continuous map

1007. $A \subseteq_s X$

1008. $\subseteq_s$:= subspace relation

1009. $A :=$ retract of X

1010. [6]

Retract

1011. `Definition`

$$A := \texttt{retract of } X \text{ if } (\exists f : X \to X)(\forall x \in X)(\forall a \in A)$$

(i) $f(x) \in A$

(ii) $f(a) = a$

1012. $f :=$ retraction

1013. $A \subseteq_s X$

1014. $\subseteq_s$:= subspace relation

1015. [6]

Absolute Retract

1016. `Definition`

$$\left(X \in \mathcal{K},\ \forall Y \in \mathcal{K} : X \subseteq Y,\ X := \text{retract of } Y\right) \rightarrow$$
$$\rightarrow \left(X := \texttt{absolute retract for } \mathcal{K}\right)$$

1017. $\mathcal{K}$:= class of topological spaces closed under homeomorphism

1018. X := topological space

1019. [6]

Abstract Simplicial Complex

1020. `Definition`

$$(A \in S \ \rightarrow \ \forall A' : A' \in S) \ \rightarrow \ (S := \texttt{abstract simplicial complex})$$

1021. $S :=$ finite $\neq \varnothing$

1022. $\varnothing \neq A' \subseteq A$

1023. [6]

Open Invitation

Review, *add* content, and `co-author` this white paper [7, 8].
Join the **Open Mathematics Collaboration**.
Send your contribution to `mplobo@uft.edu.br`.

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The **latex file** for this *white paper* together with other *supplementary files* are available in [9, 10].

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The author agrees with [8].

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The Open Mathematics Collaboration

Matheus Pereira Lobo[1,2,3] (lead author, mplobo@uft.edu.br)
https://orcid.org/0000-0003-4554-1372

[1]Federal University of Tocantins (Brazil)
[2]Federal University of Northern Tocantins (Brazil)
[3]Universidade Aberta (UAb, Portugal)