A NEW EXPLICIT FORMULA FOR BERNOULLI NUMBERS INVOLVING THE EULER NUMBER

A PREPRINT

Sumit Kumar Jha

International Institute of Information Technology
Hyderabad, India
kumarjha.sumit@research.iiit.ac.in

June 13, 2019

ABSTRACT

In this brief note, we derive a new explicit formula for Bernoulli numbers in terms of the Stirling numbers of the second kind and the Euler numbers. As a corollary of our result, we obtain an explicit formula for the Euler numbers in terms of the Stirling numbers of the second kind.

Keywords Bernoulli numbers · Stirling numbers of the second kind · Euler numbers · Polylogarithm function
AMS Classification: 11B68

There are many known explicit formulas known for the Bernoulli Numbers [1]. We prove the following.

Theorem 1. *We have*

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left(\sum_{k=1}^r (-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right), \quad (1)$$

where $S(r, k)$ denotes the Stirling numbers of the second kind, $x^{(n)} = (x)(x+1) \cdots (x+n-1)$ denotes the rising factorial, and E_r denotes the Euler number.

Proof. We begin with the following result

$$\frac{\sin n\pi}{\pi} \int_0^\infty x^{n-1} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \zeta(s) - \zeta(s, 1-n), \quad (2)$$

where $\mathbf{Li}_s(-x)$ denotes the Polylogarithm function, $\zeta(s)$ is the Riemann zeta function, and $\zeta(s, 1-n)$ is the Hurwitz zeta function. The above integral is valid for all $s \in \mathbb{C} \setminus \{1\}$, and $n \in \mathbb{R}$. This integral can be obtained from formula 3.2.1.6 in the book [2].

Plugging $n = 3/4$ in the above, and $s = -r$, a negative integer, we get

$$\int_0^\infty x^{-1/4} \frac{\mathbf{Li}_{-r}(-x)}{1+x} dx = \sqrt{2}\pi \left(\frac{B_{r+1}(1/4) - B_{r+1}}{r+1} \right). \quad (3)$$

Now, we use the following representation from the note [3]

$$\mathbf{Li}_{-r}(-x) = \sum_{k=1}^r k! S(r, k) \left(\frac{1}{1+x} \right)^{k+1} (-x)^k, \quad (4)$$

which can be easily proved using induction on r .

As a result, we have

$$\begin{aligned}
\int_0^\infty x^{-1/4} \frac{\mathbf{Li}_{-r}(-x)}{1+x} dx &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \int_0^\infty \frac{x^{k-1/4}}{(1+x)^{k+2}} dx \\
&= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \cdot \frac{\Gamma(k+3/4) \Gamma(5/4)}{\Gamma(k+2)} \\
&= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \Gamma(k+3/4) \Gamma(5/4) \\
&= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}+1\right) \cdots \left(\frac{3}{4}+k-1\right) \Gamma(3/4) \Gamma(5/4) \\
&= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}+1\right) \cdots \left(\frac{3}{4}+k-1\right) \frac{\pi}{2\sqrt{2}} \\
&= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} \frac{\pi}{2\sqrt{2}},
\end{aligned}$$

here $\Gamma(\cdot)$ is the Gamma function.

But, from [4], we have

$$\frac{B_{r+1}(1/4) - B_{r+1}}{r+1} = \frac{(-2^{-(r+1)}(1-2^{-r})B_{r+1} - 4^{-(r+1)}(r+1)E_r - B_{r+1})}{r+1}. \quad (5)$$

Thus, we have

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left(\sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right). \quad (6)$$

□

If we let $r = 2l$, an even integer, in equation (1) we immediately obtain

Corollary 1.

$$E_{2l} = -4^{2l} \sum_{k=1}^{2l} (-1)^k \cdot \frac{S(2l, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}. \quad (7)$$

Remark 1. The cases $n = 1/2$ and $n \rightarrow 1$ of equation (2) have been discussed in the note [5].

References

- 1 Gould, Henry W. "Explicit formulas for Bernoulli numbers." The American Mathematical Monthly 79.1 (1972): 44-51.
2. Handbook of Mellin Transforms by Yu. Brychkov, O. Marichev, N. Savischenko, 2019.
3. Stirling Numbers and Polylogarithms by Steven E. Landsburg. Link: <http://www.landsburg.com/query.pdf>
4. Weisstein, Eric W. "Bernoulli Polynomial." From MathWorld—A Wolfram Web Resource.
5. Jha, S.K., 2019. Two new explicit formulas for the Bernoulli Numbers. arXiv preprint arXiv:1905.11216.