A NEW DERIVATION OF $\zeta(-r) = -\frac{B_{r+1}}{r+1}$

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ABSTRACT. In this note, we give a new derivation for the fact that $\zeta(-r) = -\frac{B_{r+1}}{r+1}$ where $\zeta(s)$ represents the Riemann zeta function, and B_r represents the Bernoulli numbers. Our proof uses the well-known explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind, and the Ramanujan's master theorem to obtain an integral representation for the Riemann zeta function.

Let $\zeta(s)$ be the Riemann zeta function which is the analytic continuation of the function $f(s) = \sum_{n=1}^{\infty} n^{-s}$ with $\Re(s) > 1$ over the entire complex plane except when $s = 1$ where it has a simple pole [1].

We begin by deriving an integral expression for the Riemann zeta function using the Ramanujan's master theorem. The integral involves the polylogarithm function which is defined as

Definition 1. The *polylogarithm function* is defined by the following power series in z

$$\mathbf{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \quad (1)$$

which is valid for arbitrary complex order s , and $|z| < 1$.

Lemma 1. For all $s \in \mathbb{C} \setminus \{0\}$, we have

$$\zeta(s+1) = \frac{-1}{s} \int_0^{\infty} \frac{\mathbf{Li}_s(-x)}{(x)(1+x)} dx. \quad (2)$$

Proof. Let $H_n^{(s)} = \sum_{k=1}^n k^{-s} = \zeta(s) - \zeta(s, n+1)$ where $\zeta(s, n+1) = \sum_{m=0}^{\infty} \frac{1}{(n+1+m)^s}$ is the Hurwitz zeta function. The definition (1) directly gives us

$$\frac{\mathbf{Li}_s(z)}{1-z} = \sum_{n=1}^{\infty} H_n^{(s)} z^n$$

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which is, also,

$$\frac{\mathbf{Li}_s(-x)}{1+x} = \sum_{n=0}^{\infty} H_n^{(s)}(-x)^n. \quad (3)$$

where $H_n^{(0)} = 0$. Next, we use the Ramanujan's master theorem [2], which is

$$\int_0^{\infty} x^{n-1} (\lambda(0) - x\lambda(1) + x^2\lambda(2) - \dots) dx = \frac{\pi}{\sin(\pi s)} \lambda(-n),$$

with equation (3) to give us

$$\int_0^{\infty} x^{n-1} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \frac{\pi}{\sin n\pi} (\zeta(s) - \zeta(s, 1-n)),$$

where the integral is convergent for $0 < n < 1$. Letting $n \rightarrow 0$ on both sides of the above equation gives us required equation (2) for $\Re(s) > 1$ but can be extended to all $s \in \mathbb{C} \setminus \{1\}$ by the aid of analytic continuation. $\square$

The derivatives of the polylogarithm function follows from its power series definition

$$z \frac{\partial \mathbf{Li}_s(z)}{\partial z} = \mathbf{Li}_{s-1}(z).$$

Following can be obtained by repeated application of $z \frac{\partial}{\partial z}$ on $\mathbf{Li}_1(z) = -\ln(1-z)$,

$$\begin{aligned} \mathbf{Li}_0(z) &= \frac{z}{1-z}, \\ \mathbf{Li}_{-1}(z) &= \frac{z}{(1-z)^2}, \\ \mathbf{Li}_{-2}(z) &= \frac{z(1+z)}{(1-z)^3}, \\ \mathbf{Li}_{-2}(z) &= \frac{z(1+z)}{(1-z)^3}, \\ \mathbf{Li}_{-3}(z) &= \frac{z(1+4z+z^2)}{(1-z)^4}, \end{aligned}$$

In general, we have

$$\mathbf{Li}_{-r}(-x) = \sum_{k=1}^r k! S(r, k) \left(\frac{1}{1+x} \right)^{k+1} (-x)^k, \quad (4)$$

where $S(r, k)$ denotes Stirling numbers of the second kind, which can be defined by the following recurrence relation

$$S(r+1, k) = k S(r, k) + S(r, k-1)$$

with initial conditions $S(0, 0) = S(r, 0) = S(0, k) = 1$.

The equation (4) can be easily proved using induction on r with aid of defining recurrence relation for the Stirling numbers of the second kind.

Now plugging, $s = -r$, a negative integer, in the equation (2) gives us

$$\zeta(1-r) = \frac{1}{r} \int_0^\infty \frac{\mathbf{Li}_{-r}(-x)}{(x)(1+x)} dx. \quad (5)$$

Next, we use another important fact

$$\mathbf{Li}_{-r}(-x) = (-1)^{r-1} \mathbf{Li}_{-r}(-1/x) \quad (6)$$

which can be deduced from equation (4), and the following equivalent representation

$$\mathbf{Li}_{-r}(-x) = (-1)^{r-1} x \sum_{k=1}^r k! S(r, k) \left(\frac{1}{1+x} \right)^{k+1} (-1)^k.$$

Now, substituting $1/x$ for x in the integral (5) with equation (6) gives us

$$\begin{aligned} \zeta(1-r) &= \frac{(-1)^{r-1}}{r} \int_0^\infty \frac{\mathbf{Li}_{-r}(-x)}{1+x} dx \\ &= \frac{(-1)^{r-1}}{r} \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \int_0^\infty \frac{x^k}{(1+x)^{k+2}} dx \\ &= \frac{(-1)^{r-1}}{r} \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \frac{1}{k+1}. \end{aligned}$$

Definition 2. The *Bernoulli numbers* B_n can be defined by the following generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

We, now, use the following well-know formulae for concluding the main result of this paper, that is, $\zeta(-r) = \frac{-B_{r+1}}{r+1}$.

Theorem 1 (3). *We have*

$$B_r = \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \frac{1}{k+1}.$$

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