More Grandma Identities

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In the article [1], the author proves the following trigonometric identity (calling it the “grandma identity”) valid for integer $m \geq 1$

$$\prod_{k=1}^m 2 \cos \frac{k\pi}{2m+1} = 1.$$

In the article [2], the author proves following similar identities using (Girard) Newtons relations for symmetric polynomials, and the De Moivres formula. We prove all of these as consequences of a general proposition.

$$\prod_{k=1}^{2m-1} \sin \frac{k\pi}{4m} = \prod_{k=1}^{2m-1} \cos \frac{k\pi}{4m} = \sqrt{\frac{m}{2^{4m-3}}}. \quad (1)$$

$$\prod_{k=1}^m \sin \frac{(2k-1)\pi}{4m} = \prod_{k=1}^m \cos \frac{(2k-1)\pi}{4m} = \frac{1}{2^{\frac{2m-1}{2}}}. \quad (2)$$

$$\prod_{k=1}^m \sin \frac{k\pi}{2m+1} = \prod_{k=1}^m \cos \frac{(2k-1)\pi}{4m+2} = \frac{\sqrt{2m+1}}{2^m}. \quad (3)$$

$$\prod_{k=1}^m \cos \frac{k\pi}{2m+1} = \prod_{k=1}^m \sin \frac{(2k-1)\pi}{4m+2} = \frac{1}{2^m}. \quad (4)$$

The general proposition of which the above identities are consequences is

Proposition 1. *For all complex numbers z and all integers n , we have*

$$\sin nz = 2^{n-1} \sin z \sin \left(z + \frac{\pi}{n} \right) \sin \left(z + \frac{2\pi}{n} \right) \cdots \sin \left(z + \frac{(n-1)\pi}{n} \right). \quad (5)$$

Proof. We can write

$$\begin{aligned} \sin nz &= \frac{e^{niz} - e^{-niz}}{2i} = \frac{e^{-niz}(e^{2niz} - 1)}{2i} \\ &= \frac{e^{-niz}(e^{2iz} - 1)(e^{2iz} - e^{\frac{-2i\pi}{n}}) \cdots (e^{2iz} - e^{\frac{-2(n-1)i\pi}{n}})}{2i} \\ &= k \cdot \sin z \sin \left(z + \frac{\pi}{n} \right) \sin \left(z + \frac{2\pi}{n} \right) \cdots \sin \left(z + \frac{(n-1)\pi}{n} \right) \end{aligned}$$

where

$$k = (2i)^{n-1} e^{\frac{-i\pi}{n}(1+2+\cdots+(n-1))} = (2i)^{n-1} e^{\frac{-i\pi(n-1)}{2}} = 2^{n-1}.$$

□

1 PROOF OF IDENTITY (1)

To proceed with the proof of identity (1), we plug $n = 4m$ in (5) and let $z \rightarrow 0$ on both sides to get

$$\prod_{k=1}^{4m-1} \sin \frac{k\pi}{4m} = \frac{1}{2^{4m-1}} \lim_{z \rightarrow 0} \frac{\sin 4mz}{\sin z} = \frac{m}{2^{4m-3}}.$$

But

$$\prod_{k=1}^{4m-1} \sin \frac{k\pi}{4m} = \prod_{k=1}^{2m-1} \sin \frac{k\pi}{4m} \prod_{k=2m+1}^{4m-1} \sin \left(\pi - \frac{k\pi}{4m} \right) = \prod_{k=1}^{2m-1} \sin^2 \frac{k\pi}{4m},$$

which proves first part of (1). The second part of (1) follows from the fact

$$\cos \frac{k\pi}{4m} = \sin \left(\frac{\pi}{2} - \frac{k\pi}{4m} \right) = \sin \frac{(2m-k)\pi}{4m}.$$

2 PROOF OF IDENTITY (2)

To prove identity (2), we first let $z \rightarrow -\pi/2n$ on both sides of (5) to get

$$1 = 2^{n-1} \sin \frac{\pi}{2n} \prod_{k=1}^{n-1} \sin \left(-\frac{\pi}{2n} + \frac{k\pi}{n} \right) = 2^{n-1} \sin \frac{\pi}{2n} \prod_{k=1}^{n-1} \sin \frac{(2k-1)\pi}{2n} \quad (6)$$

Plugging $n = 2m$ in the above gives us

$$\frac{1}{2^{2m-1}} = \sin \frac{\pi}{4m} \prod_{k=1}^{2m-1} \sin \frac{(2k-1)\pi}{4m}.$$

But

$$\begin{aligned} \sin \frac{\pi}{4m} \prod_{k=1}^{2m-1} \sin \frac{(2k-1)\pi}{4m} &= \sin \frac{\pi}{4m} \prod_{k=1}^m \sin \frac{(2k-1)\pi}{4m} \prod_{k=m+1}^{2m-1} \sin \left(\pi - \frac{(2k-1)\pi}{4m} \right) \\ &= \prod_{k=1}^m \sin^2 \frac{(2k-1)\pi}{4m}. \end{aligned}$$

This proves the first part of (2). The second part of (2) follows from fact that

$$\prod_{k=1}^m \cos \frac{(2k-1)\pi}{4m} = \prod_{k=1}^m \cos \frac{(2m-2k+1)\pi}{4m} = \prod_{k=1}^m \cos \left(\frac{\pi}{2} - \frac{(2k-1)\pi}{4m} \right).$$

3 PROOF OF IDENTITY (3)

To prove identity (3), we plug $n = 2m + 1$ and let $z \rightarrow 0$ on both sides of (5) to get

$$\prod_{k=1}^{2m} \sin \frac{k\pi}{2m+1} = \frac{1}{2^{2m}} \lim_{z \rightarrow 0} \frac{\sin(2m+1)z}{\sin z} = \frac{2m+1}{2^{2m}}.$$

But

$$\prod_{k=1}^{2m} \sin \frac{k\pi}{2m+1} = \prod_{k=1}^m \sin \frac{k\pi}{2m+1} \prod_{k=m+1}^{2m} \sin \left(\pi - \frac{k\pi}{2m+1} \right) = \prod_{k=1}^m \sin^2 \frac{k\pi}{2m+1}.$$

This proves the first part of the identity (3). The second part of (3) follows from the fact that

$$\begin{aligned} \prod_{k=1}^m \cos \frac{(2k-1)\pi}{4m+2} &= \prod_{k=1}^m \sin \left(\frac{\pi}{2} - \frac{(2k-1)\pi}{4m+2} \right) \\ &= \prod_{k=1}^m \sin \frac{(m-k+1)\pi}{2m+1} = \prod_{k=1}^m \sin \frac{k\pi}{2m+1}. \end{aligned}$$

4 PROOF OF IDENTITY (4)

To prove identity (4), we plug $n = 2m + 1$ in (6) to get

$$\frac{1}{2^{2m}} = \sin \frac{\pi}{4m+2} \prod_{k=1}^{2m} \sin \frac{(2k-1)\pi}{4m+2}.$$

But

$$\begin{aligned} \sin \frac{\pi}{4m+2} \prod_{k=1}^{2m} \sin \frac{(2k-1)\pi}{4m+2} &= \sin \frac{\pi}{4m+2} \prod_{k=1}^m \sin \frac{(2k-1)\pi}{4m+2} \prod_{k=m+1}^{2m} \sin \left(\pi - \frac{(2k-1)\pi}{4m+2} \right) \\ &= \prod_{k=1}^m \sin^2 \frac{(2k-1)\pi}{4m+2}. \end{aligned}$$

This proves the first part of (4). The second part of (4) follows from the fact that

$$\prod_{k=1}^m \sin \frac{(2k-1)\pi}{4m+2} = \prod_{k=1}^m \cos \left(\frac{\pi}{2} - \frac{(2k-1)\pi}{4m+2} \right) = \prod_{k=1}^m \cos \frac{k\pi}{2m+1}.$$

REFERENCES

1. Humble, Steve. "88.62 Grandma's identity." *The Mathematical Gazette* 88, no. 513 (2004): 524-525.
2. Amrik Singh Nimbran. (2017). An Infinite Product for the Square Root of an Integer. *The American Mathematical Monthly*, 124(7), 647-650. doi:10.4169/amer.math.monthly.124.7.647