Categorical syllogisms – a set theory do-over

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Abstract

There are 36 distinct pairs of categorical premises (PCP) expressed via only the S,P,M terms and the categorical operators (or quantifiers) A,O,E,I applied to pairs of these 3 terms. Modeling such premises on a cylindrical Venn diagram or Karnaugh map with $n=3$, where the middle term M occupies one row and M', its complementary set, occupies a 2nd row, one sees that the PCPs do not act symmetrically on the M and M' rows: four P-premises (resp. S-premises) out of six act on M and only two act on M'. Only by adding premises explicitly containing the M' term, namely $A(M',)$ and $O(M',*)$ where the * stands for either S or P, one can get to 8 P-premises and 8 S-premises, with 4 P-premises, (resp. 4 S-premises) acting on the M set and 4 P-premises, (resp. 4 S-premises) acting in similar ways on the M' set. Only with the "M' premises addition" one gets to a set of 64 distinct PCPs which remains unchanged under the relabelings $M \leftrightarrow M'$, $S \leftrightarrow S'$ and $P \leftrightarrow P'$, (where S',P',M' are the complementary sets of S,P,M in a universal set U). One models the usual S,P,M terms which appear in the 3 statements of a categorical syllogism as sets. This way the two premises and the conclusion of a categorical syllogism become statements about sets. Since sets are not "distributed", do not appear in "figures", do not have "moods", etc., such concepts will not be used in this do-over of the categorical syllogisms. The 36 distinct pairs of categorical premises (PCP) which are worded using only the S,P,M terms, as well as the "extended" 64 PCPs are naturally partitioned into 5 subsets: two subsets whose PCPs do not entail any logical conclusion (LC), and three subsets whose PCPs do each entail at least one LC and thus generate a valid categorical argument (VCA). This way one obtains three classes of VCAs, each class containing some of the valid syllogisms (VS). By definition a VS is a VCA whose LC is one of the categorical operators (or quantifiers) A,O,E,I applied to the ordered pair (S,P). The LCs of the $VCA \setminus VS$ set have one of the formats $A(P,S)$, $O(P,S)$ or $I(S',P')$. Using a $P \leftrightarrow S$ relabeling one transforms a $VCA \setminus VS$ PCP which entails an $A(P,S)$ or $O(P,S)$ LC, into a PCP which has an $A(S,P)$ or $O(S,P)$ LC; if the latter PCP is a VS PCP, then one may argue that such $VCA \setminus VS$ are of no interest since they are, up to a $P \leftrightarrow S$ relabeling, identical to usual VSs. One may also argue that LCs of the format $I(S',P')$ are of no practical interest, (maybe because an $I(S',P')$ LC introduces the S', P' terms which do not appear in the (English) wording of the 64 PCPs). Such arguments, (outside of set theory), would play*

an equivalent role to the rules of valid syllogisms – whose main role is to eliminate, (under the “guise” of distribution, of “two negative premises are not allowed”, etc.), the VCA\VS set as containing only invalid syllogisms, even if all of them are valid categorical arguments. Using conversions, contrapositions and obversions, or, equivalently, set relabelings involving all the six sets S, P, M, S', P', M' , one can show that the VCAs inside each of the three VCA classes are transformed into each other, i.e., are equivalent to one another. Existential import (ei) means that one adds, to a PCP made of two universal premises, the supposition that one of the sets S, P, M, S', P', M' is not empty.

Keywords. *categorical premises • valid syllogisms • valid categorical arguments • cylindrical Venn diagram • Karnaugh map • categorical operators*

1. Introduction

By definition a categorical syllogism is made of a pair of categorical premises (PCP) to which one tacks a 3rd statement, an (S,P)-conclusion, i.e., one of the categorical operators A,O,E,I applied to the ordered pair (S,P). If the conclusion is truly entailed by the PCP one has a valid syllogism (VS), otherwise the syllogism is invalid. The 64 distinct pairs of categorical premises (PCP), (which act similarly on the M and M' subsets), can be partitioned into 5 subsets. The PCP from two of the subsets do not entail any LC. The LCs entailed by the other three PCP subsets have all fundamentally the same format: the LC singles out one subset out of the 8 subsets of U – one may call it the “LC subset” - and says that it is non empty ($\neq \emptyset$), or that either M or M' is empty except for the pinpointed LC subset, or, that, finally, one of the sets S,P,S',P' is empty except for the LC subset. (In fact in the latter case, there is a 2nd LC saying that a 2nd set out of the sets S,P,S',P' is empty except for a 2nd LC subset – nevertheless the 2nd LC subset is not independent from the 1st LC subset: for example Barbara's two LCs say that $S=SPM$, (which implies $A(S,P)$), and also that $P'=S'P'M'$. Since Barbara's premises assert also that $S \subseteq M \subseteq P$, the relation $P'=S'P'M'$ follows from $S=SPM$. Two more independent ei LCs also follow: $S \neq \emptyset$ implies $I(S,P)$, and $P' \neq \emptyset$ implies $I(S',P')$. The 5 PCP subsets are characterized as: **#1PCPs**). Both premises are particular. Obviously there are 16 such PCPs which entail no LC. **#2PCPs**). One premise is universal and one particular, and they act one on M, the middle term, and the other premise acts on M', (the complement of M in a universal set U). These are 16 more PCPs not entailing any LC. **#3PCPs**). One universal premise and one particular premise both acting on either M or M'. There are 16 such PCPs; each of their LCs is of the type: one subset out of the 8 subsets of U is $\neq \emptyset$. Since U has only 8 subsets, each one being $\neq \emptyset$ appears as an LC twice: for example, $SP'M \neq \emptyset$, meaning $O(S,P)$, is the LC of $E(M,P)I(M,S)$, Ferio/Festino/Ferison/Fresison, and also the LC of $I(M,P')E(M,S')$, Bocardo. **#4PCPs**). Two universal premises both acting on either M or M'. There are 8 such PCPs, with each LC being that either M or M' is reduced to just one subset out of its 4 subsets. Then one needs existential import (ei) on either M or M' in order to express the LC without any reference to M (or M'). (M or M' appearing in the LC would mean repeating the premises' content instead of stating the

“new knowledge” the VCA or VS “should” bring (Striker 2009: 20).) **#5PCPs**). Two universal premises, acting one on M and one on M'. The result of such a PCP is the emptying of 4 subsets of U, two subsets being located on the same column, and the other two being located on each side of that column but on different rows (one on the M row, one on the M' row). Consequently, the two LCs of such a PCP are that both sets in one of the pairs of sets, (S,P), (S',P'), (S,P'), (S',P), whose intersection is a column of U, are each reduced to one subset out of 4. For example, Barbara's PCP empties the column SP' and leaves $S=SPM$ and $P'=S'P'M'$, which leads to the following LCs: $A(S,P)$, $A(P',S')$, and via ei on S and P' to $I(S,P)$ and $I(S',P')$. Note that the two universal LCs are not independent since $A(S,P)=A(P',S')$ cfm. contraposition or set theory, but $I(S,P)$ and $I(S',P')$ are ei independent LCs.

Each VCA (and VS) LC is easily found either via a “tree like method” (which eliminates, (i.e., closes), any subset (i.e., branch), emptied by a universal premise), or, by looking at the cylindrical Venn diagram below. The VCA are partitioned into three equivalence classes, each class being generated by the PCPs from the subsets #3 to #5 above. Inside each VCA equivalence class, via a relabeling transformation of the sets S,P,M, S',P',M', (or, equivalently, via obversion, contraposition and conversion), any of the VCA can be recast (or reformulated) as any other VCA (or VS) from the same class (Radulescu 2017). The syllogistic figures, the term distribution, other rules of valid syllogisms are not used in the set theoretical treatment of the VCAs. Out of the 64 distinct PCP (or just pairs), only 32 pairs entail at least one LC and thus generate VCAs, out of which 8 are distinct VS, and 6 are distinct existential import (ei) VS. (If syllogistic figures are used, then one counts 15 VS and 9 ei VS, but this means that, e.g., the same content VS, Ferio/Festino/Ferison/Fresison, receives four different names and counts as 4 distinct VS, when in reality one deals with one PCP, $E(M,P)I(M,S)$, and one LC: $O(S,P)$. The VCA\VS subset contains 16 non ei VCAs and 18 ei VCAs. The “tree like method” is easier to apply if one first writes any premise using only the E or I statements. For any PCP in subset #3 one starts the (very short) tree with the non-empty intersection of the two sets appearing in the particular premise: in Ferio/Festino/Ferison/Fresison's case, $\emptyset \neq MS := M \cap S = MSP + MSP' = MSP'$ since the premise $E(M,P)$ says $MP = \emptyset$. Thus the LC is $MSP' \neq \emptyset$ or $O(S,P)$. (One denotes by + the union of disjoint subsets.) For any PCP in subset #4, the “LC subset” is found by “starting a tree” with either M or M' – the set which appears in both universal premises. It will result that M (or M') equals its intersection with the complements of the other two sets appearing in the two **#4PCPs** universal premises. For any PCP in subset #5 each of the two “LC subsets” can be found via two short trees, each starting with one of the “letter sets” other than M and M' and continuing by eliminating its subsets emptied by the two universal premises. For example, in the case of Barbara's PCP, $A(M,P)A(S,M)=E(M,P')E(M',S)$ start with $S=SM'+SM=SM=SPM+SP'M=SPM$ and $P'=MP'+M'P'=M'P'=S'P'M'+SP'M'=S'P'M'$. The LCs are $A(S,P)=A(P',S')$ (Barbara), and via ei on S, $I(S,P)$ (Barbari), plus, via ei on P', $I(S',P')$.

2. The Cylindrical Venn diagram (the Karnaugh map for n=3)

S'P'M	SP'M	SPM	S'PM
S'P'M'	SP'M'	SPM'	S'PM'

Fig. 1

The universal set U is graphed as a rectangle – but the left and right borders of the rectangle are glued together to generate a cylinder, so that $S'PM$ and $S'PM'$ are adjacent, and $SP'M$ and $SP'M'$ are adjacent, too – as in the usual 3-circle Venn diagram. On this “cylindrical Venn diagram” - or Karnaugh map with $n=3$ all the syllogistic conclusions are graphically obvious. It took me about a year to “invent” the cylindrical Venn diagram, only to find out - after I sent to the publisher the 1st version of a previous paper (Radulescu, 2017) – that Alan Marquand invented it in 1881, and then, in 1952 and 1953, Edward Veitch and Maurice Karnaugh used Karnaugh (-Veitch) maps for $n=3$, $n=4$, etc., to find the optimal design of digital circuits. After more than 130 years, these maps, (which have chapters dedicated to them in engineering books), apparently, never made it into logic textbooks – which are still using the 3-circle Venn diagrams.

3. New Notations

The universal operators A and E are “emptying operators”, and the particular operators I and O are “element laying” operators. $A(M,P)$ means “All M is P ”, i.e., the set $P'M := P' \cap M$ is empty. Thus, in Fig. 1, $A(M,P)$ acts on the M row, by emptying (two “horizontally adjacent” subsets) $P'M = SP'M + S'P'M$. One always list PCPs with the P -premise written first and the S -premise written after it, and inside each categorical operator the M or M' term will be listed first. Moreover, since $A(P,M)$ acts on M' as E acts on M , ($E(M,P)$ means $PM=\emptyset$, $A(P,M)$ means $PM'=\emptyset$), we'll denote $E' := E(M',*) = A(*,M)$, $I' := I(M',*) = O(*,M)$ where $*$ stands for either S or P . The above three conventions allow for a shorthand notation for the PCPs: for example EI clearly means $E(M,P)I(M,S)$, AE' means $A(M,P)E(M',S) = A(M,P)A(S,M)$, i.e., Barbara's premises, and $AA = A(M,P)A(M,S)$ are Darapti's premises. We'll also introduce two more operators acting on M' : $A' := A(M',*)$ and $O' := O(M',*)$. Now one can order all the 8 possible P -premises, (resp. all 8 possible S -premises), as vector components: $\mathbb{P}_i = \{E(M,P), E(M,P'), E(M',P), E(M',P'), I(M,P), I(M,P'), I(M',P), I(M',P')\} = (E,A,E',A', I,O,I',O')(P)$, resp., $\mathbb{S}_i = \{E(M,S), E(M,S'), E(M',S), E(M',S'), I(M,S), I(M,S'), I(M',S), I(M',S')\} = (E,A,E',A', I,O,I',O')(S)$. All the possible PCP are the components of the direct product of these two vectors $L_{ij} = \mathbb{P}_i \otimes \mathbb{S}_j$, $i,j = 1,...,8$. Without the “ M' premises addition”, there are 36 distinct PCPs partitioned in the same 5 subsets characterized in the introduction: 1. Both premises are particular, etc. Note how easy it is to decide in which one of the 5 classes a PCP falls: AA – two universal premises acting on the

M row, EE' (Celarent/Cesare) or AE' – two universal premises acting on two different rows, AO' – no LC, etc.

4. The matrix L_{ij} , $i, j = 1, \dots, 8$ and its symmetries

The “PCP matrix” $L := L_{ij} = \mathbb{P}_i \otimes \mathbb{S}_j$, $i, j = 1, 8$ contains 32 PCPs which do not entail any LC, and 32 PCPs which entail at least one LC. The latter ones appear as eight 2 by 2 sub matrices of L . The **#4PCPs** appear in $L^{(1)} := L_{ij}$, $i, j = 1, 2$, (EE, EA, AE, AA), (whose each LC is $M = \text{one of the its 4 subsets}$), and in $L^{(2)} := L_{ij}$, $i, j = 3, 4$, (E'E', E'A', A'E', A'A') - whose each LC is $M' = \text{one of the its 4 subsets}$. (See Fig. 1.) The following four 2 by 2 sub matrices all contain one particular and one universal premises, (the **#3PCPs** described in the Introduction), both acting on either M or M' and whose each LC is that one subset of U is $\neq \emptyset$. Since U has only 8 subsets and one has 16 such PCPs, it follows that the LC: some particular subset of U is $\neq \emptyset$, appears twice for each subset of U . The four sub matrices containing only **#3PCPs** are: $L^{(3)} := L_{ij}$, $i = 1, 2$, $j = 5, 6$, (EIEO, AI, AO); $L^{(4)} := L_{ij}$, $i = 3, 4$, $j = 7, 8$, (E'T', E'O', A'T', A'O'); $L^{(5)} := L_{ij}$, $i = 5, 6$, $j = 1, 2$, (IE, IA, OE, OA); $L^{(6)} := L_{ij}$, $i = 7, 8$, $j = 3, 4$, (I'E', I'A', O'E', O'A'). Finally, the **#5PCPs** appear in $L^{(7)} := L_{ij}$, $i = 1, 2$, $j = 3, 4$, (EE', EA', AE', AA') and $L^{(8)} := L_{ij}$, $i = 3, 4$, $j = 1, 2$, (E'E', E'A', A'E', A'A'). [The **#1PCPs**, (two particular premises, no LC), appear in the sub-matrix L_{ij} , $i, j = 5, 6, 7, 8$; the **#2PCPs**, (no LC), “span” another four 2 by 2 sub matrices of L .]

An $M \leftrightarrow M'$ relabeling transforms the un-primed premises, operators and LCs into the respective primed ones and vice-versa. An $S \leftrightarrow P$ relabeling transforms, for any pair (i, j) , the L_{ij} PCP and its LC into the L_{ji} PCP and its respective LC. One may group the 32 PCPs which entail at least one LC and thus generate VCAs, into 8 subsets of 4 PCPs (generating 6 VCAs per subset). The first PCP in such a group of 4 PCPs belongs to the **#4PCPs**, the following two belong to **#3PCPs** and the last one to the **#5PCPs** class of PCPs. We'll say that each of the 4 PCPs subsets is “bound to” a subset of U : the one **#4PCP**, the following two **#3PCPs** and the **#5PCP** do not act on the subset of U on which they are “bound”, but act on some of its “neighbours” in the cylindrical Venn diagram.

1. VCAs “bound to” the subset $S'P'M$:

EE=E(M,P)E(M,S)	M=S'P'M. If $M \neq \emptyset$: I(S',P'), No name
IE=I(M,P)E(M,S)	S'PM $\neq \emptyset$ or O(P,S), No name
EI=E(M,P)I(M,S)	SP'M $\neq \emptyset$ or O(S,P), Ferio/Festino/Ferison/Fresison
EE'=E(M,P)E(M',S)	S=SP'M, P=S'PM', E(S,P), Celarent/Cesare
	O(S,P) if $S \neq \emptyset$, Celaront/Cesaro; O(P,S) if $P \neq \emptyset$, No name

2. VCAs bound to the subset $SP'M$:

EA=E(M,P)E(M,S')	M=SP'M. If $M \neq \emptyset$: O(S,P), Felapton/Fesapo
IA=I(M,P)E(M,S')	SPM $\neq \emptyset$ or I(S,P), Disamis/Dimaris
EO=E(M,P)I(M,S')	S'P'M $\neq \emptyset$ or I(S',P'), No name
EA'=E(M,P)E(M',S')	S'=S'P'M, P=SPM', A(P,S)=A(S',P'),

I(S,P) if $P \neq \emptyset$, Bramantip'; I(S',P') if $S' \neq \emptyset$, No name

3. VCAs bound to the subset S'PM:

AE=E(M,P')E(M,S)	M=S'PM. If $M \neq \emptyset$: O(P,S), No name
OE=I(M,P')E(M,S)	S'PM' $\neq \emptyset$ or I(S',P'), No name
AI=E(M,P')I(M,S)	SPM' $\neq \emptyset$ or I(S,P), Darii/Datisi
AE'=E(M,P')E(M',S)	S=SPM, P'=S'PM', A(S,P), Barbara
	I(S,P) if $S \neq \emptyset$, Barbari; I(S',P') if $P' \neq \emptyset$, No name

4. VCAs bound to the subset SPM:

AA=E(M,P')E(M,S')	M=SPM. If $M \neq \emptyset$: I(S,P), Darapti
OA=I(M,P')E(M,S')	SP'M' $\neq \emptyset$ or O(S,P), Bocardo
AO=E(M,P')I(M,S')	S'PM' $\neq \emptyset$ or O(P,S), No name
AA'=E(M,P')E(M',S')	S'=SPM, P'=SP'M', E(S',P'), No name
	O(P,S) if $S' \neq \emptyset$, No name; O(S,P) if $P' \neq \emptyset$, No name

M' row VCAs:

5. VCAs bound to the subset S'P'M':

E'E=E(M',P)E(M',S)	M'=S'P'M'. If $M' \neq \emptyset$: I(S',P'), No name
I'E=I(M',P)E(M',S)	S'PM' $\neq \emptyset$ or O(P,S), No name
ET=E(M',P)I(M',S)	SP'M' $\neq \emptyset$ or O(S,P), Baroco
E'E=E(M',P)E(M,S)	S=SP'M', P=S'PM, E(S,P), Camestres/Camenes
	O(S,P) if $S \neq \emptyset$, Camestros/Camenos; O(P,S) if $P \neq \emptyset$, No name

6. VCAs bound to the subset SP'M':

E'A=E(M',P)E(M',S')	M'=SP'M'. If $M' \neq \emptyset$: O(S,P), Felapton/Fesapo'
I'A=I(M',P)E(M',S')	SPM' $\neq \emptyset$ or I(S,P), Disamis'/Dimaris'
E'O=E(M',P)I(M',S')	S'PM' $\neq \emptyset$ or I(S',P'), No name
E'A=E(M',P)E(M,S')	S'=SP'M', P=SPM, E(S',P)=A(P,S), No name
	I(S,P) if $P \neq \emptyset$, Bramantip, I(S',P') if $S' \neq \emptyset$, No name

7. VCAs bound to the subset S'PM':

A'E=E(M',P')E(M',S)	M'=S'PM'. If $M' \neq \emptyset$: O(P,S), No name
O'E=I(M',P')E(M',S)	S'PM' $\neq \emptyset$ or I(S',P'), No name
A' I'=E(M',P')I(M',S)	SPM' $\neq \emptyset$ or I(S,P), Darii'/Datisi'
A'E=E(M',P')E(M,S)	S=SPM', P'=S'PM', A(S,P)=A(P',S'), Barbara'
	I(S,P) if $S \neq \emptyset$, Barbari'; I(S',P') if $P' \neq \emptyset$, No name

8. VCAs bound to the subset SPM':

A'A=E(M',P')E(M',S')	M'=SPM'. If $M' \neq \emptyset$: I(S,P), Darapti'
O'A=I(M',P')E(M',S')	SP'M' $\neq \emptyset$ or O(S,P), Bocardo'
A'O=E(M',P')I(M',S')	S'PM' $\neq \emptyset$ or O(P,S), No name
A'A=E(M',P')E(M,S')	S'=SPM', P'=SP'M', E(S',P'), No name
	O(P,S) if $S' \neq \emptyset$, No name; O(S,P) if $P' \neq \emptyset$, No name

One can now define a “relabeling group” acting on the above VCAs subsets 1,2,...,8.

Let $p := P \leftrightarrow P'$, $s := S \leftrightarrow S'$, $m := M \leftrightarrow M'$. One can see that compositions of s,p,m generate a commutative group G with eight distinct elements: 1,s,p,m,sp,sm,pm,spm. Obviously $1=s^2=p^2=m^2=(spm)^2=(ms)^2=(ps)^2=(pm)^2$. This group acts on the above VCAs subsets 1,2,...,8, as follows:

$p(1)=3, p(2)=4, p(5)=7, p(6)=8; s(1)=2, s(3)=4, s(5)=6, s(7)=8; m(1)=5, m(2)=6, m(3)=7, m(4)=8$.

One can check that $\{G(1)\} = \{G(2)\} = \dots = \{G(8)\} = \{1,2,3,\dots,8\}$. This shows that any VCA from any of the three VCA classes can be recast as any other VCA in the same class. For example, $spm(E'E') = AA = A(M,P) A(M,S) = E(M,P')E(M,S')$ which are Darapti's premises. This means that $E'E' = E(M',P)E(M',S) = A(P,M)A(S,M)$ become Darapti's premises after an spm relabeling. $E'E'$ generates a VCA, and because M is not distributed in either premise, one “knows” that it can not be a VS. In fact one can just apply the definition of a VS as a VCA whose LC is one of the A,O,E,I operators applied to the ordered pair (S,P), and one can see that $E'E'$ is not a VS; the “distribution talk” is unnecessary. (See also Radulescu, 2017.)

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