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[knowledge base]
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Test functions and Distributions (Open Mathematics Knowledge Base)

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Abstract

In this work, we present a list of mathematical results about test functions and distributions theory for future implementation in a digital Open Mathematics Knowledge Base.

keywords: functional analysis, test functions, distributions, pure mathematics, knowledge base

The most updated version of this paper is available at
https://osf.io/xne52/download

Introduction

A. [1–4]

B. OMKB = Open Mathematics Knowledge Base (see [5])

C. *This article is constantly being updated.*

D. Test functions and Distributions (OMKB) = 108 mathematical entries (67 pages)

E. 1 entry = *notation* or *definition* or *proposition* or *theorem*

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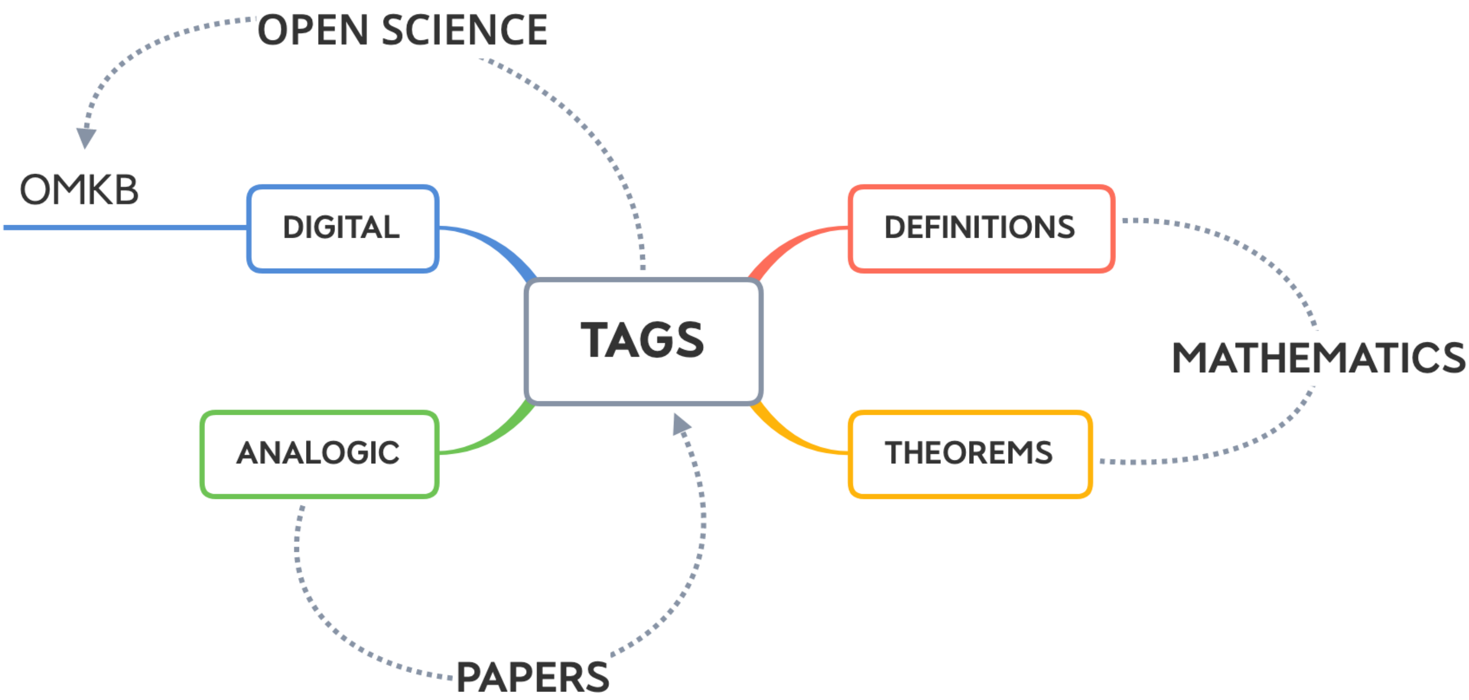
Overview

F. [6]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Important: guidelines for advanced search

G. In order to maximize the efficiency while using this document, note the following guidelines for **advanced search** in *Acrobat Reader*.

H. Acrobat Reader → Preferences → Search → Range of words for proximity searches ≈ 10

I. Acrobat Reader → open the search window → advanced settings → show more options → select a folder where the PDF is located → check the proximity box → choose match all of the words

#approximate identity #convolution #definition #sequence of functions #test functions space

1. The term approximate identity on $\mathbb{R}^n$ will denote a sequence of functions
$$h_j(x) = j^n h(jx), \;\; j = 1, 2, \cdots,$$
where $h \in \mathcal{D}(\mathbb{R}^n), h \geq 0$, and $\int_{\mathbb{R}^n} h(x)dx = 1$

#approximate identity #convolution #space of distributions #test functions space #theorem

2. If $(h_j)_{j\geq 1}$ is an approximate identity on $\mathbb{R}^n$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, and $u \in \mathcal{D}'(\mathbb{R}^n)$ $\Longrightarrow$

   (a) $\lim_{j\to\infty} \phi * h_j = \phi$ in $\mathcal{D}(\mathbb{R}^n)$

   (b) $\lim_{j\to\infty} u * h_j = u$ in $\mathcal{D}'(\mathbb{R}^n)$

#compact subset #notation

3. K = compact subset

#compact support #continuously differentiable #definition #Fréchet space #test functions space

4. Let K be a compact on Ω, $\mathcal{D}_K(\Omega) = \left\{ f \in C^\infty(\Omega) \middle| \operatorname{supp}(f) \subset K \right\}$
5. $\mathcal{D}_K(\Omega)$ is a subspace of $C^\infty(\Omega)$
6. $\mathcal{D}_K(\Omega)$ is a Fréchet space
7. $C_0^\infty(\Omega) = \left\{ f \in C^\infty(\Omega) \middle| \operatorname{supp}(f) \text{ is compact and } \operatorname{supp}(f) \subset \Omega \right\}$

#continuous function #proposition #space of distributions #support of distributions

8. If $f \in C^0(\Omega)$ and $T \in \mathcal{D}'(\Omega) \implies \mathrm{supp}(f) = \mathrm{supp}(T_f)$ where $T_f(\phi) = \int_\Omega \phi(x) f(x) dx$, for all $\phi \in \mathcal{D}(\Omega)$

#continuous linear mapping #continuously differentiable #notation #test functions space

9. L is a continuous linear mapping of $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$

#continuous linear mapping #convolution #space of distributions #test functions space #theorem

10. If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $L\phi = u * \phi$ with $\phi \in \mathcal{D}(\mathbb{R}^n) \implies L$ is a continuous linear mapping of $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$ which satisfies $\tau_x L = L\tau_x$, with $x \in \mathbb{R}^n$

11. If L is a continuous linear mapping of $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{C}(\mathbb{R}^n)$ which satisfies $\tau_x L = L\tau_x$, with $x \in \mathbb{R}^n \implies$ there is an unique $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $L\phi = u * \phi$ with $\phi \in \mathcal{D}(\mathbb{R}^n)$

#continuously differentiable #definition #Fréchet space #support

12. $f \in C^k(\Omega) \Leftrightarrow \exists \partial^\alpha f$ and it's continuous, $|\alpha| \le k$
13. $f : \Omega \to \mathbb{C}$ continuous, supp $(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$
14. $C^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{C} | f \text{ has continuous partial derivatives of all orders} \right\}$
15. $C^\infty(\Omega)$ is a Fréchet space with the Heine-Borel property

#continuously differentiable #notation #open subset #partial derivatives #test functions space

16. $\Omega \neq \varnothing$ is a nonempty open subset of $\mathbb{R}^n$
17. $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}_+^n$
18. $\partial^\alpha = \alpha$-order partial derivatives, with $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdot \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$, and order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$
19. For any $k = 0, 1, \ldots, \infty$, let $C^k(\Omega)$ denote the vector space of all k-times continuously differentiable complex-valued functions on Ω
20. $f \in C^0(\Omega) =$ continuous function
21. $C_0^\infty(\Omega)$ or $C_c^\infty(\Omega)$ is the set of all test functions on Ω

#continuously differentiable functions #convergence #proposition #sequence #test functions space

22. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence on $\mathcal{D}(\Omega)$ and $f_j \longrightarrow 0$ on $\mathcal{D}(\Omega) \iff$

    (a) $\exists K \subset \Omega$ compact such that $\operatorname{supp}(f_j) \subset K$, $\forall j \in \mathbb{N}$

    (b) For all $\alpha-$multi-index, $\partial^\alpha f_j \longrightarrow 0$ uniformly on K, $\forall j \in \mathbb{N}$

23. If $\Omega_1, \cdots, \Omega_m \subseteq \mathbb{R}^n$ are open and $\varphi \in C_0^\infty\left(\bigcup_{j=1}^m \Omega_j\right) \Longrightarrow \exists \varphi_j \in C_0^\infty(\Omega_j)$ such that $\varphi = \varphi_1 + \cdots + \varphi_m$

24. If $\Omega_1, \cdots, \Omega_m \subseteq \mathbb{R}^n$ are open and $K \subseteq \bigcup_{j=1}^m \Omega_j \Longrightarrow \exists \varphi_j \in C_0^\infty(\Omega_j)$, $\varphi_j \geq 0$, $\sum_{j=1}^m \varphi_j \leq 1$ with $\sum_{j=1}^m \varphi_j = 1$ on a neighborhood of K

#continuously differentiable functions #convergence #sequence #test functions space #theorem

25. T is a linear mapping of $\mathcal{D}(\Omega)$ into a locally convex space $Y \Longrightarrow$ each of the following properties implies the others

    (a) T is continuous

    (b) T is bounded

    (c) If $\phi_j \longrightarrow 0$ in $\mathcal{D}(\Omega) \Longrightarrow T\phi_j \longrightarrow 0$ in Y

    (d) The restrictions of T to every $\mathcal{D}_K(\Omega) \subset \mathcal{D}(\Omega)$ are continuous

#continuously differentiable functions #differential operator #proposition #test functions space

26. Every differential operator D^α is a continuous mapping of $\mathcal{D}(\Omega)$ into $\mathcal{D}(\Omega)$

#convergence #Heine-Borel property #sequence #test functions space #theorem #topology

27. (a) A convex balanced subset $V \subset \mathcal{D}(\Omega)$ is open $\iff V \in \beta$

    (b) τ_K of any $\mathcal{D}_K(\Omega) \subset \mathcal{D}(\Omega)$ coincides with the subspace topology that $\mathcal{D}_K(\Omega)$ inherits from $\mathcal{D}(\Omega)$

    (c) If E is a bounded subset of $\mathcal{D}(\Omega) \implies E \subset \mathcal{D}_K(\Omega)$ for some $K \subset \Omega$, and $\exists M_N < \infty$ such that every $\phi \in E$ satisfies

$$\|\phi\| \leq M_N \ \ (N = 0, 1, \cdots)$$

    (d) $\mathcal{D}(\Omega)$ has the Heine-Borel property

    (e) If $(\phi_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}(\Omega) \implies (\phi_j)_{j \in \mathbb{N}} \subset \mathcal{D}_K(\Omega)$ for some compact $K \subset \Omega$, and

$$\lim_{j,k \to \infty} \|\phi_j - \phi_k\|_N = 0 \ \ (N = 0, 1, \cdots)$$

    (f) if $\phi_j \longrightarrow 0$ in the topology of $\mathcal{D}(\Omega) \implies \exists K \subset \Omega$ compact such that $\text{supp}(\phi_i) \subset K$, and $D^\alpha \phi_j \longrightarrow 0$ uniformly as $i \longrightarrow \infty$ for every multi-index α

    (g) In $\mathcal{D}(\Omega)$ every Cauchy sequence converges

#convergence #notation #sequence

28. $\varphi_j \longrightarrow \varphi$ = the sequence φ_j converges to φ

#convergence in distribution #definition #sequences of distributions #test functions space

29. $(T_j)_{j\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. The sequence $T_j \longrightarrow T$ on $\mathcal{D}'(\Omega) \iff$ $T_j(\varphi) \longrightarrow T(\varphi)$ on $\mathbb{C}$, $\forall \varphi \in C_0^\infty(\Omega)$

#convergence in distribution #sequences of distributions #theorem

30. $(T_j)_{j\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$. If exists $T(\phi) \in \mathbb{R}$ such that $T_j(\phi) \longrightarrow T(\phi)$ for all $\phi \in \mathcal{D}(\Omega) \Longrightarrow T \in \mathcal{D}'(\Omega)$ and $\partial^\alpha T_j \longrightarrow \partial^\alpha T$ on $\mathcal{D}'(\Omega)$, for all $\alpha-$ multi-index

31. $(T_j)_{j\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ and $(\varphi_j)_{j\in\mathbb{N}} \subset C^\infty(\Omega)$. If $T_j \longrightarrow T$ on $\mathcal{D}'(\Omega)$ and $\varphi_j \longrightarrow \varphi$ on $C^\infty(\Omega) \Longrightarrow \varphi_j T_j \longrightarrow \varphi T$ on $\mathcal{D}'(\Omega)$

#convergence in test functions space #definition #sequences #test functions space

32. $(\varphi_j)_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. The sequence $\varphi_j \longrightarrow \varphi$ on $C_0^\infty(\Omega)$ if

    (a) $\exists K \subset \Omega$ compact such that $\text{supp}(\varphi_j) \subset K$, $\forall j \in \mathbb{N}$

    (b) $\forall \alpha \in \mathbb{Z}_+^n$, $\partial^\alpha \varphi_j \longrightarrow \partial^\alpha \varphi$ uniformly

#convergence of test functions #distribution #sequences #test functions space #theorem

33. Let $T : C_0^\infty(\Omega) \to \mathbb{C}$ be a linear map. The following properties are equivalent

    (a) $T \in \mathcal{D}'(\Omega)$

    (b) If $(\varphi_j)_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ and $\varphi_j \longrightarrow 0$ on $C_0^\infty(\Omega) \Longrightarrow T(\varphi_j) \longrightarrow 0$ (on $\mathbb{C}$)

#convolution #definition #reflection of functions #test functions space #translation of functions

34. Let f be a function in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. The functions $\tau_x f, \widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ are defined by

    - $\tau_x f(y) = f(y - x)$, for all $y \in \mathbb{R}^n$
    - $\widetilde{f}(y) = f(-y)$, for all $y \in \mathbb{R}^n$

35. If $f \in \mathcal{D}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \implies$

    (a) $\tau_x f \in \mathcal{D}(\mathbb{R}^n)$

    (b) $\widetilde{f} \in \mathcal{D}(\mathbb{R}^n)$

36. If $x, y, 0 \in \mathbb{R}^n$ and $\tau_x f, \widetilde{f} : \mathbb{R}^n \to \mathbb{R} \implies$

    (a) $\tau_x \tau_y = \tau_{x+y} = \tau_y \tau_x$

    (b) $(\tau_x f)\widetilde{} = \tau_{-x} \widetilde{f}$

    (c) $\tau_0 f = f$

#convolution #definition #space of distributions #test functions space #translation

37. $u \in \mathcal{D}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The function $\tau_x u : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ is defined by

$$\tau_x u(\phi) = u(\tau_{-x}\phi),$$

for all $\phi \in \mathcal{D}(\mathbb{R}^n)$

#convolution #proposition #space of distributions #translation

38. If $u \in \mathcal{D}'(\mathbb{R}^n) \implies \tau_x u \in \mathcal{D}'(\mathbb{R}^n)$

39. If $u \in \mathcal{D}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and a multi-index $\alpha \implies D^\alpha(\tau_x u) = \tau_x(D^\alpha u)$

#convolution of a distribution and a continuously differentiable function #definition

40. If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, their convolution $(u * \phi)$ is defined by

$$(u * \phi)(x) = u(\tau_x \widetilde{\phi}),$$

for all $x \in \mathbb{R}^n$

#convolution of a distribution and a continuously differentiable function and a test function #theorem

41. Suposse $u \in \mathcal{D}'(\mathbb{R}^n)$ has compact support, $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$ $\Longrightarrow$

    (a) $\tau_x(u * \phi) = (\tau_x u) * \phi = u * (\tau_x \phi)$ if $x \in \mathbb{R}^n$

    (b) $u * \phi \in C^\infty(\mathbb{R}^n)$ and

    $$D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$$

    for every multi-index α

    (c) $u * \psi \in \mathcal{D}(\mathbb{R}^n)$

    (d) $u * (\phi * \psi) = (u * \phi) * \psi = (u * \psi) * \phi$

#convolution of a distribution and a test function #definition

42. If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$, their convolution $(u * \phi)$ is defined by

$$(u * \phi)(x) = u(\tau_x \widetilde{\phi}),$$

for all $x \in \mathbb{R}^n$

#convolution of a distribution and two test functions #theorem #translation

43. If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi, \psi \in \mathcal{D}(\mathbb{R}^n) \implies$

    (a) $\tau_x(u * \phi) = (\tau_x u) * \phi = u * (\tau_x \phi), \;\; \forall x \in \mathbb{R}^n$

    (b) $(u * \phi) \in C^\infty(\mathbb{R}^n)$ and

    $$D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$$

    for every multi-index α

    (c) $u * (\phi * \psi) = (u * \phi) * \psi$

#convolution of a locally integrable function and a test function #definition

44. If f is a locally integrable function and $g \in \mathcal{D}(\mathbb{R}^n) \implies$ (46) can be written as
$$(f * g)(x) = f(\tau_x \widetilde{g})$$

#convolution of complex functions #definition

45. If f and g are complex functions in $\mathbb{R}^n$, their convolution $f * g$ is defined by
$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy,$$
for all $x \in \mathbb{R}^n$

46. If f and g are as (45) $\Longrightarrow$
$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\widetilde{g}(y-x)dy = \int_{\mathbb{R}^n} f(y)\tau_x\widetilde{g}(y)dy,$$
for all $x \in \mathbb{R}^n$

#convolution of distributions #Dirac measure #support of distributions #theorem

47. $u, v, w \in \mathcal{D}'(\mathbb{R}^n)$

    (a) If at least one u, v has compact support $\Longrightarrow u * v = v * u$

    (b) If $\operatorname{supp}(u)$ and $\operatorname{supp}(v)$ and if at least one of these is compact $\Longrightarrow \operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v)$

    (c) If at least two of the supports $\operatorname{supp}(u)$, $\operatorname{supp}(v)$, $\operatorname{supp}(w)$ are compact $\Longrightarrow (u * v) * w = u * (v * w)$

    (d) If δ is the Dirac measure and a multi-index $\alpha \Longrightarrow D^\alpha u = (D^\alpha \delta) * u$

    (e) If at least one of the sets $\operatorname{supp}(u)$, $\operatorname{supp}(v)$ is compact $\Longrightarrow D^\alpha(u * v) = (D^\alpha u) * v = u * (D^\alpha v)$, for every multi-index α

#convolution of two distributions and a test function #definition #support of distributions

48. If $u, v \in \mathcal{D}'(\mathbb{R}^n)$, and at least one of these two distributions has compact support, define

$$L\phi = u * (v * \phi), \ \ \phi \in \mathcal{D}(\mathbb{R}^n)$$

#definition #derivative de a distribution #space of distributions

49. Let α be a multi-index and $T \in \mathcal{D}'(\Omega)$. The formula defines the α-th derivative of T
$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi),$$
$\forall \phi \in \mathcal{D}(\Omega)$

#definition #Dirac measure #distribution of infinite order

50. The Dirac measure δ is the distribution defined by

$$< \delta, \phi > = \delta_0(\phi) = \phi(0),$$

for every $\phi \in \mathcal{D}(\Omega)$

51. $I = (0, 1) \subset \mathbb{R}$. T is a distribution of infinite order defined by

$$T(\phi) = \sum_{j=1}^{\infty} \phi^{(j)} \left(\frac{1}{j} \right),$$

$\phi \in \mathcal{D}(\Omega)$.

#definition #distribution #locally integrable function #test functions space

52. Let $f \in L^1_{\text{loc}}(\Omega)$ be a function defined on $\Omega \subset \mathbb{R}^n$. We associate with f a distribution $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ defined by

$$< T_f, \phi > = T_f(\phi) \; = \int_\Omega f(x)\phi(x)dx,$$

$\forall \phi \in \mathcal{D}(\Omega)$

#definition #distribution #multiplication of a continuously differentiable function and a distribution

53. $f \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. The distribution $fT : \mathcal{D}(\Omega) \to \mathbb{R}$ is defined by

$$(fT)(\phi) = T(f\phi),$$

$\forall \phi \in \mathcal{D}(\Omega)$

#definition #Fréchet space #F-space #Heine-Borel property

54. (X, τ) is a F-space if τ is induced by a complete invariant metric d
55. (X, τ) is a Fréchet space if X is a locally convex F-space
56. (X, τ) has the Heine-Borel property if every closed and bounded subset of X is compact

#definition #Fréchet space #test functions space # topology

57. Let Ω be a nonempty open set in $\mathbb{R}^n$

    (a) For every compact $K \subset \Omega$, τ_K denotes the Fréchet space topology of $\mathcal{D}_K(\Omega)$

    (b) β is the collection of all convex balanced sets $W \subset \mathcal{D}(\Omega)$ such that $\mathcal{D}_K(\Omega) \cap W \in \tau_K$ for every compact $K \subset \Omega$

    (c) τ is the collection of all unions of sets of the form $\phi + W$, with $\phi \in \mathcal{D}(\Omega)$ and $W \in \beta$

#definition #locally integrable function #test functions space

58. $L^1_{\text{loc}}(\Omega) = \left\{ f : \Omega \to \mathbb{C} \,\middle|\, f\varphi \in L_1(\Omega), \forall\, \varphi \in C_0^\infty(\Omega) \right\}$

#definition #space of distributions #support of distributions #test functions space #vanishes

59. $T \in \mathcal{D}'(\Omega)$, T vanishes in $\widetilde{\Omega} \iff \widetilde{\Omega}$ is an open of Ω and $T(\phi) = 0$ for every $\phi \in \mathcal{D}(\Omega)$

60. Let Υ be the union of all open $\widetilde{\Omega} \subset \Omega$ in which T vanishes. The support of T is defined by
$$\text{supp}(T) = \Omega - \Upsilon$$

61. The $\text{supp}(T)$ is a closed set

#definition #test functions space

62. $\phi \in \mathcal{D}(\Omega) \Longleftrightarrow \phi \in C_0^\infty(\Omega)$ and $\mathrm{supp}(\phi)$ is a compact subset of Ω

63. $\mathcal{D}(\Omega)$ is not metrizable

64. $\mathcal{D}(\Omega) = \bigcup_{K \subset \Omega} \mathcal{D}_K(\Omega)$

#derivative de a distribution #notation

65. $D^\alpha T$ = the α-th distributional derivative of T

#Dirac measure #notation #open interval

66. δ = the Dirac measure

67. $I = (0, 1)$ is an open interval on $\mathbb{R}$

#distribution #derivative de a distribution #derivative de a locally integrable function #proposition

68. If $T \in \mathcal{D}'(\Omega) \implies D^\alpha T \in \mathcal{D}'(\Omega)$
69. If $f \in L^1_{\text{loc}}(\Omega)$ and $D^\alpha f$ also exists in the classical sense and is locally integrable $\implies D^\alpha f$ is also distribution
70. $T_{D^\alpha f}(\phi) = \int_\Omega \phi(x)(D^\alpha f)(x)dx$, for all $\phi \in L^1_{\text{loc}}(\Omega)$ and $T_{D^\alpha f} \in \mathcal{D}'(\Omega)$
71. If f has continuous partial derivatives of all orders up to N and $|\alpha| \leq N$ $\implies D^\alpha T_f = T_{D^\alpha f}$
72. Any distribution has derivatives of all orders.
73. $T \in \mathcal{D}'(\Omega)$ and for all $j, k \in \mathbb{N}$,

$$\frac{\partial^2 T}{\partial x_j \partial x_k} = \frac{\partial^2 T}{\partial x_k \partial x_j}$$

#distribution #linear functional #space of distributions #test functions space #theorem

74. If T is a linear functional on $\mathcal{D}(\Omega) \Longrightarrow$ the following conditions are equivalent

    (a) $T \in \mathcal{D}'(\Omega)$

    (b) To every compact $K \subset \Omega$, $\exists N > 0$ and $C < \infty$ such that

    $$|T(\phi)| \leq C\|\phi\|_N$$

    holds for every $\phi \in \mathcal{D}_K(\Omega)$

#distribution #locally integrable function #proposition #space of distributions

75. T is a distribution on $\Omega \iff T \in \mathcal{D}'(\Omega)$

76. $L^1_{\mathrm{loc}}(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$

#distribution #locally integrable functions #space of distributions #test functions space #proposition

77. If $f, g \in L^1_{\text{loc}}(\Omega)$ and $T_f(\phi) = T_g(\phi)$, $\forall \phi \in C_0^\infty(\Omega) \implies f = g$ almost everywhere on Ω

78. If $T \in \mathcal{D}'(\Omega)$ such that $T(\varphi) \geq 0$ for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0 \implies T$ is a positive measure

#distribution #multiplication of a continuously differentiable function and a distribution #proposition

79. If $f \in C^{\infty}(\Omega)$ and $T \in \mathcal{D}'(\Omega) \implies fT \in \mathcal{D}'(\Omega)$

#distribution #space of distributions #test functions space #theorem

80. $T \in \mathcal{D}'(\Omega) \Longleftrightarrow T : \mathcal{D}(\Omega) \to \mathbb{C}$ is a linear map and for every compact $K \subseteq \Omega$, $\exists C_K > 0, N_K \in \mathbb{Z}_+$ such that

$$|T(\phi)| \leq C_K \sum_{|\alpha| \leq N_K} \sup_{x \in K} |\partial^\alpha \phi(x)|,$$

$\forall \phi \in \mathcal{D}_K(\Omega)$

#distributions as derivatives #space of distributions #theorem

81. If $T \in \mathcal{D}'(\Omega)$ and K is a compact of $\Omega \implies \exists f \in C^0(\Omega)$ and a $\alpha-$multi-index such that

$$T(\phi) = (-1)^{|\alpha|} \int_{\Omega} f(x)(D^{\alpha}\phi)(x)dx,$$

for every $\phi \in \mathcal{D}_K(\Omega)$

#distributions as derivatives #space of distributions #support #theorem

82. Let K be a compact, $\widetilde{\Omega}$ and Ω are open in $\mathbb{R}^n$, and $K \subset \widetilde{\Omega} \subset \Omega$. If $T \in \mathcal{D}'(\Omega)$, $K = \text{supp}(T)$, and T has order $N \implies \exists f_\beta$ continuous functions in Ω (one for each multi-index β with $\beta_i \leq N+2$ for $i = 1, \cdots, n$) with supports in $\widetilde{\Omega}$, such that

$$T = \sum_\beta D^\beta f_\beta$$

83. If T is as (82) and $\phi \in \mathcal{D}(\Omega) \implies$

$$T(\phi) = \sum_\beta (-1)^{|\beta|} \int_\Omega f_\beta(x)(D^\beta \phi)(x)dx$$

#distributions as derivatives #space of distributions #support #theorem

84. $T \in \mathcal{D}'(\Omega)$ and $\exists g_\alpha$ continuous functions on Ω, one for each multi-index α, such that

    (a) each compact $K \subset \Omega$ intersects the supports of only finitely many g_α

    (b) $T = \sum_\alpha D^\alpha g_\alpha$

    If T has finite order $\implies$ the functions g_α can be chosen so that only finitely many are different from 0

#functions space #locally integrable function #measurable #proposition

85. $f \in L^1_{\text{loc}}(\Omega) \iff f : \Omega \to \mathbb{C}$ is measurable and $f|_K \in L_1(K), \forall\, K \subset \Omega$, K compact

86. $f : \Omega \to \mathbb{C}$ is measurable $\iff \text{Re}(f)$ and $\text{Im}(f)$ are measurable

87. $X \neq \varnothing$, $g : X \to \mathbb{R}$. Let g be Lebesgue measurable $\iff \{x \in X : g(x) > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

88. $\int_K |f|\, dx < +\infty, \forall K \subset \Omega,\ K$ compact $\iff \int_\Omega |f\varphi|\, dx < +\infty$, $\forall \varphi \in C_0^\infty(\Omega)$

89. $f\varphi \in L_1(\Omega) \iff \int_\Omega |f\varphi|\, dx < +\infty$

#functions space #locally integrable function #notation

90. $\mathrm{Re}(f)$ = the real part of f

91. $\mathrm{Im}(f)$ = the imaginary part of f

92. $L^1_{\mathrm{loc}}(\Omega)$ = the set of all locally integrable functions on Ω

93. $L_1(\Omega)$ = the function space defined using the 1-norm $\|f\| \equiv \left(\int_\Omega |f| dx\right) < +\infty$

#local base #locally convex #test functions space #theorem #topology

94. (a) τ is a topology on $C_0^\infty(\Omega)$ and β is a local base of τ

    (b) $(C_0^\infty(\Omega), \tau)$ is a locally convex topological vector space

#local base #locally convex #test functions space #theorem #topology

95. τ is a topology in $\mathcal{D}(\Omega)$ and β is a local base for τ

96. τ makes $\mathcal{D}(\Omega)$ into a locally convex topological vector space

#natural numbers #non-negatives integers #notation #real numbers

97. $\mathbb{N}$ = the set of natural numbers

98. $\mathbb{Z}_+$ = the set of non-negative integers numbers

99. $\mathbb{R}$ = the set of real numbers

#notation #sequence

100. $(\varphi_j)_{j\in\mathbb{N}}$ = a sequence

#notation #space of distributions

101. $\mathcal{D}'(\Omega)$ = the continuous dual space of $C_0^\infty(\Omega)$ with the topology of uniform convergence on bounded subsets of $C_0^\infty(\Omega)$(space of distributions on Ω) for all $j \in \mathbb{N}$

#notation #support

102. supp (f) = the support of f

#notation #test functions space

103. $\mathcal{D}(\Omega)$ = The test functions space

#notation #topological vector space

104. (X, τ) = topological vector space with topology τ

#support of a distribution #support of a test function #theorem

105. $T \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$

   (a) If $\operatorname{supp}(\phi) \bigcap \operatorname{supp}(T) = \varnothing \implies T(\phi) = 0$

   (b) If $\operatorname{supp}(T) = \varnothing \implies T = 0$

   (c) If $\psi \in C^\infty(\Omega)$ and $\psi = 1$ in some open set V containing $\operatorname{supp}(T) \implies \psi T = T$

   (d) If $\operatorname{supp}(T)$ is a compact subset of $\Omega \implies \exists \psi \in \mathcal{D}(\Omega)$ such that $\psi = 1$ in some open set containing $\operatorname{supp}(T)$

   (e) If $\operatorname{supp}(T)$ is a compact subset of $\Omega \implies T$ has finite order. Further, T extends in an unique way to a continuous linear functional on $C^\infty(\Omega)$

#support of distributions #theorem

106. If $T \in \mathcal{D}'(\Omega)$, $p \in \Omega$, $\operatorname{supp}(T) = \{p\}$, and T has order $N \implies \exists c_\alpha$ constants such that

$$T = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \delta_p,$$

where δ_p is the evaluation functional defined by $\delta_p(\phi) = \phi(p)$

107. Every distribution the form

$$T = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \delta_p,$$

has $\{p\}$ for its support (unless $c_\alpha = 0$ for all α)

#support of distributions #theorem #vanishes

108. If Υ is as (60) $\Longrightarrow T$ vanishes in Υ

Open Invitation

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Open Science

The **latex file** for this paper together with other *supplementary files* are available [6].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [7] following the structure from [5], please cite it accordingly [8]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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