[knowledge base]

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Modules over Rings

Open Mathematics Collaboration^{*†}

March 23, 2021

Abstract

MODULES over RINGS and its underlying definitions are presented in this white paper (knowledge base).

keywords: modules, rings, abstract algebra, knowledge base

The most updated version of this white paper is available at

<https://osf.io/xjsc7/download>

Open Mathematics Knowledge Base

<http://omkb.org>

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Definition

1. Module over R (left and right R -modules)
2. Left module over R (left R -module)

$$(M, R, +, \cdot)$$

- (a) $(M, +) :=$ commutative group
- (b) Closure under (left) scalar multiplication:
 $\forall k \in R, \forall v \in M, kv \in M$
- (c) Scalar (left) multiplication identity: $(1 \in R, v \in M) \rightarrow (1v = v)$
- (d) Associativity of (left) scalar multiplication:
 $\forall j, k \in R, v \in M, (jk)v = j(kv)$
- (e) (left) Distributivity of 1 scalar over 2 vectors:
 $\forall k \in R, v, w \in M, k(v + w) = kv + kw$
- (f) (left) Distributivity of 2 scalars over 1 vector:
 $\forall j, k \in R, v \in M, (j + k)v = jv + kv$

$M, R :=$ sets

$(R, +, \cdot) :=$ ring

$+, \cdot :=$ binary operations on M, R

$\cdot : R \times M \rightarrow M$

(scalar multiplication between elements of R and M)

[1, 2]

3. Right module over R (right R -module)

$$(M, R, +, \cdot)$$

- (a) $(M, +) :=$ commutative group
- (b) Closure under (right) scalar multiplication:
 $\forall k \in R, \forall v \in M, vk \in M$
- (c) Scalar (right) multiplication identity: $(1 \in R, v \in M) \rightarrow (v1 = v)$
- (d) Associativity of (right) scalar multiplication:
 $\forall j, k \in R, v \in M, v(jk) = (vj)k$
- (e) (right) Distributivity of 1 scalar over 2 vectors:
 $\forall k \in R, v, w \in M, (v + w)k = vk + wk$
- (f) (right) Distributivity of 2 scalars over 1 vector:
 $\forall j, k \in R, v \in M, v(j + k) = vj + vk$

$M, R :=$ sets

$(R, +, \cdot) :=$ ring

$+, \cdot :=$ binary operations on M, R

$\cdot : R \times M \rightarrow M$

(scalar multiplication between elements of R and M)

$[1, 2]$

Prerequisites

4. Group

$$(G, \star)$$

(a) Associativity: $\forall x, y, z \in G, (x \star y) \star z = x \star (y \star z)$

(b) Identity: $\exists e \in G : \forall x \in G, e \star x = x \star e = x$

(c) Inverse: $\forall x \in G \exists y \in G : x \star y = y \star x = e$

$G :=$ set

$\star :=$ binary operation

[1]

5. Commutative group (Abelian)

$$G_b$$

$$\forall g_1, g_2 \in G_b, g_1 g_2 = g_2 g_1$$

$G_b :=$ group

[2]

6. Ring

$$(R, +, \cdot)$$

(a) $(R, +) :=$ commutative group

(b) $(R, \cdot) :=$ monoid

(c) Multiplication is distributive over addition in R

$R :=$ set

$+, \cdot :=$ binary operations on R (addition and multiplication)

[1, 2]

7. Binary operation

$$\star : S \times S \rightarrow S$$

$S :=$ set

$S \times S :=$ Cartesian product

[1]

8. Distributive

$$\forall x, y, z \in R : x \cdot (y + z) = x \cdot y + x \cdot z \quad \text{left distributive}$$

$$\forall x, y, z \in R : (y + z) \cdot x = y \cdot x + z \cdot x \quad \text{right distributive}$$

[1, 2]

9. Semigroup

$$\mathcal{S} = (S, \circ)$$

$S :=$ non-empty set

$\circ :=$ associative binary operation

[4]

10. **Monoid** $:=$ a semigroup $\mathcal{S} = (S, \circ)$ that contains an identity $e \in S$ such that

$$\forall x \in S : e \circ x = x \circ e = x$$

[4]

11. Associative binary operation

$$\forall x, y, z \in S : x \circ (y \circ z) = (x \circ y) \circ z$$

$\circ :=$ binary operation

[4]

12. Cartesian product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$A, B :=$ sets

$A \times B :=$ Cartesian product

$(a, b) :=$ ordered pair

[3]

13. Ordered pair

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$a :=$ first coordinate

$b :=$ second coordinate

[1, 3]

Open Invitation

*Review, add content, and **co-author** this white *white paper* [5, 6].*

*Join the **Open Mathematics Collaboration**.*

Send your contribution to `mplobo@uft.edu.br`.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [7].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [8], please cite it accordingly [9]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

Acknowledgements

+ **Center for Open Science**

<https://cos.io>

+ **Open Science Framework**

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