Analysis of wave propagation in cylinders and rods made of functionally graded piezoelectric material

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Abstract

In this paper we study from both theoretical and numerical points of view the problem of design and optimization of functionally graded piezoelectric material (FGPM) transducers. The governing equations for FGPM are solved using numerical scheme based on the polynomial approach. The lowest frequencies for circular and rectangular rings with different radius-to-thickness ratios are calculated to study the guided wave characteristics. It is assumed that elastic stiffnesses, piezoelectric constants and dielectric permittivities of FGPM vary in an arbitrary way inside a structure. The fundamental mode starts propagating at the phase speed that is determined numerically. Finally, two-dimensional numerical simulations are performed for various parameters to control the accuracy of the solution.

Keywords: Piezoelectricity, functionally graded piezoelectric materials, numerical solutions, integral equations

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1 Introduction

Functionally graded piezoelectric material (FGPM) is the composite material designed to have specific properties needed for different applications. Advantages of this new kind of materials can improve the reliability and life span of the piezoelectric devices [1]. Wave propagation in FGPM structures has received a lot of attention in the past ten years. Previous studies were focused essentially on simple semi-infinite and one-dimensional structures, such as horizontally infinite plates and axially infinite hollow cylinders. However, there is a need to handle more complicated and realistic problems. The advantages of FGPM transducers versus traditional transducers are presented in [2, 3].

Many computational methods have been developed to solve the wave propagation in complex FGPM structures. In [4] an approach based on double orthogonal polynomial series for solving the wave propagation problem in a two-dimensional FGPM structure, which is FGPM ring with a rectangular cross-section, is proposed. Effects of various physical parameters on dispersion curves and cutoff frequencies in a Y-cut quartz plate with elastic or functionally graded top layer are studied in [5]. Viscoelastic shear horizontal (SH) wave propagation in FGM plates and laminated plates is investigated in [6].

The investigations of the last decade focus on wave propagation in piezoelectric transducers with various cross-sections which has been studied in numerous publications. For instance, we could refer the papers [7–12]. Multi-layered systems with layers made in different materials including both solid and fluid layers were addressed as well [13]. Recently, acoustic wave propagation in poroelastic hollow cylinder filled with and surrounded by the same fluid considering the seismo-electric effect was studied in [14]. The authors of [15] study the surface effect on wave propagation in an anisotropic elastic cylinder. Electro-magnetic wave propagation through the subwavelength aperture of a dielectric cylindrical metal-coated waveguide is studied in [16, 17]. Zhang et al. [20] studied magneto-electric effect on waves in functionally graded piezoelectric-piezomagnetic (FGPP) fan-shaped cylindrical structures by exploiting the double Legendre orthogonal polynomial method.

In this paper we study from both theoretical and numerical points of view the problem of design and optimization of solid and hollow cylinders and rods made of FGPM. Up to our best knowledge, few analytical studies about the waves in such waveguide structures were made. Majority of the literature considers different numerical methods such as the finite element method. These methods are usually very time-consuming for simulations of waves propagation in 3D structures. Therefore, a semi-analytic method that directly incorporates the boundary conditions into the governing dynamic equations of motion and numerical scheme based on the polynomial approach is developed.

2 Methodology

We consider a cylindrical structure in the cylindrical coordinate system (r, θ, z) . In the wave propagation problems considered in this study, the external forces and electric charges are considered to be zero. The dynamic equations of motion are

$$\sigma_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (1)$$

$$D_i = e_{ikl}S_{kl} + \epsilon_{ik}E_k, \quad E_i = -\phi_{,i} \quad (2)$$

Here σ_{ij} and D_i are stress tensor and electrical displacement vector and c_{ijkl} , e_{kij} , and ϵ_{ik} are elastic and piezoelectric coefficients, and dielectric permittivity, respectively. Index after comma denotes partial differentiation with respect to the index coordinate.

Mechanical stresses are related to displacements through the classical equations of motion in cylindrical coordinates:

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\partial \sigma_{r\theta}}{r\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - \rho \frac{\partial^2 u_r(r, \theta, t)}{\partial t^2} &= 0, \\ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} - \frac{\partial \sigma_{rr}}{\partial r} &= 0, \\ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} - \frac{\partial \sigma_{rr}}{\partial r} &= 0, \end{aligned} \quad (3)$$

The harmonic wave solutions for displacements are formulated as:

$$\begin{aligned} u(r, \theta, z, t) &= \exp(ikz - i\omega t)U(r, \theta), \\ v(r, \theta, z, t) &= \exp(ikz - i\omega t)V(r, \theta), \\ w(r, \theta, z, t) &= \exp(ikz - i\omega t)W(r, \theta) \end{aligned} \tag{4}$$

The piezoelectric effect is characterized by the coupling between mechanical and electrical properties of the material in equations (3). Characteristic equations are obtained from equations (1-3) in the analytical form using the variable separation technique. Solving these equations directly is a difficult task and they are solved numerically. To determine form of the boundary conditions, we consider the method of boundary integral equations for the case of constant surface potential [18]. Natural traction-free and open-circuit boundary conditions (i.e. mechanical stresses and electric displacement components set to zero) on all surfaces of cylinders and rods are considered in this study. An asymptotic numerical solution is obtained using the power function expansion technique [19].

3 Numerical results

Numerical comparison with the reference results for an elastic homogeneous rectangular rod is performed to check the validity of the given method. Upon this comparison perfect agreement was found between different sets of dispersion curves and other characteristics. The most common technique that allows to identify and understand quickly what propagating modes should be used is the calculation of cutoff frequencies. The ability to reliable and accurate calculation of cutoff frequencies has an important role in the analysis of waves, especially for hollow cylinders where circumferential resonances of longitudinal and flexural modes may occur.

In the numerical examples given below two types of piezoelectric materials are considered: PZT-4 and LiNbO₃. The corresponding material parameters are given in Table 1 [4]. We consider the piezoelectric transducers of the shape described in the preceding section and assume their surfaces are insulated.

Material	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
	Units: $10^{10} N/m^2$								
PZT-4	13.9	7.4	7.4	13.9	7.4	11.5	2.56	2.56	3.05
LiNbO3	20.5	11.4	7.5	19.6	8.1	17.9	5.3	5.5	6.3
Material	e_{15}	e_{24}	e_{31}	e_{32}	e_{33}	ϵ_{11}	ϵ_{22}	ϵ_{33}	ρ
	Units: C/m^2					Units: $10^{-11} F/m$			Units: kg/m^3
PZT-4	12.7	12.7	-5.2	-5.2	15.1	650	650	560	7500
LiNbO3	6.7	5.9	-1.0	-0.9	8.5	230	255	120	4650

Table 1: Physical parameters of the considered piezoelectric materials

To demonstrate the performance of our approach Figure 1 displays the lowest frequencies for different thickness to radius values for the thin FGPM cylinder composed of both PZT-4 and LiNbO3 materials (inner and outer surface, respectively).

Figure 1: The lowest frequency to thickness to radius ratios for the thin FGPM cylinder

Wave propagation in a FGPM solid bar with a rectangular section with the same parameters is illustrated in Figure 2. Normalized frequencies obtain the largest values at thickness to radius equal to 0.5 and decrease at smaller and larger values. With increasing the thickness further the frequency decreased.

Figure 2: The lowest frequency to thickness to radius ratios for the thin FGPM bar

The wave velocity is dependent on the frequency (i.e. dispersion). Table 2 shows the frequency values for different for real wavenumbers that correspond to the lowest 6 propagating modes in the FGPM cylinder. Geometry parameter of the waveguide is chosen as 0.4 in this case.

Wavenumber	1	2	3	4	5	6
2	1.23	2.75	4.56	4.98	9.34	11.2
5	2.98	4.27	7.51	8.23	12.29	14.62
10	5.11	8.23	11.03	11.94	16.83	17.69

Table 2: Frequency versus wavenumber for the lowest 6 propagating modes

4 Conclusions

FGPM transducers are widely used in the many engineering fields. We have studied the problem of design and optimization of these transducers. Our method is based on a numerical scheme using the polynomial approach. The lowest frequencies for circular and rectangular rings with different radius-to-thickness ratios have been calculated. Through numerical comparison with the available results for a homogeneous rectangular rod, the validity of the present approach is confirmed. In the future we will extend this work to compare our results to finite element calculations.

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