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A semigroup with a left identity and left inverse is a group

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Abstract

We translate the proof of the theorem stated in the title, accomplished by Prover9, into a human readable form.

keywords: prover9, proof assistant, semigroup, left identity, left inverse, group, abstract algebra

The most updated version of this white paper is available at

https://osf.io/wsbne/download

Introduction

1. The theorem (3) is a very strong result since it only requires *associativity*, *left identity*, and *left inverse* in order that a pair $(S, \cdot)$ has all properties of a **group**.

2. This white paper was written for a homework problem in the `Automated Reasoning` course during my PhD program at UAb [1].

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Theorem

3. If for all $x \in S$,

$$(\mathcal{S} = (S, \cdot) \coloneqq \text{semigroup}) \wedge (1 \cdot x = x) \wedge (x' \cdot x = 1)),$$

then $\mathcal{S}$ is a group.

Results from Prover9

Step 1

4. Input

```
assign(max_seconds, 30).

% group axioms
formulas(sos).
(x * y) * z = x * (y * z).
1 * x = x.
x' * x = 1.
end_of_list.

formulas(goals).
x * 1 = x.
end_of_list.
```

5. Output

```
1 x * 1 = x # label(non_clause) # label(goal).  [goal].
2 (x * y) * z = x * (y * z).  [assumption].
3 1 * x = x.  [assumption].
4 x' * x = 1.  [assumption].
5 c1 * 1 != c1.  [deny(1)].
```

```
6 x' * (x * y) = y.
[para(4(a,1),2(a,1,1)),rewrite([3(2)]),flip(a)].
9 x'' * 1 = x.  [para(4(a,1),6(a,1,2))].
10 x'' * y = x * y.  [para(6(a,1),6(a,1,2))].
11 x * 1 = x.  [back_rewrite(9),rewrite([10(4)])].
12 $F.  [resolve(11,a,5,a)].
```

Step 2

6. Input

```
assign(max_seconds, 30).

% group axioms
formulas(sos).
(x * y) * z = x * (y * z).
1 * x = x.
x' * x = 1.
x * 1 = x.
end_of_list.

formulas(goals).
x * x' = 1.
end_of_list.
```

7. Output

```
1 x * x' = 1 # label(non_clause) # label(goal).  [goal].
2 (x * y) * z = x * (y * z).  [assumption].
3 1 * x = x.  [assumption].
4 x' * x = 1.  [assumption].
5 x * 1 = x.  [assumption].
6 c1 * c1' != 1.  [deny(1)].
```

```
7 x' * (x * y) = y.
[para(4(a,1),2(a,1,1)),rewrite([3(2)]),flip(a)].
10 x'' = x.  [para(4(a,1),7(a,1,2)),rewrite([5(4)])].
11 x * x' = 1.  [para(10(a,1),4(a,1,1))].
12 $F.  [resolve(11,a,6,a)].
```

Results translated into human form

8. In this section, I present literally the translation of Prover9's proof for theorem (3) in a way we humans understand more clearly.

Step 1

9. In item (5), line 5, `deny(1)` means assuming $x_1 \cdot 1 \neq x_1$.

10. In item (5), line 6, `para(4(a,1),2(a,1,1))` means substituting $x'x$ into (xy) in the associativity property, $(xy)z = x(yz)$, and then using $x'x = 1$.

11. $(x'x)z = x'(xz)$

12. $1 \cdot z = x'(xz)$

13. In item (5), line 6, `rewrite([3(2)])` means rewriting (12) using $1 \cdot x = x$.

14. $z = x'(xz)$

15. In item (5), line 6, `flip(a)` means writing (14) as $x'(xz) = z$.

16. Replacing z for y, the result is

$$x'(xy) = y.$$

17. In item (5), line 9, `para(4(a,1),6(a,1,2))` means substituting $x'x$ into (xy) in (16), and then using $x'x = 1$.

18. $x''(x'x) = x$

19. $x'' \cdot 1 = x$

20. In item (5), line 10, `para(6(a,1),6(a,1,2))` means substituting $x'(xy)$ into (xy) in $x'(xy) = y$, and then using $x'(xy) = y$.

21. $x''(x'(xy)) = xy$

22. $x''y = xy$

23. In item (5), line 11, `back_rewrite(9)` and `rewrite([10(4)])` mean the following.

24. From item (5), line 9, we have $x'' \cdot 1 = x$.

25. From item (5), line 10, we have $x''y = xy$.

26. In (25), let $y = 1$, then $x'' \cdot 1 = x \cdot 1$.

27. Using (24) in (26) and flipping sides, $x \cdot 1 = x$.

28. In item (5), line 12, `resolve(11,a,6,a)` means that (9) and (27) lead to a contradiction.

29. Therefore, $x \cdot 1 = x$. $\square$

Step 2

30. Item (7), line 6, results in

$$x_1 x_1' \neq 1.$$

31. Item (7), line 7, leads to the following:

    (a) Inserting $x'x$ into (xy) in $(xy)z = x(yz)$, and using $x'x = 1$,

$$(x'x)z = x'(xz),$$

$$1 \cdot z = x'(xz);$$

(b) Rewriting it using $1 \cdot x = x$ and converting z to y,

$$y = x'(xy);$$

(c) Finally, flipping it,

$$x'(xy) = y.$$

32. Item (7), line 10, leads to

(a) Converting (xy) into $x'x$ in $x'(xy) = y$, and using $x'x = 1$,

$$x''(x'x) = x,$$

$$x'' \cdot 1 = x;$$

(b) Rewriting it using $x \cdot 1 = x$,

$$x'' = x.$$

33. Item (7), line 11, results in

(a) Converting x' into x'' (and x into x') in $x'x = 1$, and using $x'' = x$,

$$x''x' = 1,$$

$$xx' = 1.$$

34. Finally, item (7), line 12, concludes that (33.a) contradicts (30).

35. Therefore, $xx' = 1$. $\square$

How would a human prove (3)?

36. The purpose of this section is to prove theorem (3) supposing a human had done it in its own right.

37. <u>Givens</u>

(a) $\mathcal{S} = (S, \cdot) :=$ semigroup

(b) $\forall x \in S : 1 \cdot x = x$

(c) $\forall x \in S\ \exists x' \in S : x' \cdot x = 1$

38. We need to prove that $\mathcal{S} = (S, \cdot)$ is a group.

39. First, we show that $\mathcal{S}$ has *right identity*, namely, $\forall x \in S : x \cdot 1 = x$.

40. Second, we show that $\mathcal{S}$ has *right inverse*, namely, $\forall x \in S\ \exists x' \in S : x \cdot x' = 1$.

41. Both (39) and (40) are proved by **contradiction**.

42. Note that we use `juxtaposition`, i.e., $x \cdot y = xy$.

First

43. Let $x, y \in S$.

44. Suppose $x_1 1 \neq x_1$.

45. Recall that a **semigroup** is *associative*, then $(x'x)y = x'(xy)$.

46. Hereafter, due to (45), we omit the parenthesis.

47. Using (37.c) in (45), $1y = x'xy$.

48. Using (37.b) in (47), $x'xy = y$.

49. Let's calculate $x''x'x$.

50. Using (48) in (49), $x''x'x = x$.

51. Since $x'x = 1$, then $x''1 = x$.

52. Multiplying (51) by y on the right, $x''1y = xy$.

53. $1y = y$, then $x''y = xy$.

54. For $y = 1$, $x''1 = x1$.

55. Using (51), $x1 = x$.

56. Therefore, since x is arbitrary, $\mathcal{S}$ has a **right identity**.

Second

57. Let $x, y \in S$.

58. Suppose $x_1 x_1' \neq 1$.

59. Due to (37.c) and (37.b), $x'xy = y$.

60. Let's calculate $x''x'x$.

61. Using (59) in (60), $x''x'x = x$.

62. Since $x'x = 1$, $x''1 = x$.

63. Since $x''1 = x''$,
$$x'' = x.$$

64. Multiplying (63) by x' on the right, $x''x' = xx'$.

65. Applying $x'x = 1$ in $x''x'$,
$$x''x' = 1.$$

66. (63) and (65) lead to
$$xx' = 1.$$

67. Therefore, since x is arbitrary, $\mathcal{S}$ has **right inverse**.

68. In conclusion, $\mathcal{S} = (S, \cdot)$ is a group. $\square$

Final Remarks

69. We presented and translated Prover9's proof of theorem (3) into a human readable style, and then proved the theorem in a natural human form.

Open Invitation

Review, *add* content, and **co-author** this *white paper* [2, 3].
Join the **Open Mathematics Collaboration**.
Send your contribution to `mplobo@uft.edu.br`.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [4].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [5], please cite it accordingly [6]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

Acknowledgements

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Agreement

70. All authors **agree** with [3].

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