Taking quartic roots of 2

THINH D. NGUYEN*

Moscow State University
kosmofarmer@gmail.com

July 22, 2018

Abstract

Taking square roots of -1 modulo composite numbers was shown to be hard in our previous work [5]. This short note only discusses the same phenomenon of taking quartic roots of 2 modulo prime numbers. Diligent readers of Prasolov and I. F. Sharygin can realize how effective each problem, each chapter helps shape our mathematical reasoning.

I. BACKGROUND, NOTATION AND CLAIMS

While working with modular polynomials, nothing can be more interesting and also striking than raising the degree of them. Previously, we discussed the equation $x^2 \equiv -1 \pmod n$ where n can be any natural number. Now, we want to deal with quartic equation of the form $x^4 \equiv 2 \pmod p$ where p is a prime number. The long story of this equation and variants has begun since 1749. Around that time, Euler conjectured the necessary and sufficient condition of p for the existence of a quartic root of 2 modulo p . His book *Tractatus de numerorum doctrina capita sedecim quae supersunt* [1] was published in 1849, one hundred years after he wrote it. In this book, he conjectured that 2 is a fourth power residue (mod p) iff. p is of the form $x^2 + 64y^2$. For example, 2 is a fourth power residue (mod 73), since $73 = 3^2 + 64$.

Gauss proved this conjecture in 1828 in his first monograph [2] on biquadratic reciprocity. Historically, Euler's conjecture was widespread among mathematicians long before officially printed. Together with Fermat's $4n + 1$ theorem, we can see that such a prime p can be uniquely represented as $a^2 + b^2$, up to the swapping of a and b . Parity of a and b must be different as well. So, Gauss's

*Perebor

condition can be stated as that the even one among a and b must be divisible by 8. H. Davenport wrote an excellent book [3] on arithmetics, which contains, in much details, various topics.

Modern mathematics has since developed so much fruitful ideas to solve polynomials both for real roots and modular roots. Zhi-Hong Sun [4] extends the condition for other values of natural number $m \neq 2$ to be quartic residue mod p .

Gauss's condition can be seen as a polynomial-time reduction between computational problems. Some definitions are necessary.

i. Definitions

QUARTIC-2-MODULUS = $\{p \mid p \text{ is congruent to } 1 \text{ modulo } 4 \text{ and } 2 \text{ is a quartic residue mod } p\}$

FEASIBLE-QUAD-DIOPHANTINE = $\{ \langle n, d \rangle \mid \text{the Diophantine equation } x^2 + dy^2 = n \text{ is feasible} \}$

PROPERTY-OF-DIOPHANTINE-SOLUTION = $\{n \mid \text{the Diophantine equation } x^2 + y^2 = n \text{ is satisfied by a solution } (a, b) \text{ where } a \text{ or } b \text{ is divisible by } 8\}$

So, now the following claims are immediate from Gauss's condition.

ii. Claims

Claim 1: QUARTIC-2-MODULUS $\leq_p$ FEASIBLE-QUAD-DIOPHANTINE

Claim 2: QUARTIC-2-MODULUS $\leq_p$ PROPERTY-OF-DIOPHANTINE-SOLUTION

If you have the ability to solve Diophantine equation, you have enough capabilities to deal with much of modular arithmetics. Even an algorithm to account for the existence of a solution to Diophantine equation is of vastly achievements.

REFERENCES

- [1] Euler, Leonhard (1849), *Tractatus de numerorum doctrina capita sedecim quae supersunt*, Comment. Arithmet. 2
- [2] Gauss, Carl Friedrich (1828), *Theoria residuorum biquadraticorum, Commentatio prima*, Gttingen: Comment. Soc. regiae sci, Gttingen 6
- [3] H. Davenport, *The Higher Arithmetic: An Introduction to the Theory of Numbers*, Cambridge University Press

- [4] Zhi-Hong Sun, *Quartic residues and binary quadratic forms*, Journal of Number theory 113 pp.10-52, Elsevier, 2005
- [5] Thinh D. Nguyen, *Taking square roots of -1*, <https://osf.io/5fmhd/>, 2018