MONOTONICITY AND COMPLETE MONOTONICITY OF TWO FUNCTIONS CONSTITUTED VIA THREE DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION

FENG QI

Dedicated to people facing and battling COVID-19

ABSTRACT. In the paper, by convolution theorem of the Laplace transforms, a monotonicity rule for the ratio of two Laplace transforms, Bernstein's theorem for completely monotonic functions, and other analytic techniques, the author

- (1) presents the decreasing monotonicity of a ratio constituted via three derivatives of a function involving trigamma function;
- (2) discovers necessary and sufficient conditions for a function constituted via three derivatives of a function involving trigamma function to be completely monotonic.

These results conform previous guesses posed by the author.

CONTENTS

1. Motivations	1
2. Lemmas	3
3. Decreasing monotonicity	5
4. Necessary and sufficient conditions of complete monotonicity	7
5. Remarks	7
References	9

1. MOTIVATIONS

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

and its logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ are respectively called Euler's gamma function and digamma function. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for $k \in \{0\} \cup \mathbb{N}$ are known as polygamma functions, where $\mathbb{N}$ denotes the set of all positive integers.

Recall from Chapter XIII in [3], Chapter 1 in [12], and Chapter IV in [13] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies

2010 *Mathematics Subject Classification.* Primary 33B15; Secondary 26A48, 26A51, 26D07, 33B10, 44A10, 44A35.

Key words and phrases. monotonicity; complete monotonicity; trigamma function; derivative; difference; ratio; Laplace transform; convolution theorem; monotonicity rule; Bernstein's theorem; exponential function; inequality; guess.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

$(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$, then we call $f(x)$ a completely monotonic function on I .

There are a number of papers and mathematicians dedicated to investigation of complete monotonicity of some functions involving the gamma and polygamma functions. For more information and details, please refer to the papers [2, 9, 11, 15] and closely related references therein.

Let

$$G(x) = x[x\psi'(x) - 1] - \frac{1}{2} = x^2 \left[\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} \right], \quad x \in (0, \infty).$$

It is easy to obtain that

$$G'(x) = x^2 \psi''(x) + 2x\psi'(x) - 1$$

and

$$G^{(\ell)}(x) = x^2 \psi^{(\ell+1)}(x) + 2\ell x \psi^{(\ell)}(x) + \ell(\ell-1) \psi^{(\ell-1)}(x)$$

for $\ell \geq 2$. For $k \in \{0\} \cup \mathbb{N}$ and $\theta_k, \tau_k \in \mathbb{R}$, let

$$\mathcal{G}_{k, \theta_k}(x) = G^{(2k+1)}(x) + \theta_k [G^{(k)}(x)]^2 \quad (1.1)$$

and

$$\mathfrak{G}_{k, \tau_k}(x) = \frac{G^{(2k+1)}(x)}{[(-1)^k G^{(k)}(x)]^{\tau_k}} \quad (1.2)$$

on $(0, \infty)$. In [7, Theorem 3.1 and Theorem 4.1], the author discovered the following necessary and sufficient conditions, limits, and double inequality:

- (1) if and only if $\theta_k \geq \frac{3(2k+2)!}{k!(k+1)!}$, the function $\mathcal{G}_{k, \theta_k}(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\theta_k \leq 0$, the function $-\mathcal{G}_{k, \theta_k}(x)$ is completely monotonic on $(0, \infty)$;
- (3) if and only if $\tau_k \geq 2$, the function $\mathfrak{G}_{k, \tau_k}(x)$ is decreasing on $(0, \infty)$;
- (4) if $\tau_k \leq 1$, the function $\mathfrak{G}_{k, \tau_k}(x)$ is increasing on $(0, \infty)$;
- (5) only if

$$\tau_k \leq \begin{cases} \psi'(1), & k = 0 \\ -\frac{\psi'''(1)}{\psi'(1)\psi''(1)}, & k = 1 \\ \frac{k-1}{k} \frac{\psi^{(k-1)}(1)\psi^{(2k+1)}(1)}{\psi^{(k)}(1)\psi^{(2k)}(1)}, & k \geq 2, \end{cases}$$

the function $\mathfrak{G}_{k, \tau_k}(x)$ is increasing on $(0, \infty)$;

- (6) the limits

$$\lim_{x \rightarrow 0^+} \mathfrak{G}_{k, \tau_k}(x) = \begin{cases} -2^{\tau_0}, & k = 0 \\ 6\psi''(1), & k = 1 \\ \frac{2(2k+1)}{(k-1)^{\tau_k} k^{\tau_k-1}} \frac{\psi^{(2k)}(1)}{|\psi^{(k-1)}(1)|}, & k \geq 2 \end{cases}$$

and

$$\lim_{x \rightarrow \infty} \mathfrak{G}_{k, \tau_k}(x) = \begin{cases} -\infty, & \tau_k > 2 \\ -\frac{3(2k+2)!}{k!(k+1)!}, & \tau_k = 2 \\ 0, & \tau_k < 2 \end{cases}$$

are valid;

(7) the double inequality

$$-\frac{3(2k+2)!}{k!(k+1)!} < \mathfrak{G}_{k,2}(x) < \begin{cases} -4, & k=0 \\ 6\psi''(1), & k=1 \\ \frac{2(2k+1)}{(k-1)^2k} \frac{\psi^{(2k)}(1)}{|\psi^{(k-1)}(1)|}, & k \geq 2 \end{cases}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.

In [7, Remark 3.1], basing on the proof of necessary conditions in [7, Theorem 3.1], the author guessed that the function

$$\mathcal{G}_k(x) = \frac{G^{(2k+2)}(x)}{G^{(k)}(x)G^{(k+1)}(x)}, \quad k \in \{0\} \cup \mathbb{N}$$

should be decreasing on $(0, \infty)$ and that the double inequality

$$-\frac{6(2k+2)!}{k!(k+1)!} < \mathcal{G}_k(x) < \begin{cases} -4\psi'(1), & k=0 \\ -\frac{6\psi'''(1)}{\psi'(1)}, & k=1 \\ \frac{2(2k+1)}{k^2(k-1)} \frac{\psi^{(2k+1)}(1)}{\psi^{(k-1)}(1)\psi^{(k)}(1)}, & k \geq 2 \end{cases} \quad (1.3)$$

should be valid and sharp in the sense that the lower and upper bounds cannot be replaced by any bigger and small numbers.

For $m, n \in \{0\} \cup \mathbb{N}$, let

$$\mathcal{G}_{m,n}(x) = \frac{G^{(m+n+1)}(x)}{G^{(m)}(x)G^{(n)}(x)} \quad (1.4)$$

and

$$\mathcal{G}_{m,n;\lambda_{m,n}}(x) = G^{(m+n+1)}(x) + \lambda_{m,n}G^{(m)}(x)G^{(n)}(x) \quad (1.5)$$

on $(0, \infty)$. It is clear that

$$\mathcal{G}_{m,n}(x) = \mathcal{G}_{n,m}(x), \quad \mathcal{G}_{m,n;\lambda_{m,n}}(x) = \mathcal{G}_{n,m;\lambda_{n,m}}(x), \\ \mathcal{G}_{k,k+1}(x) = \mathcal{G}_k(x), \quad \mathcal{G}_{k,k}(x) = \mathfrak{G}_{k,\tau_k}(x), \quad \mathcal{G}_{k,k;\lambda_{k,k}}(x) = \mathcal{G}_{k,\theta_k}(x)$$

with $\tau_k = 2$ and $\lambda_{k,k} = \theta_k$. This means that the functions $\mathcal{G}_{m,n}(x)$ and $\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ are generalizations of the ones $\mathcal{G}_{k,\theta_k}(x)$ and $\mathfrak{G}_{k,\tau_k}(x)$ defined in (1.1) and (1.2).

In this paper, we mainly

- (1) prove that the function $\mathcal{G}_{m,n}(x)$ for $m, n \in \{0\} \cup \mathbb{N}$ is decreasing on $(0, \infty)$;
- (2) discover necessary and sufficient conditions on $\lambda_{m,n}$ such that the functions $\pm(-1)^{m+n+1}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ are completely monotonic on $(0, \infty)$.

The decreasing monotonicity of the function $\mathcal{G}_{m,n}(x)$ for $m, n \in \{0\} \cup \mathbb{N}$ on $(0, \infty)$ confirms that the above guesses posed by the author in [7, Remark 3.1] are true.

2. LEMMAS

The following lemmas are necessary in this paper.

Lemma 2.1 (Convolution theorem for the Laplace transforms [13, pp. 91–92]). *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \geq 0$ such that $|f_k(t)| \leq M_k e^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u)f_2(t-u)du \right] e^{-st}dt = \int_0^\infty f_1(u)e^{-su}du \int_0^\infty f_2(v)e^{-sv}dv.$$

Lemma 2.2 ([14, Lemma 4]). *Let the functions $A(t)$ and $B(t) \neq 0$ be defined on $(0, \infty)$ such that their Laplace transforms $\int_0^\infty A(t)e^{-xt}dt$ and $\int_0^\infty B(t)e^{-xt}dt$ exist. If the ratio $\frac{A(t)}{B(t)}$ is increasing, then the ratio $\frac{\int_0^\infty A(t)e^{-xt}dt}{\int_0^\infty B(t)e^{-xt}dt}$ is decreasing on $(0, \infty)$.*

Lemma 2.3. *Let*

$$w(t) = \begin{cases} \frac{e^t[(t-2)e^t + t + 2]}{(e^t - 1)^3}, & t \neq 0; \\ \frac{1}{6}, & t = 0. \end{cases}$$

Then the following conclusions are valid:

- (1) *the function $w(t)$ is infinitely differentiable, positive, and even on $(-\infty, \infty)$, is increasing on $(-\infty, 0)$, and is decreasing on $(0, \infty)$;*
- (2) *the function $w(t)$ is logarithmically concave on $(-\infty, \infty)$;*
- (3) *for fixed $s \in (0, 1)$, the ratio $\frac{w^s(t)}{w(st)}$ is even in $t \in (-\infty, \infty)$ and decreasing in t from $(0, \infty)$ onto $(0, 6^{1-s})$.*

Proof. The infinite differentiability, positivity, even property, monotonicity, and logarithmic concavity of the function $w(t)$ on $(-\infty, \infty)$ have been proved in [7, Lemma 2.1].

Since the function $w(t)$ is even on $(-\infty, \infty)$, it is immediate that the ratio $\frac{w^s(t)}{w(st)}$ is even on $(-\infty, \infty)$.

It is straightforward that

$$\lim_{t \rightarrow 0} \frac{w^s(t)}{w(st)} = \frac{\lim_{t \rightarrow 0} w^s(t)}{\lim_{t \rightarrow 0} w(st)} = \frac{(1/6)^s}{1/6} = 6^{1-s}$$

and

$$\lim_{t \rightarrow \infty} \frac{w^s(t)}{w(st)} = \frac{\lim_{t \rightarrow \infty} [e^t w(t)/t]^s}{\lim_{t \rightarrow \infty} [e^{st} w(st)/st]} \lim_{t \rightarrow \infty} \frac{e^{st}/st}{(e^t/t)^s} = 0,$$

where we used the limit $\lim_{t \rightarrow \infty} \frac{e^t w(t)}{t} = 1$.

The first derivative of the ratio $\frac{w^s(t)}{w(st)}$ is

$$\frac{d}{dt} \left[\frac{w^s(t)}{w(st)} \right] = \frac{sw^s(t)}{w(st)} \left[\frac{w'(t)}{w(t)} - \frac{w'(st)}{w(st)} \right].$$

Accordingly, for obtaining monotonicity of the ratio $\frac{w^s(t)}{w(st)}$, it is sufficient to show that the ratio $\frac{w'(t)}{w(t)}$ is decreasing on $(0, \infty)$. For this, it is sufficient to show that the function $w(t)$ is logarithmically concave on $(0, \infty)$. The proof of Lemma 2.3 is complete. $\square$

Lemma 2.4 ([7, Lemma 2.2]). *For $k \in \{0\} \cup \mathbb{N}$, the function $(-1)^k G^{(k)}(x)$ is completely monotonic on $(0, \infty)$, with the limits*

$$\lim_{x \rightarrow 0^+} [(-1)^k G^{(k)}(x)] = \begin{cases} \frac{1}{2}, & k = 0 \\ 1, & k = 1 \\ (-1)^k k(k-1)\psi^{(k-1)}(1), & k \geq 2 \end{cases} \quad (2.1)$$

and

$$\lim_{x \rightarrow \infty} [(-1)^k x^{k+1} G^{(k)}(x)] = \frac{k!}{6}. \quad (2.2)$$

Lemma 2.5 (Bernstein's theorem [13, p. 161, Theorem 12b]). *A function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\sigma(t), \quad x \in (0, \infty), \quad (2.3)$$

where $\sigma(s)$ is non-decreasing and the integral in (2.3) converges for $x \in (0, \infty)$.

3. DECREASING MONOTONICITY

In this section, we prove that the function $\mathcal{G}_{m,n}(x)$ defined in (1.4) is decreasing.

Theorem 3.1. *For $m, n \in \{0\} \cup \mathbb{N}$, the function $\mathcal{G}_{m,n}(x)$ in (1.4) is decreasing in $x \in (0, \infty)$ and maps from $(0, \infty)$,*

- (1) *if $(m, n) = (0, 0)$, onto the interval $(-6, -4)$;*
- (2) *if $(m, n) \in \{(1, 0), (0, 1)\}$, onto the interval $(-12, -4\psi'(1))$;*
- (3) *if $(m, n) \in \{(2, 0), (0, 2)\}$, onto the interval $(-18, \frac{6\psi''(1)}{\psi'(1)})$;*
- (4) *if $(m, n) = (1, 1)$, onto the interval $(-36, 6\psi''(1))$;*
- (5) *if $(m, n) \in \{(2, 1), (1, 2)\}$, onto the interval $(-72, -\frac{6\psi'''(1)}{\psi'(1)})$;*
- (6) *if $m, n \geq 2$, onto the interval*

$$\left(-\frac{6(m+n+1)!}{m!n!}, \frac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)} \right).$$

Consequently, for $m, n \in \{0\} \cup \mathbb{N}$, the double inequality

$$-\frac{6(m+n+1)!}{m!n!} < \mathcal{G}_{m,n}(x) < \begin{cases} -4, & (m, n) = (0, 0) \\ -4\psi'(1), & (m, n) \in \{(1, 0), (0, 1)\} \\ \frac{6\psi''(1)}{\psi'(1)}, & (m, n) \in \{(2, 0), (0, 2)\} \\ 6\psi''(1), & (m, n) = (1, 1) \\ -\frac{6\psi'''(1)}{\psi'(1)}, & (m, n) \in \{(2, 1), (1, 2)\} \\ \frac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}, & m, n \geq 2 \end{cases} \quad (3.1)$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

Proof. Since $\mathcal{G}_{m,n}(x) = \mathcal{G}_{n,m}(x)$, for convenience, without loss of generality, we assume $m \geq n \geq 0$ in this proof.

In the proof of [8, Theorem 4], the author derived an integral representation

$$G(x) = \int_0^\infty w(t)e^{-xt} dt. \quad (3.2)$$

Then the function $\mathcal{G}_{m,n}(x)$ can be reformulated as

$$\begin{aligned} \mathcal{G}_{m,n}(x) &= -\frac{\int_0^\infty t^{m+n+1} w(t) e^{-xt} dt}{\int_0^\infty t^m w(t) e^{-xt} dt \int_0^\infty t^n w(t) e^{-xt} dt} \\ &= -\frac{\int_0^\infty t^{m+n+1} w(t) e^{-xt} dt}{\int_0^\infty \left[\int_0^t u^m (t-u)^n w(u) w(t-u) du \right] e^{-xt} dt}, \end{aligned}$$

where we used Lemma 2.1. Further, by Lemma 2.2, it is sufficient to prove that the ratio

$$\begin{aligned} W_{m,n}(t) &= \frac{t^{m+n+1}w(t)}{\int_0^t u^m(t-u)^n w(u)w(t-u)du} \\ &= \frac{w(t)}{\int_0^1 s^m(1-s)^n w(st)w((1-s)t)ds} \\ &= \frac{1}{\int_0^1 s^m(1-s)^n \frac{w(st)}{w^s(t)} \frac{w((1-s)t)}{w^{1-s}(t)} ds} \end{aligned}$$

is decreasing on $(0, \infty)$, where we changed the variable $u = st$. For this, it is sufficient to verify that the ratio $\frac{w^s(t)}{w(st)}$ is decreasing in $t \in (0, \infty)$ for fixed $s \in (0, 1)$. This requirement has been verified in Lemma 2.3. As a result, the function $\mathcal{G}_{m,n}(x)$ is decreasing on $(0, \infty)$.

Making use of the limits (2.1) and (2.2) in Lemma 2.4 yields

$$\begin{aligned} \lim_{x \rightarrow 0^+} \mathcal{G}_{m,n}(x) &= \frac{\lim_{x \rightarrow 0^+} G^{(m+n+1)}(x)}{\lim_{x \rightarrow 0^+} G^{(m)}(x) \lim_{x \rightarrow 0^+} G^{(n)}(x)} \\ &= \begin{cases} \frac{\lim_{x \rightarrow 0^+} G'(x)}{\lim_{x \rightarrow 0^+} G(x) \lim_{x \rightarrow 0^+} G(x)}, & (m, n) = (0, 0) \\ \frac{\lim_{x \rightarrow 0^+} G''(x)}{\lim_{x \rightarrow 0^+} G'(x) \lim_{x \rightarrow 0^+} G(x)}, & (m, n) = (1, 0) \\ \frac{\lim_{x \rightarrow 0^+} G'''(x)}{\lim_{x \rightarrow 0^+} G''(x) \lim_{x \rightarrow 0^+} G(x)}, & (m, n) = (2, 0) \\ \frac{\lim_{x \rightarrow 0^+} G'''(x)}{\lim_{x \rightarrow 0^+} G'(x) \lim_{x \rightarrow 0^+} G'(x)}, & (m, n) = (1, 1) \\ \frac{\lim_{x \rightarrow 0^+} G^{(4)}(x)}{\lim_{x \rightarrow 0^+} G''(x) \lim_{x \rightarrow 0^+} G'(x)}, & (m, n) = (2, 1) \\ \frac{\lim_{x \rightarrow 0^+} G^{(m+n+1)}(x)}{\lim_{x \rightarrow 0^+} G^{(m)}(x) \lim_{x \rightarrow 0^+} G^{(n)}(x)}, & m \geq n \geq 2 \end{cases} \\ &= \begin{cases} -4, & (m, n) = (0, 0) \\ -4\psi'(1), & (m, n) = (1, 0) \\ \frac{6\psi''(1)}{\psi'(1)}, & (m, n) = (2, 0) \\ 6\psi''(1), & (m, n) = (1, 1) \\ -\frac{6\psi'''(1)}{\psi'(1)}, & (m, n) = (2, 1) \\ \frac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}, & m \geq n \geq 2 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow \infty} \mathcal{G}_{m,n}(x) &= -\frac{\lim_{x \rightarrow \infty} [(-1)^{m+n+1} x^{m+n+2} G^{(m+n+1)}(x)]}{\lim_{x \rightarrow \infty} [(-1)^m x^{m+1} G^{(m)}(x)] \lim_{x \rightarrow \infty} [(-1)^n x^{n+1} G^{(n)}(x)]} \\ &= -\frac{6(m+n+1)!}{m!n!}. \end{aligned}$$

Therefore, those ‘‘onto’’ properties and the double inequality (3.1) and its sharpness follow readily. The proof of Theorem 3.1 is complete. $\square$

4. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE MONOTONICITY

In this section, we discover necessary and sufficient conditions on $\lambda_{m,n}$ such that the functions $\pm(-1)^{m+n+1}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ defined in (1.5) are completely monotonic.

Theorem 4.1. *For $m, n \in \{0\} \cup \mathbb{N}$,*

- (1) *if and only if $\lambda_{m,n} \leq 0$, the function $(-1)^{m+n+1}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0, \infty)$;*
- (2) *if and only if $\lambda_{m,n} \geq \frac{6(m+n+1)!}{m!n!}$, the function $(-1)^{m+n}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0, \infty)$.*

Proof. By the integral representation (3.2), we can write $\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ in (1.5) as

$$\begin{aligned} \mathcal{G}_{m,n;\lambda_{m,n}}(x) &= (-1)^{m+n+1} \left[\int_0^\infty t^{m+n+1} w(t) e^{-xt} dt \right. \\ &\quad \left. - \lambda_{m,n} \int_0^\infty t^m w(t) e^{-xt} dt \int_0^\infty t^n w(t) e^{-xt} dt \right]. \end{aligned}$$

Utilizing Lemma 2.1 results in

$$\begin{aligned} &(-1)^{m+n+1} \mathcal{G}_{m,n;\lambda_{m,n}}(x) \\ &= \int_0^\infty t^{m+n+1} w(t) e^{-xt} dt - \lambda_{m,n} \int_0^\infty \left[\int_0^t u^m (t-u)^n w(u) w(t-u) du \right] e^{-xt} dt \\ &= \int_0^\infty \left[t^{m+n+1} w(t) - \lambda_{m,n} \int_0^t u^m (t-u)^n w(u) w(t-u) du \right] e^{-xt} dt \\ &= \int_0^\infty \left[1 - \lambda_{m,n} \frac{\int_0^t u^m (t-u)^n w(u) w(t-u) du}{t^{m+n+1} w(t)} \right] t^{m+n+1} w(t) e^{-xt} dt \\ &= \int_0^\infty \left[1 - \frac{\lambda_{m,n}}{W_{m,n}(t)} \right] t^{m+n+1} w(t) e^{-xt} dt. \end{aligned}$$

In the proof of Theorem 3.1 in last section, the function $W_{m,n}(t)$ has been proved to be decreasing on $(0, \infty)$. Direct computation arrives at

$$\begin{aligned} \lim_{t \rightarrow 0} W_{m,n}(t) &= \frac{\lim_{t \rightarrow 0} w(t)}{\int_0^1 s^m (1-s)^n \lim_{t \rightarrow 0} [w(st)w((1-s)t)] ds} \\ &= \frac{6}{\int_0^1 s^m (1-s)^n ds} = \frac{6}{B(m+1, n+1)} = \frac{6(m+n+1)!}{m!n!} \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} W_{m,n}(t) &= \frac{\lim_{t \rightarrow \infty} \frac{e^t w(t)}{t}}{\int_0^1 s^m (1-s)^n \lim_{t \rightarrow \infty} \frac{e^{st} w(st)}{st} \lim_{t \rightarrow \infty} \frac{e^{(1-s)t} w((1-s)t)}{(1-s)t} ds} \lim_{t \rightarrow \infty} \frac{1}{s(1-s)t} \\ &= 0, \end{aligned}$$

where we used the limit $\lim_{t \rightarrow \infty} \frac{e^t w(t)}{t} = 1$. Accordingly, by virtue of Lemma 2.5, if and only if $\lambda_{m,n} \leq 0$, the function $(-1)^{m+n+1}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0, \infty)$; if and only if $\lambda_{m,n} \geq \frac{6(m+n+1)!}{m!n!}$, the function $(-1)^{m+n}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0, \infty)$. The proof of Theorem 4.1 is complete. $\square$

5. REMARKS

Finally, we list several remarks on our main results and their proofs.

Remark 5.1. The third conclusion in Lemma 2.3 in this paper generalizes a result in [7, Lemma 2.1]: the ratio $\frac{w(2t)}{w^2(t)}$ is even on $(-\infty, \infty)$, increasing from $(-\infty, 0)$ onto $(0, 6)$, and decreasing from $(0, \infty)$ onto $(0, 6)$.

Remark 5.2. The double inequality (3.1) is a generalization of the double inequality (1.3) which was guessed in [7, Remark 3.1] by the author.

Remark 5.3. For $m, n \in \{0\} \cup \mathbb{N}$, direct differentiation gives

$$\mathcal{G}'_{m,n}(x) = \frac{G^{(m+n+2)}(x)[G^{(m)}(x)G^{(n)}(x)] - G^{(m+n+1)}(x)[G^{(m)}(x)G^{(n)}(x)]'}{[G^{(m)}(x)G^{(n)}(x)]^2}$$

on $(0, \infty)$. The decreasing monotonicity of $\mathcal{G}_{m,n}(x)$ in Theorem 3.1 implies that, for $m, n \in \{0\} \cup \mathbb{N}$, the inequality

$$G^{(m+n+1)}(x)[G^{(m)}(x)G^{(n)}(x)]' - G^{(m+n+2)}(x)[G^{(m)}(x)G^{(n)}(x)] > 0,$$

equivalently,

$$\frac{[G^{(m)}(x)G^{(n)}(x)]'}{G^{(m)}(x)G^{(n)}(x)} < \frac{G^{(m+n+2)}(x)}{G^{(m+n+1)}(x)},$$

is valid on $(0, \infty)$.

We guess that, for $m, n \in \{0\} \cup \mathbb{N}$, the function

$$G^{(m+n+1)}(x)[G^{(m)}(x)G^{(n)}(x)]' - G^{(m+n+2)}(x)[G^{(m)}(x)G^{(n)}(x)]$$

is completely monotonic in $x \in (0, \infty)$.

One can also consider necessary and sufficient conditions on $\Upsilon_{m,n} \in \mathbb{R}$ for $m, n \in \{0\} \cup \mathbb{N}$ such that the function

$$G^{(m+n+1)}(x)[G^{(m)}(x)G^{(n)}(x)]' - \Upsilon_{m,n}G^{(m+n+2)}(x)[G^{(m)}(x)G^{(n)}(x)]$$

and its negativity are respectively completely monotonic on $(0, \infty)$.

Remark 5.4. Direct computation gives

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \frac{G^{(m+n+2)}(x)}{G^{(m+1)}(x)G^{(n)}(x) + G^{(m)}(x)G^{(n+1)}(x)} \\ &= - \lim_{x \rightarrow 0^+} \frac{(-1)^{m+n+2}G^{(m+n+2)}(x)}{\left(\begin{aligned} & [(-1)^{m+1}G^{(m+1)}(x)][(-1)^nG^{(n)}(x)] \\ & + [(-1)^mG^{(m)}(x)][(-1)^{n+1}G^{(n+1)}(x)] \end{aligned} \right)} \\ &= - \begin{cases} \lim_{x \rightarrow 0^+} \frac{(-1)^2G''(x)}{[-G'(x)]G(x) + G(x)[(-1)G'(x)]}, & (m, n) = (0, 0) \\ \lim_{x \rightarrow 0^+} \frac{(-1)^3G'''(x)}{[(-1)^2G''(x)]G(x) + [-G'(x)][-G'(x)]}, & (m, n) \in \{(1, 0), (0, 1)\} \\ \lim_{x \rightarrow 0^+} \frac{(-1)^4G^{(4)}(x)}{[(-1)^3G'''(x)]G(x) + G''(x)[-G'(x)]}, & (m, n) \in \{(2, 0), (0, 2)\} \\ \lim_{x \rightarrow 0^+} \frac{(-1)^4G^{(4)}(x)}{[(-1)^2G''(x)][-G'(x)] + [-G'(x)][(-1)^2G''(x)]}, & (m, n) = (1, 1) \\ \lim_{x \rightarrow 0^+} \frac{(-1)^5G^{(5)}(x)}{\left(\begin{aligned} & [(-1)^3G'''(x)][-G'(x)] \\ & + [(-1)^2G''(x)][(-1)^2G''(x)] \end{aligned} \right)}, & (m, n) \in \{(2, 1), (1, 2)\} \\ \lim_{x \rightarrow 0^+} \frac{(-1)^{m+n+2}G^{(m+n+2)}(x)}{\left(\begin{aligned} & [(-1)^{m+1}G^{(m+1)}(x)][(-1)^nG^{(n)}(x)] \\ & + [(-1)^mG^{(m)}(x)][(-1)^{n+1}G^{(n+1)}(x)] \end{aligned} \right)}, & m, n \geq 2 \end{cases} \end{aligned}$$

$$= \begin{cases} -2\psi'(1), & (m, n) = (0, 0) \\ \frac{6\psi''(1)}{1 + \psi'(1)}, & (m, n) \in \{(1, 0), (0, 1)\} \\ \frac{12\psi'''(1)}{3\psi''(1) - 2\psi'(1)}, & (m, n) \in \{(2, 0), (0, 2)\} \\ -\frac{3\psi'''(1)}{\psi'(1)}, & (m, n) = (1, 1) \\ \frac{10\psi^{(4)}(1)}{2[\psi'(1)]^2 - 3\psi''(1)}, & (m, n) \in \{(2, 1), (1, 2)\} \\ \frac{(m+n+2)(m+n+1)\psi^{(m+n+1)}(1)}{\binom{mn[(m+1)(n-1)\psi^{(m)}(1)\psi^{(n-1)}(1)]}{+(m-1)(n+1)\psi^{(m-1)}(1)\psi^{(n)}(1)}}, & m, n \geq 2 \end{cases}$$

and

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{G^{(m+n+2)}(x)}{G^{(m+1)}(x)G^{(n)}(x) + G^{(m)}(x)G^{(n+1)}(x)} \\ &= - \lim_{x \rightarrow \infty} \frac{(-1)^{m+n+2}x^{m+n+3}G^{(m+n+2)}(x)}{\left(\begin{aligned} & [(-1)^{m+1}x^{m+2}G^{(m+1)}(x)][(-1)^n x^{n+1}G^{(n)}(x)] \\ & + [(-1)^m x^{m+1}G^{(m)}(x)][(-1)^{n+1}x^{n+2}G^{(n+1)}(x)] \end{aligned} \right)} \\ &= - \frac{(m+n+2)!/6}{[(m+1)!/6](n!/6) + (m!/6)[(n+1)!/6]} = - \frac{6(m+n+1)!}{m!n!}, \end{aligned}$$

where we used the limits (2.1) and (2.2) in Lemma 2.4. It is easy to verify that

$$\frac{6(m+n+1)!}{m!n!} \geq \begin{cases} -2\psi'(1), & (m, n) = (0, 0); \\ \frac{6\psi''(1)}{1 + \psi'(1)}, & (m, n) \in \{(1, 0), (0, 1)\}; \\ \frac{12\psi'''(1)}{3\psi''(1) - 2\psi'(1)}, & (m, n) \in \{(2, 0), (0, 2)\}; \\ -\frac{3\psi'''(1)}{\psi'(1)}, & (m, n) = (1, 1); \\ \frac{10\psi^{(4)}(1)}{2[\psi'(1)]^2 - 3\psi''(1)}, & (m, n) \in \{(2, 1), (1, 2)\}. \end{cases}$$

Motivated by these inequalities and other hints, we guess that, for $m, n \in \{0\} \cup \mathbb{N}$, the function

$$\frac{G^{(m+n+2)}(x)}{G^{(m+1)}(x)G^{(n)}(x) + G^{(m)}(x)G^{(n+1)}(x)}$$

should be decreasing on $(0, \infty)$.

Remark 5.5. This paper is the seventh one in a series of articles including [4, 5, 6, 7, 8, 10].

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
- [2] C. Berg, E. Massa, and A. P. Peron, *A family of entire functions connecting the Bessel function J_1 and the Lambert W function*, Constr. Approx. **52** (2020), in press; available online at <https://doi.org/10.1007/s00365-020-09499-x>.

- [3] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at <https://doi.org/10.1007/978-94-017-1043-5>.
- [4] F. Qi, *Complete monotonicity and monotonicity of two functions defined by two derivatives of a function involving trigamma function*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02993723>.
- [5] F. Qi, *Lower bound of sectional curvature of manifold of beta distributions and complete monotonicity of functions involving polygamma functions*, MDPI Preprints **2020**, 2020110315, 22 pages; available online at <https://doi.org/10.20944/preprints202011.0315.v1>.
- [6] F. Qi, *Monotonicity of a ratio involving trigamma and tetragamma functions*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/5rfb8>.
- [7] F. Qi, *Necessary and sufficient conditions for two functions defined by two derivatives of a function involving trigamma function to be completely monotonic or monotonic*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/6ar4p>.
- [8] F. Qi, *Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold*, São Paulo J. Math. Sci. **15** (2021), in press; available online at <https://doi.org/10.1007/s40863-020-00193-1>.
- [9] F. Qi and B.-N. Guo, *From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions*, J. Math. Anal. Appl. **493** (2021), no. 1, Article 124478, 19 pages; available online at <https://doi.org/10.1016/j.jmaa.2020.124478>.
- [10] F. Qi, L.-X. Han, and H.-P. Yin, *Monotonicity and complete monotonicity of two functions defined by three derivatives of a function involving trigamma function*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02998203>.
- [11] F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, *Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions*, Appl. Anal. Discrete Math. **14** (2020), no. 2, 512–527; available online at <https://doi.org/10.2298/AADM191111033Q>.
- [12] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <https://doi.org/10.1515/9783110269338>.
- [13] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
- [14] Z.-H. Yang and J.-F. Tian, *Monotonicity and inequalities for the gamma function*, J. Inequal. Appl. **2017**, Paper No. 317, 15 pages; available online at <https://doi.org/10.1186/s13660-017-1591-9>.
- [15] Z.-H. Yang, J.-F. Tian, and M.-H. Ha, *A new asymptotic expansion of a ratio of two gamma functions and complete monotonicity for its remainder*, Proc. Amer. Math. Soc. **148** (2020), no. 5, 2163–2178; available online at <https://doi.org/10.1090/proc/14917>.

Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: <https://qifeng618.wordpress.com>, <https://orcid.org/0000-0001-6239-2968>