Comparing the Rightmost Group Seat Number for K Groups of People Under Random Seating Assignment

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Abstract

Suppose there are N seats labeled 1 to N from left to right. There are K groups of people that need to be seated. Denote the rightmost seating number of each group by $M_1, M_2, \dots, M_K$. This article studies the probability that a group k has the smallest value among $\{M_l, l = 1, \dots, K\}$ and its relationship with the group sizes $L_1, L_2, \dots, L_K$. In particular, it is shown that when the groups are non-overlapping, the group with the fewest number of people tends to have the smallest rightmost value of M_k . This result can be generalized to the case of overlapping groups under some assumptions. However, the result does not generalize to all cases and we demonstrate this with counter examples.

1 Problem Definition

Suppose there are N seats labeled 1 to N from left to right. There are K groups of people that need to be seated. To ensure everybody has a seat, the total number of people has to be less than N . Assuming the K groups are non-overlapping, we denote the rightmost seating number of each group by $M_1, M_2, \dots, M_K$. This article studies the probability that a group k has the smallest value among $\{M_l, l = 1, \dots, K\}$ and its relationship with the group sizes $L_1, L_2, \dots, L_K$. Later in the article, we extend the analysis to the case of overlapping groups.

1.1 Illustrative Example

Figure 1 gives an illustration of the problem of interest. In this case we have altogether 9 people from 3 groups to be seated in 9 seats: group 1 (black) consists of 2 individuals, group 2 (green) group consists of 3 individuals and group 3 (blue) consists of 4 individuals. Figure 1 gives an instance of a random

seating arrangement. M_1, M_2, M_3 denote the index of the rightmost seat of the groups 1, 2 and 3 respectively. As observed in this instance, there is a correlation between the group size and the values of M_k (small group sizes tend to have smaller value of M_k) which is intuitive under random seating arrangement.

Figure 1: Illustration for Three (Colored) Groups of People with 9 Chairs. M_1, M_2, M_3 are the respective rightmost seats for each group, where M_1 is the leftmost of these three seats.

2 Non-Overlapping Groups

Definition 2.1. Let $S_k, k = 1 \leq k \leq K$ be K non-overlapping groups each with L_k members respectively to be seated. For each group $k, k = 1, \dots, K$, let

$M_k \doteq$ seat index that are furthest to the right taken by members in group k .

2.1 Comparing M_i

Lemma 1. Let S_1, S_2 be two groups of people with L_1 and L_2 members respectively. If $S_1 \cap S_2 = \emptyset$, then the rightmost seating numbers M_1 and M_2 under random seating assignment have the property that

$$P(M_1 < M_2) = \frac{L_2}{L_1 + L_2}.$$

Proof. Suppose the seats are indexed by $1, 2, \dots, L_1 + L_2$. It is clear that each person in the two groups has an equal probability taking the rightmost seat which is the seat indexed by $L_1 + L_2$. Since everybody takes a different seat, these events are exclusive and hence the probabilities are additive. Therefore, event that $M_1 < M_2$ is equivalent to the event that a member in S_2 takes the seat indexed by $L_1 + L_2$, which is $L_2/(L_1 + L_2)$ since S_2 have L_2 members. $\square$

Lemma 2. Let $S_i, i = 1, \dots, K$, be K non-overlapping groups of people with L_i members respectively. Let $L = \sum_{k=1}^K L_k$ be the total number of people in K groups. Then for $K \geq 2$

$$P(M_k = \max(M_1, M_2, \dots, M_K)) = \frac{L_k}{L}. \quad (1)$$

$$P(M_1 < M_2 < \dots < M_K) = \prod_{k=2}^K \frac{L_k}{N_k}, \text{ where } N_k = \sum_{j=1}^k L_j. \quad (2)$$

Proof. Equation (1) can be easily shown using the same argument as in the proof of Lemma 1. In this case, everybody has the same probability taking the rightmost seat of all available seats under random seating. Since group k has L_k members, the probability that one of its member taking the rightmost seat is L_k/L .

We show Equation (2) by mathematical induction. From Lemma 1, $P(M_1 < M_2) = L_2/(L_1 + L_2)$, so Equation (2) is true when $K = 2$. Now suppose this is true for $n = K$, we show that this is true for $n = K + 1$.

It is necessary to show that the ordering of $M_1, M_2, \dots, M_K$ is independent of the seating arrangement of group S_{K+1} . This is because random seating can be done sequentially by seating members in group S_{K+1} first and then seating the members in the other groups. Therefore after L_{K+1} seats are taken by members in S_{K+1} , when other groups are being seated using the remaining seats, the probability that $M_1 < M_2 < \dots < M_K$ remains constant regardless where members in S_{K+1} are seated. Therefore, the event that $M_1 < M_2 \dots M_K < M_{K+1}$ is identical to $M_1 < M_2 < \dots < M_K$ and $M_{K+1} = \max_{k=1}^K M_k$, which are two independent events. Therefore,

$$\begin{aligned} & P(M_1 < M_2 < \dots < M_K < M_{K+1}) \\ &= P(M_1 < M_2 < \dots < M_K) P\left(M_{K+1} = \max_{1 \leq k \leq K} M_k\right) \\ &= \prod_{k=1}^K \frac{L_k}{N_k} \cdot \frac{L_{K+1}}{N_{K+1}}, \end{aligned}$$

where the last step in the above is due to Equation (2) for $n = K$ by mathematical induction and Equation (1). This validates Equation (1) for $n = K + 1$. $\square$

In the Illustrative Example 1.1, under random seating assignment,

$$P(M_1 < M_2 < M_3) = \frac{L_1}{L_1 + L_2} \cdot \frac{L_3}{L_1 + L_2 + L_3} = \frac{4}{6} \cdot \frac{3}{9} = \frac{2}{9}.$$

2.2 Group Index with the Smallest M_i

For K non-overlapping groups $S_1, S_2, \dots, S_K$, let Y be the group index such that M_Y is the smallest of all $M_i, i = 1, \dots, K$. We refer Y as the winning group index. From Lemma 1 we can easily obtain the following result.

Corollary 1. *Let S_1, S_2 be two non-overlapping groups with L_1 and L_2 members respectively. Under random seating assignment,*

$$P(Y = 1) = P(M_1 < M_2) = \frac{L_2}{L_1 + L_2},$$

Furthermore if $L_1 < L_2$, then $P(Y = 1) > P(Y = 2)$ and hence $P(Y = 1) > 1/2$ which implies that Y favors the group with less members.

The following corollary extends the result for three non-overlapping groups S_1, S_2, S_3 .

Lemma 3. *Let S_1, S_2, S_3 be three non-overlapping groups with L_1, L_2 and L_3 members respectively, and let $L = L_1 + L_2 + L_3$ be the total number of members in all three groups. Under random seating assignment,*

$$P(Y = 1) = P(M_1 < M_2, M_1 < M_3) = \frac{L_2 L_3}{L} \left(\frac{1}{L_1 + L_2} + \frac{1}{L_1 + L_3} \right). \quad (3)$$

Furthermore,

$$P(Y = 1) > \frac{L_2 L_3}{(L_1 + L_2)(L_1 + L_3)} = \frac{L_2 L_3}{(L - L_2)(L - L_3)} \quad (4)$$

If $L_1 \leq L_2 \leq L_3$, then

$$P(Y = 1) \geq P(Y = 2) \geq P(Y = 3), \quad (5)$$

where the equality holds under the case of equal group size. In other words, Equation (5) implies the random variable Y favors the group with least members.

Proof. Notice $Y = 1$ implies that M_1 is the smallest among M_1, M_2, M_3 . Since the groups are non-overlapping, therefore we only need to consider the following two cases: $M_1 < M_2 < M_3$ and $M_1 < M_3 < M_2$. Applying Lemma 2 to the three groups S_1, S_2 and S_3 , we have

$$P(M_1 < M_2 < M_3) = \frac{L_2 L_3}{(L_1 + L_2)L}, \quad P(M_1 < M_3 < M_2) = \frac{L_2 L_3}{(L_1 + L_3)L}.$$

Therefore Equation (3) is true by summing up the above two probabilities.

We now show the bound in Equation (4). From Equation (3), we have

$$P(Y = 1) = \frac{L_2 L_3 (2L_1 + L_2 + L_3)}{(L_1 + L_2)(L_1 + L_3)(L_1 + L_2 + L_3)} > \frac{L_2 L_3}{(L_1 + L_2)(L_1 + L_3)}.$$

Finally, we show $P(Y = 1)$ is the largest probabilities among $P(Y = j), j = 1, 2, 3$ if $L_1 \leq L_2 \leq L_3$. For this, we only need to show $P(Y = 1) \geq P(Y = 2)$ if $L_1 \leq L_2$ since likewise $P(Y = 1) \geq P(Y = 3)$ if $L_1 \leq L_3$. This can be seen by directly comparing $P(Y = 1)$ and $P(Y = 2)$ using Equation (3). From Equation (3),

$$\begin{aligned} P(Y = 1) &= \frac{L_3}{L} \left(\frac{L_2}{L_1 + L_2} + \frac{L_2}{L_1 + L_3} \right), \\ P(Y = 2) &= \frac{L_3}{L} \left(\frac{L_1}{L_2 + L_1} + \frac{L_1}{L_2 + L_3} \right). \end{aligned}$$

Comparing these two probabilities, it is easy to see that if $L_1 \geq L_2$,

$$\frac{L_2}{L_1 + L_2} \geq \frac{L_1}{L_1 + L_2} \quad \text{and} \quad \frac{L_2}{L_1 + L_3} \geq \frac{L_1}{L_2 + L_3}.$$

Therefore $P(Y = 1) \geq P(Y = 2)$ and the equality holds only when $L_1 = L_2$. $\square$

Using similar proof in the corollary above, we can further extend the results on Y to an arbitrary number of K non-overlapping group. We shall use recursion to accomplish the task.

Theorem 1. *Let $S_k, k = 1, \dots, K$, be K non-overlapping groups, each with L_i members respectively, to be seated under random seating assignment. Let $L = \sum_{k=1}^K L_k$ be the total number of people in the K groups and*

(i) *For any $k - 1$ groups $S_{i_1}, S_{i_2}, \dots, S_{i_{k-1}}$, define*

$$p_k(S_1; \{S_{i_1}, \dots, S_{i_{k-1}}\}) \doteq P(M_1 < M_{i_j}, j = 1, \dots, k-1), \quad (6)$$

that is, the probability that M_1 is the smallest rightmost seating number for the k groups $S_1, S_{i_1}, \dots, S_{i_{k-1}}$. Then

$$p(Y = 1) \doteq p_K(S_1; \{S_2, \dots, S_K\}) = \sum_{k=2}^K \frac{L_k}{L} p_{K-1}(S_1; \{S_2, \dots, S_K\} \setminus \{S_k\}), \quad (7)$$

where this recursion can be terminated at $K = 2$ with $p_2(S_i; S_j) = \frac{L_j}{L_i + L_j}$.

(ii) *If $L_i \leq L_j$, then $P(Y = i) \geq P(Y = j)$, where the strictly inequality holds when $L_i < L_j$. This implies that Y favors the group with the smaller size.*

(iii) *For $K \geq 3$,*

$$p(Y = 1) > (K - 2)! \prod_{k=2}^K \frac{L_k}{L - L_k}. \quad (8)$$

Proof. (i) Notice $Y = 1$ implies M_1 is the smallest of all $M_k, k = 1, \dots, K$. We can split the case of $Y = 1$ into the following $K - 1$ cases where in each case, $M_k, j = 2, \dots, K$ is the largest of $M_1, M_2, \dots, M_K$. Since members in the K groups are non-overlapping, these cases are exclusive and thus the probabilities are additive. From Lemma 2 Equation (1), the probability of each case, that is, M_k is the largest among all $M_1, M_2, \dots, M_K$, is L_k/L . Notice that random seating can be done sequentially by first seating members in group S_k randomly, and then seat members in the remaining groups $S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_K$ randomly. Therefore given M_k is the largest of all $M_1, \dots, M_K$, $Y = 1$ is identical to the event $M_1 < M_j, j = 2, \dots, K, j \neq k$, which is independent of the value of M_k . This implies

$$p(Y = 1) = \sum_{k=2}^K P(M_k \text{ is the largest}) P(M_1 < M_j, j = 2, \dots, K, j \neq k),$$

which leads to Equation (7) using the definition of p_k in Equation (6).

(ii) Without loss of generality we assume $L_1 \leq L_2$ and show the inequality for $i = 1$ and $j = 2$. First, when $L_1 = L_2$, due to the symmetry in the definition Y regarding to group S_1 and S_2 , under random seating, it is easy to see $P(Y = 1) = P(M_1 < M_2, M_1 < M_3, \dots, M_1 < M_K)$ and $P(Y = 2) = P(M_2 < M_1, M_2 < M_3, \dots, M_2 < M_K)$ are identical. Now we show the strict inequality when $L_1 < L_2$. Let us split S_2 into two subgroups: subgroup S'_2 with size L_1 and subgroup S''_2 with size $L_2 - L_1$. Define the rightmost seat number for members in S'_2 and S''_2 by M'_2 and M''_2 accordingly. Since the size of subgroup S'_2 is identical to group S_1 ,

$$P(M_1 < M'_2, M_1 < M_3, \dots, M_1 < M_K) = P(M'_2 < M_1, M'_2 < M_3, \dots, M'_2 < M_K). \quad (9)$$

Furthermore, since $S'_2 \subset S_2$, $M'_2 \leq M_2$ and therefore $M_1 < M'_2$ implies $M_1 < M_2$. Hence

$$\begin{aligned} P(Y = 1) &= P(M_1 < M_2, M_1 < M_3, \dots, M_1 < M_K) \\ &\geq P(M_1 < M'_2, M_1 < M_3, \dots, M_1 < M_K). \end{aligned} \quad (10)$$

Similarly, $M_2 < M_j, j \neq 2$ implies $M'_2 < M_j$, and hence

$$\begin{aligned} P(M'_2 < M_1, M'_2 < M_3, \dots, M'_2 < M_K) &\geq P(M_2 < M_1, M_2 < M_3, \dots, M_2 < M_K) \\ &= P(Y = 2). \end{aligned} \quad (11)$$

Combine all the equality and inequalities above (9, 10 and 11), we have $P(Y = 1) \geq P(Y = 2)$. This is a strict inequality when $L_1 < L_2$ since M_2 is not always equal to M'_2 when the set S''_2 is non-empty.

(iii). We shall use induction to show the Inequality (8). From Lemma 3 Equation (4), we have shown that the inequality is true for $K = 3$. Now assume that the inequality is true for $n = K$ and we show this is true also for $n = K + 1$. From Equation (7), for $n = K + 1$, using the inequality assumption (Inequality 8) when $n = K$,

$$\begin{aligned} P(Y = 1) &> (K - 2)! \sum_{k=2}^{K+1} \frac{L_k}{L} \prod_{j=2, j \neq k}^{K+1} \frac{L_j}{L - L_k - L_j} \\ &= (K - 2)! \sum_{k=2}^{K+1} \frac{L_k}{L} \prod_{j=2, j \neq k}^{K+1} \frac{L_j}{L - L_j} \\ &> (K - 2)! \sum_{k=2}^{K+1} \frac{L - L_k}{L} \prod_{j=2}^{K+1} \frac{L_j}{L - L_j} \\ &= (K - 2)! \prod_{j=2}^{K+1} \frac{L_j}{L - L_j} \cdot \sum_{k=2}^{K+1} \frac{L - L_k}{L} \\ &= (K - 2)! \prod_{j=2}^{K+1} \frac{L_j}{L - L_j} \left(K - \frac{1}{L} \sum_{k=2}^{K+1} L_k \right) > (K - 1)! \prod_{j=2}^{K+1} \frac{L_j}{L - L_j}, \end{aligned}$$

which demonstrates the inequality for $n = K + 1$. $\square$

In the Illustrative Example 1.1, under random seating assignment, the lower bound from Equation 8 for $P(Y = 1)$ is $3/9 \cdot 4/9 = 4/27$. We know from the previous section that the actual probability is $2/9$, which is greater than $4/27$.

3 Extensions to Overlapping Groups

In this section, we extend our study to the case of possibly K overlapping groups, that is, there exist members that belong to two or more groups. In the non-overlapping case, for each group, we find the the rightmost seating number for members in each of the K groups and compare these numbers for the K groups. However, in this case we need to modify comparisons due to possible ties from seats taken by members with non-exclusive membership.

Let us consider the case of two groups S_1 and S_2 with an intersection $S = S_1 \cap S_2 \neq \emptyset$. When a member in S takes the rightmost seat, $M_1 = M_2$ which produces a tie. To resolve the tie, we only consider the seats for those individuals that have exclusive membership, that is, individuals not in S . In the following, we formally define a strict comparison between the seating numbers for members of two groups S_1 and S_2 , and define a winning group if the group wins the comparison.

Definition 3.1. Let S_1, S_2 be two groups of people with L_1 and L_2 members respectively. Let the S be the intersection of $S_1 \cap S_2$ with size l , and $S'_1 = S_1 \setminus S, S'_2 = S_2 \setminus S$ be subsets of S_1 and S_2 respectively after removing the intersection. Denote the rightmost seating number for members in S'_1 and S'_2 by $\widetilde{M}_1$ and $\widetilde{M}_2$ respectively. Then we define a strict comparison operator, $\prec$, between seats taken by members in S_1 and S_2 as follows:

$$S_1 \prec S_2 \doteq \widetilde{M}_1 < \widetilde{M}_2. \quad (12)$$

Furthermore, define the winning group label Y by $Y = 1$ if $S_1 \prec S_2$ and $Y = 2$ if $S_2 \prec S_1$.

Lemma 4. For two groups S_1, S_2 with sizes L_1, L_2 members and an intersection of size l , under random seating assignment,

$$P(Y = 1) = P(S_1 \prec S_2) = \frac{L_1 - l}{L_1 + L_2 - 2l}$$

Furthermore when $L_1 < L_2$,

$$P(Y = 1) > \frac{1}{2}.$$

Proof. Let $S'_1 = S_1$ and $S'_2 = S_2 \setminus S$. It is clear S'_1, S'_2 are non-overlapping with sizes $L_1 - l, L_2 - l$ respectively. Applying Lemma 1 on S'_1 and S'_2 gives the desired result. This in fact generalizes Lemma 1 to the case of two possible non-overlapping groups. $\square$

In the following we generalize the comparison operator $\prec$ to an arbitrary K groups and define a winning group label Y .

Definition 3.2. Let $S_i, i = 1, \dots, K$, be K possibly overlapping groups of people with L_i members respectively. For any subset of k groups, $S_{i_1}, S_{i_2}, \dots, S_{i_k}$, $k = 2, \dots, K$, define

$$S_{i_1} \prec \{S_{i_2}, \dots, S_{i_k}\} \text{ if and only if } S_{i_1} \prec S_{i_j}, j = 2, \dots, k.$$

Define the winning label Y as follows:

$$Y = k \text{ if and only if } S_k \prec \{S_1, S_2, \dots, S_{k-1}, S_{k+1}, \dots, S_K\}.$$

Lemma 5. *If none of the K groups $S_1, \dots, S_K$ are identical, then for every seating arrangement, there exists a winning group with label Y .*

Proof. For each $k, k = 1, \dots, K$, let M_k be the rightmost seating number for members in group k . It is obvious that in the case of non-overlapping group, $M_1, \dots, M_K$ are all distinct, and the winning group is the group with the smallest M_k among all $M_1, \dots, M_K$.

When the groups are over-lapping, we may have ties in $M_k, k = 1, \dots, K$. Let the smallest of these seating numbers be $M^{(1)}$ and without loss of generality, assume that the individual taking seat $M^{(1)}$ belong to groups $S_1, S_2, \dots, S_{k_1}$. If $k_1 = 1$, then we have already found the winning group. If not we continue the following process.

Let $W^{(0)} = \{S_1, S_2, \dots, S_K\}$ and $W^{(1)} = \{S_1, S_2, \dots, S_{k_1}\}$. Then for any $S_i \in W^{(0)}$ and $S_j \in W^{(1)}$, we have $S_i \prec S_j$. Therefore, if the winning group exist, it must be an element of $W^{(1)}$. Let the seats taken by members of groups in $W^{(1)}$ be $\mathcal{L}^{(1)}$ with the largest seat number be $M^{(1)}$. Eliminating the individual taking seat $M^{(1)}$ from $\mathcal{L}^{(1)}$, for each of the k_1 group in $W^{(1)}$, we find out the rightmost seating numbers of remaining members in the group. Let the smallest of these k_1 numbers be $M^{(2)}$. It is easy to see that $M^{(2)} < M^{(1)}$, and the winning group, if it exists, must have the (new) rightmost seating number $M^{(2)}$. Let $W^{(2)}$ be the subset of $W^{(1)}$ whose (new) rightmost seating number is $M^{(2)}$. If $W^{(2)}$ consists only one group, we stop here. Otherwise, we can eliminate the individual taking seat $M^{(2)}$ and repeat this process of elimination and finding new rightmost seating numbers for the groups in $W^{(2)}$. In this way, we created a sequence of groups that are getting smaller until we only have one group left $W^{(1)} \supset W^{(2)} \supset \dots \supset W^{(J)}$. Since none of the two groups $S_i, i = 1, \dots, K$ are identical, the final set of group $W^{(J)}$ with one element exists. In fact, this final group in $W^{(J)}$ must be the winning group by construction since it wins all the comparisons with other groups. □

Now we study the distribution of winning group label Y in the general case of K possibly overlapping groups. In particular, we are interested in generalizing Lemma 4 to the case of K groups. Unfortunately, for the most general case, it is not true that the winning group label Y always favor the the group with the fewest number of people.

Theorem 2. Let $S_i, i = 1, \dots, K$ be K possibly overlapping groups each with L_i members respectively. Assume $L_1 < L_2$, then the following is not always true,

$$p(S_1 \prec \{S_2, S_3, \dots, S_K\}) \not\geq P(S_2 \prec \{S_1, S_3, \dots, S_K\}).$$

or equivalently

$$P(Y = 1) \not\geq P(Y = 2). \quad (13)$$

This implies that, for K subsets $S_1, \dots, S_K$ have sizes $L_1 \leq \dots \leq L_K$, the winning group label Y does not always favor the smallest group.

Proof. We construct a counter-example to demonstrate this. Suppose that we have altogether 8 individuals labeled by 1 to 8. Define three overlapping groups as follows: $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5, 6\}$, $S_3 = \{2, 3, 7\}$. It is clear that S_1 has fewer members than S_2 . We compute $P(Y = 1) = P(S_1 \prec S_2, S_1 \prec S_3)$ in the following manner. First, it is easy to see that the event $S_1 \prec S_2, S_1 \prec S_3$ is equivalent to the fact that $\{2, 3\} \prec \{4, 5, 6\}$ and $\{1\} \prec \{7\}$, which are two independent events. Therefore, $P(Y = 1) = 3/5 \cdot 1/2 = 3/10$. The calculation of $P(Y = 2)$ is more complicated. Notice that

$$P(Y = 2) = P(\{4, 5, 6\} \prec \{2, 3\}; \{1, 4, 5, 6\} \prec \{2, 3, 7\}).$$

Instead of direct computation, we compute the following probability

$$\begin{aligned} & P(\{4, 5, 6\} \prec \{2, 3\}; \{2, 3, 7\} \prec \{1, 4, 5, 6\}) \\ &= P(\{4, 5, 6\} \prec \{2, 3\} \prec \{1\}) - P((\{4, 5, 6\} \prec \{2, 3\} \prec \{1\}) \prec \{7\}) \\ &= \frac{1}{6} \cdot \frac{2}{5} - \frac{1}{7} \cdot \frac{1}{6} \cdot \frac{2}{5} = \frac{2}{35}. \end{aligned}$$

Therefore, $P(Y = 2) = P(\{4, 5, 6\} \prec \{2, 3\}) - P(\{4, 5, 6\} \prec \{2, 3\}; \{2, 3, 7\} \prec \{1, 4, 5, 6\}) = 2/5 - 2/35 = 12/35$. Since $P(Y = 1) = 3/10 < P(Y = 2) = 12/35$, this example shows the statement in Equation (13). $\square$

From the counter example in the proof above, we can intuitively interpret the reason that the winning label Y may not always favor the group with the fewest members as follows. When there are a lot of overlap between two groups, there are a lot of competition between them to be the winning group and hence each of their winning probabilities is diluted (for example, S_1 and S_3 in the example). In this case, another group with more members (S_2 for example) may become more likely to the winning group. Although we do not have the general conclusion that the winning label Y favors the set with fewer members, we can still show that in some special instances this is true.

Theorem 3. Let $S_i, i = 1, \dots, K$ be K groups with L_i members respectively. Assume S_1 and S_2 are disjoint with the rest of $K - 2$ groups $S_3, \dots, S_K$, which implies that none of the members in S_1 or S_2 belong to the remaining groups. If $L_1 \leq L_2$ then

$$p(S_1 \prec S_2, S_1 \prec S_3, \dots, S_1 \prec S_K) \geq P(S_2 \prec S_1, S_2 \prec S_3, \dots, S_2 \prec S_K),$$

or equivalently

$$P(Y = 1) \geq P(Y = 2), \quad (14)$$

where the equality holds if and only if $L_1 = L_2$. This implies that, the winning group label Y favor the group S_1 with the smaller size.

Proof. We first assume S_1 and S_2 are non-overlapping. Let S'_2 be a subset of S_2 such that the size of S'_2 is L_1 . Define $S''_2 = S_2 \setminus S'_2$. It is easy to see by the definition of S'_2 that

$$p(S_1 \prec S'_2, S_1 \prec S_3, \dots, S_1 \prec S_K) = P(S'_2 \prec S_1, S'_2 \prec S_3, \dots, S'_2 \prec S_K).$$

Since $S'_2 \subset S_2$, it is obvious that $S'_2 \prec S_2$. Therefore,

$$P(S'_2 \prec S_1, S'_2 \prec S_3, \dots, S'_2 \prec S_K) \geq P(S_2 \prec S_1, S_2 \prec S_3, \dots, S_2 \prec S_K),$$

and

$$p(S_1 \prec S_2, S_1 \prec S_3, \dots, S_1 \prec S_K) \geq p(S_1 \prec S'_2, S_1 \prec S_3, \dots, S_1 \prec S_K)$$

Combining the above two inequalities, the result now follows. The equality is only hold when the rightmost seat for members in S_2 and S'_2 are always identical which implies $L_1 = L_2$. $\square$

4 Conclusion

In this article, we randomly arrange K groups of people and compare the values of M_k among each other. We show that when the groups are non-overlapping, the group with the fewest number of people tends to have the smallest rightmost value of M_k . We also consider the generalization of this result to the case of non-overlapping groups.