RIGID INDEPENDENCE

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Abstract. We extend the notion of independent sets to rigid independent sets. We choose the term *rigid* instead of *stable* since it reflects the **rigidity** of a block of concrete rather than the **stability** of a standing block.

I. Background and notation

Given an undirected graph $G = (V, E)$, an independent set is a subset of vertices $S \subseteq V$ without any adjacent pair. An independent set S is called *rigid* if its cardinality is at least 2 and it is **impossible** to replace any vertex $v \in S$ by another vertex $u \notin S$ so that $S \setminus \{v\} \cup \{u\}$ is again a maximal independent set. Clearly, rigidity implies maximality. We choose the term *rigid* to describe our notion. It is intuitive to imagine a block of concrete is standing firm unaffected by small variations.

Definition. (RIGID INDEPENDENT problem)

Input. An undirected graph $G(V, E)$, a natural number k

Output. YES if G has a rigid independent set of cardinality k , NO otherwise

Claim. We have that $\mathbf{X3C} \leq_p \text{RIGID INDEPENDENT}$

In this claim, $\mathbf{X3C}$ is the well-known exact cover by 3-sets which hardness was proven by [1].

II. Details of the reduction

Given an $\mathbf{X3C}$ instance, we have a universe set U and a collection C of 3-subsets of U . Each element of C is a subset of U containing 3 elements from the universe. We build our graph as follows.

For each element $u_i \in U$, we create an edge in our graph, namely (u_i, v_i) . Note that all these edges are disjoint.

We then proceed to create gadgets for each 3-set $s \in C$. For each such set, we create a new vertex for it. If s contains u_1, u_2, u_3 , we connect s to them. We also create for s two new adjacent vertices, namely t_{s1} and t_{s2} . We insist that no pair of t vertices are adjacent, and this is crucial in what follows. We set $k = |U| + |U|/3 + 2(|C| - \frac{|U|}{3})$. We have finished our construction. Now, we prove the correctness of our reduction.

Sketch of proof:

First, note that all the rigid independent set of our newly constructed graph must include all the v 's vertices. For otherwise, if u_i is in the supposed rigid set, we replace it with v_i to violate the requirement of rigidity. If u_i is not in the supposed rigid set, we pick an arbitrary vertex in this supposed set, and replace it with v_i . Since the only adjacent vertex of v_i is u_i , the new set is again independent, violating the rigidity.

Second, for each cluster of 3 vertices s, t_{s1}, t_{s2} , either only s or both the t 's vertices must be included in a rigid independent set of our newly constructed graph. If none is included, we are safe to add one of the t 's vertices and remain independent after excluding any other vertex in the set. If exactly one of

the t 's vertex is included, we can swap it out and swap in the other t vertex. And surely, if s is included, none of the two t 's vertices is included.

For clarity in notation, we suppose $|\mathbf{U}| = n$. We now prove that our reduction works in both directions as usual.

For one direction, if the **X3C** instance is a YES instance, we construct a rigid independent set of required cardinality for our newly constructed graph. As noted above, we need to take all the v 's vertices. So, we have used up to n vertices. For the next $n/3$ vertices to be included, we pick the same amount of s 's vertices that are included in the solution to the **X3C** instance. The remaining $2(|\mathbf{C}| - \frac{n}{3})$ vertices are the pairs of t 's vertices of unused s 's vertices. The rigidity of this set is obvious. We cannot swap an s vertex for an s vertex outside the chosen set, because the new one is adjacent to its own 2 t 's vertices. Also, swapping a v vertex with its corresponding u vertex is hopeless to stay independent, due to the fact that the set of chosen s 's vertices is a cover of the original **X3C** instance. Swapping out an s vertex, it is impossible to swap in any of its 3 adjacent u 's vertices. Lastly, swapping out an s vertex, it is also impossible to add just one of its t 's vertex while maintaining maximality.

Conversely, if our newly constructed graph possesses a rigid independent set of the required cardinality, we shall construct an exact cover for the original **X3C** instance. Clearly, this rigid set must contain the n v 's vertices as noted above. Taking in the second note above, we can conclude that it must include exactly $\frac{|\mathbf{U}|}{3}$ s 's vertices, and all the pairs of t 's vertices of the unused s 's vertices. If the set of chosen s 's vertices is a cover for the original instance, it is obviously an exact cover, due to the cardinalities of the two sets $\mathbf{U}$ and itself. So, we only need to prove that it is indeed a cover. Assume for the sake of contradiction, some u is not covered. We can now violate the assumed rigidity by simply swapping u with its corresponding v .

III. Conclusion

Garey and Johnson [1] shapes their theory based on previous primal works of Cook, Levin and Karp. Johnson [2] moves on with the guide to this theory. As long as we study a mathematical conjecture, we should encourage ourselves to have moderate amount of labour hours on popular maths books like these. Then, reading some articles on theory of computing like [3] is a good practice. Having all those done, we could think of the ultimate final for all mathematics sciences.

REFERENCE

1. Michael R. Garey, David S. Johnson, **Computers and Intractability: A Guide to the Theory of NP-Completeness**
2. David S. Johnson, **The NP-Completeness Column: An Ongoing Guide**
3. Phan Dinh Dieu, Le Cong Thanh, Le Tuan Hoa, **Average Polynomial Time Complexity of Some NP-Complete Problems**, Theor. Comput. Sci. 46(3): 219-237 (1986)