Path in directed graph with time duration at each vertex

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1. Introduction

In this section, we give the definition of our problem.

Given

- a directed graph $G = (V, E, d)$, where $d: V \rightarrow I(Q_0^+ \cup \{+\infty\})$ (here Q_0^+ denotes the set of non-negative rational numbers and $I(Q_0^+ \cup \{+\infty\})$ the set of intervals, bounded or unbounded above, with non-negative rational bounds) is a function associating with each vertex $v \in V$ an interval $d(v) = [a, b]$ for some $a \in Q_0^+, b \in Q_0^+ \cup \{+\infty\}$ and $a \leq b$,
- two vertices $s, t \in V$ and
- an integer h encoded in binary,

We have to *decide* whether or not there exist

- a path in G , possibly with repeated vertices and edges, $(v_0, v_1, \dots, v_{n-1}, v_n)$, with $v_0 = s$ and $v_n = t$ and
- a list of values $d_0, \dots, d_n \in Q_0^+$, such that $\sum_{i=0}^n d_i = h$ and for all $i = 0, \dots, n$, $d_i \in d(v_i)$.

Intuitively, we have to find a path in G , possibly where we get to the same vertices/edges also more than once, and where we remain in each vertex for a non-negative rational amount of time allowed by the minimum/maximum duration function, such that the overall time of the path equals h .

This can be solved easily in **PSPACE**. We want to show that it is in **NP** (we already know it is **NP-hard**). This is not trivial to prove, as we may have $h \in \Theta(2^n)$, for instance. Thus the required path may have length exponential in both $|V|$ and in the binary encoding of h .

2. Hardness of our problem

If zero-width durations are allowed, then the PARTITION PROBLEM can be reduced to our problem: Order the input numbers x_i arbitrarily. Make a path with a vertex for each number x_i , with zero-width interval $[x_i, x_i]$. Prepend s to this path, and append t . Add edges directed from every lower-numbered vertex to every higher-numbered others, and set $h = t/2$ where t is the sum of all input numbers.

3. Polynomial-time verifiable certificate

This solution is by Gerhard Woeginger.

In order to prove that this problem belongs to **NP**, we provide a polynomial-size certificate, and then we show how to check it in deterministic polynomial time.

The certificate is just the following: a set of integers $\{x_{u,v} \mid (u,v) \in E\}$. Intuitively, $x_{u,v}$ is the number of times the solution path traverses (u,v) .

We now describe the verification algorithm.

1. We consider the subset E' of edges of G , $E' := \{(u,v) \in E \mid x_{u,v} > 0\}$. We check whether E' induces a strongly (undirected) connected subgraph of G .
2. We check whether
 - $\sum_{(u,v) \in E'} x_{u,v} = \sum_{(v,w) \in E'} x_{v,w}$, for all $v \in V \setminus \{s, t\}$;
 - $\sum_{(u,s) \in E'} x_{u,s} = \sum_{(s,w) \in E'} x_{s,w} - 1$;
 - $\sum_{(u,t) \in E'} x_{u,t} = \sum_{(t,w) \in E'} x_{t,w} + 1$.
3. For all $v \in V \setminus \{s\}$, we define $y_v := \sum_{(u,v) \in E'} x_{u,v}$, i.e., the number of times the solution path gets into v . Moreover, $y_s := \sum_{(s,u) \in E'} x_{s,u}$.
4. We check whether there exist real values z_v , for every $v \in V$, such that
 - $d_{\min}(v)y_v \leq z_v \leq d_{\max}(v)y_v$ (here $d_{\min}$ and $d_{\max}$ denote resp. the lower and the upper bound of the rational interval $d(v)$, and
 - $\sum_{v \in V} z_v = h$.

We now sketch the correctness:

Steps 1. and 2. together check that the values $x_{u,v}$ for the arcs specify a directed Eulerian path from s to t (we refer to [1, section 1.3])

Steps 3. and 4. calculate z_v , for all $v \in V$, which is the total waiting time of the path on the node v . We observe that the (in)equalities of step 4. form a linear program (LP), which can be solved in deterministic polynomial time (e.g., using the ellipsoid algorithm).

References

- [1] *Dieter Jungnickel, Graphs, Networks and Algorithms, 3rd edition, 2008*