LANGUAGES AND THEIR TWO HALVES

Thinh D. Nguyen

Email: kosmofarmer@yandex.com

Abstract. *We show that the hardness of a language and its two halves can be drastically different.*

I. PROBLEM STATEMENT

Let $L \subseteq \Sigma^*$,

$$\begin{aligned}\frac{1}{2}L &= \{w \in \Sigma^* | \exists v \in \Sigma^* \, wv \in L \wedge |w| = |v|\} \\ L\frac{1}{2} &= \{w \in \Sigma^* | \exists v \in \Sigma^* \, vw \in L \wedge |w| = |v|\}\end{aligned}$$

We shall prove that (1) there exists languages L such that $L \in P$, but $\frac{1}{2}L, L\frac{1}{2}$ are **NP**-complete, (2) there exists languages L such that L is **NP**-complete, but $\frac{1}{2}L, L\frac{1}{2}$ are in P .

II. THE LANGUAGE OF PUZZLES AND THEIR ANSWER KEY

This problem reminds us of the fact that **NP** problems are "hard to solve but easy to verify". When you have a solution to any **NP** problem, you can check it in polynomial time.

So, pick an **NP**-complete language such as **HAMPATH** i.e. "The set of all directed graphs that have Hamiltonian paths". Define L to be the language of all strings where the first half is a problem in **HAMPATH** and the second half is a solution to the problem. We then add filler symbols to make sure the two halves have the same length.

Now, L is in P because the solution to an **NP**-complete problem can be checked in polynomial time. But $\frac{1}{2}L$ is **NP**-complete because **HAMPATH** is an **NP**-complete language.

III. THE LANGUAGE OF PUZZLE-PAIRS THAT CONTAIN EACH OTHER'S SOLUTIONS

Making sure that **both** halves of the language are **NP**-complete is a little trickier, but doable. The idea is to make a language of strings P_1xP_2y where P_1 and P_2 are **NP**-hard problems, x and y together give a solution key for P_1 and P_2 , but when separated x and y give no useful information. Then you can quickly confirm P_1xP_2y (the language is in P), but P_1x and P_2y are just hard problems (both in **NP**-complete).

Let L be the language of all strings with the following properties: The first half is a graph G in **HAMPATH** and an isomorphism $G \rightarrow H$. The second half is a graph H in **HAMPATH** and a Hamiltonian path in G .

If you know G, H , an isomorphism $G \rightarrow H$, and a Hamiltonian path in G , you can easily confirm whether G and H are Hamiltonian in polynomial time. Hence L is in P .

If you only know the first half of a string in L , however, you only have a graph G and an isomorphism with another (unknown) graph. Deciding whether G is in **HAMPATH** is an **NP**-complete problem, hence $\frac{1}{2}L$ is **NP**-complete. If you only know the second half of a string in L , then you have a

graph H and a Hamiltonian path for a *different (unknown) graph* G . Without knowing the isomorphism $G \rightarrow H$, we can argue that knowing this Hamiltonian path does not help decide whether H is in Hamiltonian. Hence $L_{\frac{1}{2}}$ is **NP**-complete.

IV. THE LANGUAGE OF SIMPLE STRINGS COMBINED WITH A COMPLICATED RULE

To make both halves easy (in **P**) but the combined string hard (**NP**-complete), the idea is to find a language consisting of pairs of objects, where each object is separately not too special (one can decide whether it's well-formed in polynomial time), but the rule for pairing them up is complicated and deciding whether the pairing is valid is **NP**-complete. Hence the language of whole strings will be **NP**-complete even if the halves are both in **P**.

The **SUBGRAPH-ISO** problem is: Given two graphs G and H , decide whether G is isomorphic to a *subgraph* of H . This problem is **NP**-complete. Therefore, let L be the set of strings which first half is a graph G and the second half is a graph H containing a subgraph isomorphic to G . We pad the strings as necessary so that both halves have the same length.

Then L is **NP**-complete, but the first and second halves of L are just the set of all strings corresponding to valid graphs. That's decidable in **P**.

V. CONCLUSION

Garey and Johnson [1] shape their theory based on previous primal works of Cook, Levin and Karp. Johnson [2] moves on with the guide to this theory. As long as we study a mathematical conjecture, we should encourage ourselves to have moderate amount of labour hours on popular mathematics books like these. Then, reading some articles on theory of computing like [3] is a good practice. Having all those done, we could think of the ultimate final for all mathematics sciences.

REFERENCE

1. Michael R. Garey, David S. Johnson, **Computers and Intractability: A Guide to the Theory of NP-Completeness**
2. David S. Johnson, **The NP-Completeness Column: An Ongoing Guide**
3. Phan Dinh Dieu, Le Cong Thanh, Le Tuan Hoa, **Average Polynomial Time Complexity of Some NP-Complete Problems**, Theor. Comput. Sci. 46(3): 219-237 (1986)