Geometric realization of locally regular heptagonal toroidal polyhedra.

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ABSTRACT. The article presents a geometric realization of toroidal polyhedra whose faces are flat heptagons. As an example, we have analyzed in detail a polyhedron of the 7th genus, consisting of 72 heptagonal faces. This solution allows realization for the number of faces $f \geq 24$ of the genus $g \geq 3$. Polyhedra in this case are closed orientable surfaces with flat faces without self-intersections, consisting of polygons and embedded in a three-dimensional space.

1. Introduction

At present, problems related to the geometric realization of toroidal polyhedra remain quite relevant. Polyhedra, each face of which is a p -gons are of much interest in this area. It should be noted that faces can be convex and non-convex polygons. For cases of hexagonal faces, numerous solutions, described in various articles [2, 4], are presented.

Before construction, for convenience, preliminary calculations were performed to determine the characteristics of the desired polyhedra by methods from the field of topology and combinatorics. According to Euler's theorem, the vertices, edges, and faces of a polyhedron of the genus- g are related by the equation:

$$v - e + f = \chi = 2 - 2g \quad (1)$$

where χ is the Euler characteristic.

For cases where the degrees of all vertices are $\deg(v) = 3$, the equality $3v = 2e$ holds [1]. In this case, we consider the graph embedding in the surface of the genus – g . The degrees of the vertices are abbreviated as - q .

Since for regular polyhedra of degree of all vertices $q = 3$ and the number of vertices of all faces for the cases under consideration are the same ($p = 7$), the number of vertices and edges can be written as $v = \frac{f \cdot p}{3}$, $e = \frac{f \cdot p}{2}$. Substituting these values in equation (1), we obtain the following formulas for calculations

$$f = 12g - 12 \quad (3)$$

$$g = \frac{f + 12}{12} \quad (4)$$

The calculation results for some of the first values are summarized in table 1.

Table 1

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p
2	12	28	42	7
3	24	56	84	7
4	36	84	126	7
5	48	112	168	7
6	60	140	210	7
7	72	168	252	7
8	84	196	294	7
9	96	224	336	7
10	108	252	378	7
11	120	280	420	7
12	132	308	462	7
13	144	336	504	7
14	156	364	546	7

The polyhedron at $g = 2$ was first introduced and described in the article "A locally regular heptagon-dodecahedron embedded in 3-space" by Jorg M. Wills [3]. Also, a lot of work in this area was done by David I. McCooey, who described new combinations for $g = 2$. In addition, he first introduced an realization for $g = 3, 5, 11$ [5, 6].

2. Polyhedra description

A distinctive feature of the solution presented in the article is the possibility of geometric realization for all cases of this series at $f \geq 24$ and $g \geq 3$. For ease of understanding and representation, polyhedra are divided into condition al segments.

Segments are those sections whose angle (α_1) changes depending on the type of polyhedron and in these examples $\alpha_1 = 360/(g - 1)$ (Fig. 5). Also, polyhedra are symmetrical when rotated by an angle α_1 around the OZ axis.

As a detailed example, we will analyze the case $(p, q; g)(7, 3; 7)$ in two different constructions, for the other cases, images and brief descriptions are provided, since they are made by analogy. For clarity of general designs, some angles between edges were made small, but they can be seen when considering face layouts on the plane and on general isometric views.

Polyhedron (7, 3; 7)-V1.

This polyhedron has six horizontal 2-fold figure axes and one 6-fold figure axis (Fig. 5), hence the polyhedron has rotational symmetry. The symmetry group for Schoenflies notations - D6.

The angles between the segments $\alpha_1 = 60^\circ$ and the center point O (0:0:0) are located at the intersection of the figure axes. The polyhedron faces are divided into 6 identical types, 12 of each type (Fig. 4).

Fig. 1. General views (7, 3; 7)-V1.

Fig. 2. Sections with an enlarged scale (7, 3; 7)-V1.

Fig. 3. Sections with enlarged scale and transparent faces (7, 3; 7)-V1.

Fig. 4. Layout of faces on the plane (7, 3; 7)-V1.

Fig. 5. Dividing the polyhedron into segments (7, 3; 7)-V1.

Equations of face planes (7, 3; 7)-V1.

Face planes 1 (Fig. 4).

$$\begin{aligned}
 x + y + 360 &= 0 \\
 x(\sqrt{3} - 1) + y(\sqrt{3} + 1) + 720 &= 0 \\
 x(\sqrt{3} + 1) - y(\sqrt{3} - 1) + 720 &= 0 \\
 x(\sqrt{3} + 1) + y(\sqrt{3} - 1) + 720 &= 0 \\
 x(\sqrt{3} - 1) - y(\sqrt{3} + 1) + 720 &= 0 \\
 x - y + 360 &= 0 \\
 x + y - 360 &= 0 \\
 x(\sqrt{3} - 1) + y(\sqrt{3} + 1) - 720 &= 0 \\
 x(\sqrt{3} + 1) - y(\sqrt{3} - 1) - 720 &= 0 \\
 x(\sqrt{3} + 1) + y(\sqrt{3} - 1) - 720 &= 0 \\
 x(\sqrt{3} - 1) - y(\sqrt{3} + 1) - 720 &= 0 \\
 x - y - 360 &= 0
 \end{aligned}$$

Face planes 2 (Fig. 4).

$$\begin{aligned}
 x + y + 200 &= 0 \\
 x(\sqrt{3} - 1) + y(\sqrt{3} + 1) + 400 &= 0 \\
 x(\sqrt{3} + 1) - y(\sqrt{3} - 1) + 400 &= 0 \\
 x(\sqrt{3} + 1) + y(\sqrt{3} - 1) + 400 &= 0 \\
 x(\sqrt{3} - 1) - y(\sqrt{3} + 1) + 400 &= 0 \\
 x - y + 200 &= 0 \\
 x + y - 200 &= 0 \\
 x(\sqrt{3} - 1) + y(\sqrt{3} + 1) - 400 &= 0 \\
 x(\sqrt{3} + 1) - y(\sqrt{3} - 1) - 400 &= 0 \\
 x(\sqrt{3} + 1) + y(\sqrt{3} - 1) - 400 &= 0 \\
 x(\sqrt{3} - 1) - y(\sqrt{3} + 1) - 400 &= 0 \\
 x - y - 200 &= 0
 \end{aligned}$$

Face planes 3 (Fig. 4).

$$\begin{aligned}
 z - 30 &= 0 \\
 z + 30 &= 0
 \end{aligned}$$

Face planes 4 (Fig. 4).

$$\begin{aligned}
 x + 4z - 220 &= 0 \\
 x - \sqrt{3}y + 8z - 440 &= 0 \\
 x + \sqrt{3}y - 8z + 440 &= 0 \\
 x - 4z + 220 &= 0 \\
 x - \sqrt{3}y - 8z + 440 &= 0 \\
 x + \sqrt{3}y + 8z - 440 &= 0 \\
 x + 4z + 220 &= 0 \\
 x - \sqrt{3}y + 8z + 440 &= 0 \\
 x + \sqrt{3}y - 8z - 440 &= 0 \\
 x - 4z - 220 &= 0 \\
 x - \sqrt{3}y - 8z - 440 &= 0 \\
 x + \sqrt{3}y + 8z + 440 &= 0
 \end{aligned}$$

Face planes 5 (Fig. 4).

$$\begin{aligned}
 y + z + 254 &= 0 \\
 y - z + 254 &= 0 \\
 y + 2z + \sqrt{3}x + 508 &= 0 \\
 y - 2z + \sqrt{3}x + 508 &= 0 \\
 2z - y + \sqrt{3}x + 508 &= 0 \\
 \sqrt{3}x - 2z - y + 508 &= 0 \\
 y - z - 254 &= 0 \\
 y + z - 254 &= 0 \\
 y - 2z + \sqrt{3}x - 508 &= 0 \\
 y + 2z + \sqrt{3}x - 508 &= 0 \\
 \sqrt{3}x - 2z - y - 508 &= 0 \\
 2z - y + \sqrt{3}x - 508 &= 0
 \end{aligned}$$

Face planes 6 (Fig. 4).

$$\begin{aligned}
 3x + 2y + 436 &= 0 \\
 3x - 2y - 436 &= 0 \\
 x(2 \cdot \sqrt{3} + 3) - y(3 \cdot \sqrt{3} - 2) + 872 &= 0 \\
 x(2 \cdot \sqrt{3} - 3) + y(3 \cdot \sqrt{3} + 2) + 872 &= 0 \\
 x(2 \cdot \sqrt{3} - 3) - y(3 \cdot \sqrt{3} + 2) + 872 &= 0 \\
 x(2 \cdot \sqrt{3} + 3) + y(3 \cdot \sqrt{3} - 2) + 872 &= 0 \\
 3x + 2y - 436 &= 0 \\
 3x - 2y + 436 &= 0 \\
 x(2 \cdot \sqrt{3} + 3) - y(3 \cdot \sqrt{3} - 2) - 872 &= 0 \\
 x(2 \cdot \sqrt{3} - 3) + y(3 \cdot \sqrt{3} + 2) - 872 &= 0 \\
 x(2 \cdot \sqrt{3} - 3) - y(3 \cdot \sqrt{3} + 2) - 872 &= 0 \\
 x(2 \cdot \sqrt{3} + 3) + y(3 \cdot \sqrt{3} - 2) - 872 &= 0
 \end{aligned}$$

Coordinates of polyhedron vertices (7, 3; 7)-V1.

The coordinates of the polyhedron vertices are given in the Cartesian coordinate system and divided into 2 columns with the results of symbolic expressions and decimals. For decimals, the last digit is rounded.

Table 2

№	Coordinates X		Coordinates Y		Coordinates Z	
	Symbolic expression	Decimals	Symbolic expression	Decimals	Symbolic expression	Decimals
1	$-(146\sqrt{3} - 34)$	-218,8794179051	$-(34\sqrt{3} + 146)$	-204,8897274573	38	38
2	$-(130\sqrt{3} - 50)$	-175,1666049840	$-(50\sqrt{3} + 130)$	-216,602540378	30	30
3	$-(400 - 200\sqrt{3})$	-53,5898384862	$-(200\sqrt{3} - 200)$	-146,410161514	30	30
4	$-(100\sqrt{3} - 100)$	-73,2050807569	$-(300 - 100\sqrt{3})$	-126,794919243	$(80 - 25\sqrt{3})$	36,698729811
5	$-(100\sqrt{3})$	-173,2050807569	-100	-100	55	55
6	$-(82\sqrt{3} + 18)$	-160,0281662206	$-(82 - 18\sqrt{3})$	-50,823085464	64	64
7	$-(148\sqrt{3} - 26)$	-230,3435195202	$-(26\sqrt{3} + 148)$	-193,033320997	42	42
8	$-(146\sqrt{3} + 34)$	-286,8794179051	$-(146 - 34\sqrt{3})$	-87,110272543	-38	-38
9	$-(130\sqrt{3} + 50)$	-275,1666049840	$-(130 - 50\sqrt{3})$	-43,397459622	-30	-30
10	$-(500 - 200\sqrt{3})$	-153,5898384862	$(200 - 100\sqrt{3})$	26,794919243	-30	-30
11	$-(200\sqrt{3} - 200)$	-146,4101615138	0	0	$-(80 - 25\sqrt{3})$	-36,698729811
12	$-(100\sqrt{3})$	-173,2050807569	-100	-100	-55	-55
13	$-(82\sqrt{3} - 18)$	-124,0281662206	$-(18\sqrt{3} + 82)$	-113,176914536	-64	-64
14	$-(148\sqrt{3} + 26)$	-282,3435195202	$-(148 - 26\sqrt{3})$	-102,966679003	-42	-42
15	$-(142\sqrt{3} + 22)$	-267,9512146748	$(142 - 22\sqrt{3})$	103,894882234	-30	-30
16	$-(142\sqrt{3} + 38)$	-283,9512146748	$(142 - 38\sqrt{3})$	76,182069312	-30	-30
17	$-(180\sqrt{3})$	-311,7691453624	180	180	-106	-106
18	$-(146\sqrt{3} - 34)$	-218,8794179051	$(34\sqrt{3} + 146)$	204,889727457	-38	-38
19	$-(148\sqrt{3} - 26)$	-230,3435195202	$(26\sqrt{3} + 148)$	193,033320997	-42	-42
20	$-(127\sqrt{3} - 12)$	-207,9704525612	$(12\sqrt{3} + 127)$	147,784609691	0	0
21	$-(127\sqrt{3} + 12)$	-231,9704525612	$(127 - 12\sqrt{3})$	106,215390309	0	0
22	$-(142\sqrt{3} - 22)$	-223,9512146748	$(22\sqrt{3} + 142)$	180,105117767	30	30
23	$-(142\sqrt{3} - 38)$	-207,9512146748	$(38\sqrt{3} + 142)$	207,817930688	30	30
24	$-(180\sqrt{3})$	-311,7691453624	180	180	106	106
25	$-(146\sqrt{3} + 34)$	-286,8794179051	$(146 - 34\sqrt{3})$	87,110272543	38	38
26	$-(148\sqrt{3} + 26)$	-282,3435195202	$(148 - 26\sqrt{3})$	102,966679003	42	42
27	$-(100\sqrt{3})$	-173,2050807569	100	100	-55	-55
28	$-(100\sqrt{3})$	-173,2050807569	100	100	55	55
29	$-(200\sqrt{3} - 200)$	-146,4101615138	0	0	$(80 - 25\sqrt{3})$	36,698729811
30	$-(82\sqrt{3} + 18)$	-160,0281662206	$(82 - 18\sqrt{3})$	50,823085464	-30	-30
31	$-(82\sqrt{3} + 18)$	-160,0281662206	$(82 - 18\sqrt{3})$	50,823085464	-64	-64
32	$-(130\sqrt{3} + 50)$	-275,1666049840	$(130 - 50\sqrt{3})$	43,397459622	30	30
33	$-(360\sqrt{3} - 360)$	-263,5382907248	0	0	-30	-30
34	$-(360\sqrt{3} - 360)$	-263,5382907248	0	0	30	30
35	$-(82\sqrt{3} - 18)$	-124,0281662206	$(18\sqrt{3} + 82)$	113,176914536	30	30
36	$-(82\sqrt{3} - 18)$	-124,0281662206	$(18\sqrt{3} + 82)$	113,176914536	64	64
37	$-(100\sqrt{3} - 100)$	-73,2050807569	$(300 - 100\sqrt{3})$	126,794919243	$-(80 - 25\sqrt{3})$	-36,698729811
38	$-(100\sqrt{3} - 100)$	-73,2050807569	$(300 - 100\sqrt{3})$	126,794919243	$(80 - 25\sqrt{3})$	36,698729811
39	-100	-100	$(300\sqrt{3} - 400)$	119,615242271	30	30
40	$-(142\sqrt{3} - 38)$	-207,9512146748	$-(38\sqrt{3} + 142)$	-207,817930688	-30	-30
41	$-(142\sqrt{3} - 22)$	-223,9512146748	$-(22\sqrt{3} + 142)$	-180,105117767	-30	-30
42	$-(109\sqrt{3})$	-188,7935380250	-109	-109	-30	-30
43	$-(82\sqrt{3} - 18)$	-124,0281662206	$-(18\sqrt{3} + 82)$	-113,176914536	-30	-30
44	-100	-100	$-(300\sqrt{3} - 400)$	-119,615242271	-30	-30
45	-100	-100	-260	-260	-30	-30
46	$-(180\sqrt{3} - 180)$	-131,7691453624	$-(540 - 180\sqrt{3})$	-228,230854638	-30	-30
47	$-(109\sqrt{3})$	-188,7935380250	109	109	-30	-30
48	0	0	360	360	106	106
49	0	0	360	360	-106	-106
50	-76	-76	284	284	-30	-30
51	$-(180\sqrt{3} - 180)$	-131,7691453624	$(540 - 180\sqrt{3})$	228,230854638	-30	-30
52	$-(180\sqrt{3} - 180)$	-131,7691453624	$(540 - 180\sqrt{3})$	228,230854638	30	30

№	Coordinates X		Coordinates Y		Coordinates Z	
53	-100	-100	260	260	30	30
54	-68	-68	292	292	38	38
55	44	44	284	284	30	30
56	76	76	284	284	30	30
57	-52	-52	296	296	42	42
58	-24	-24	254	254	0	0
59	24	24	254	254	0	0
60	$-(180\sqrt{3} - 180)$	-131,7691453624	$-(540 - 180\sqrt{3})$	-228,230854638	30	30
61	-76	-76	-284	-284	30	30
62	-44	-44	-284	-284	30	30
63	0	0	-218	-218	30	30
64	-36	-36	-164	-164	30	30
65	$-(109\sqrt{3})$	-188,7935380250	-109	-109	30	30
66	$-(82\sqrt{3} + 18)$	-160,0281662206	$-(82 - 18\sqrt{3})$	-50,823085464	30	30
67	$-(500 - 200\sqrt{3})$	-153,5898384862	$-(200 - 100\sqrt{3})$	-26,794919243	30	30
68	$-(142\sqrt{3} + 38)$	-283,9512146748	$-(142 - 38\sqrt{3})$	-76,182069312	30	30
69	$-(142\sqrt{3} + 22)$	-267,9512146748	$-(142 - 22\sqrt{3})$	-103,894882234	30	30
70	24	24	-254	-254	0	0
71	52	52	-296	-296	42	42
72	-36	-36	-164	-164	64	64
73	0	0	-218	-218	-30	-30
74	44	44	-284	-284	-30	-30
75	100	100	260	260	-30	-30
76	68	68	292	292	-38	-38
77	$(180\sqrt{3} - 180)$	131,7691453624	$(540 - 180\sqrt{3})$	228,230854638	30	30
78	$(180\sqrt{3} - 180)$	131,7691453624	$(540 - 180\sqrt{3})$	228,230854638	-30	-30
79	0	0	-360	-360	106	106
80	68	68	-292	-292	38	38
81	-24	-24	-254	-254	0	0
82	$-(100\sqrt{3} - 100)$	-73,2050807569	$-(300 - 100\sqrt{3})$	-126,794919243	$-(80 - 25\sqrt{3})$	-36,698729811
83	76	76	-284	-284	-30	-30
84	0	0	-360	-360	-106	-106
85	-68	-68	-292	-292	-38	-38
86	-52	-52	-296	-296	-42	-42
87	36	36	164	164	30	30
88	36	36	164	164	64	64
89	0	0	200	200	55	55
90	0	0	200	200	-55	-55
91	$(100\sqrt{3} - 100)$	73,2050807569	$(300 - 100\sqrt{3})$	126,794919243	$-(80 - 25\sqrt{3})$	-36,698729811
92	$(100\sqrt{3} - 100)$	73,2050807569	$(300 - 100\sqrt{3})$	126,794919243	$(80 - 25\sqrt{3})$	36,698729811
93	$(400 - 200\sqrt{3})$	53,5898384862	$(200\sqrt{3} - 200)$	146,410161514	30	30
94	$(127\sqrt{3} - 12)$	207,9704525612	$-(12\sqrt{3} + 127)$	-147,784609691	0	0
95	$(148\sqrt{3} - 26)$	230,3435195202	$-(26\sqrt{3} + 148)$	-193,033320997	-42	-42
96	$(82\sqrt{3} + 18)$	160,0281662206	$-(82 - 18\sqrt{3})$	-50,823085464	-64	-64
97	$(82\sqrt{3} + 18)$	160,0281662206	$-(82 - 18\sqrt{3})$	-50,823085464	-30	-30
98	$(109\sqrt{3})$	188,7935380250	-109	-109	-30	-30
99	$(109\sqrt{3})$	188,7935380250	-109	-109	30	30
100	$(142\sqrt{3} - 22)$	223,9512146748	$-(22\sqrt{3} + 142)$	-180,105117767	30	30
101	$(180\sqrt{3})$	311,7691453624	180	180	106	106
102	$(180\sqrt{3})$	311,7691453624	180	180	-106	-106
103	$(142\sqrt{3} - 38)$	207,9512146748	$(38\sqrt{3} + 142)$	207,817930688	-30	-30
104	$(130\sqrt{3} - 50)$	175,1666049840	$(50\sqrt{3} + 130)$	216,602540378	30	30
105	$(146\sqrt{3} - 34)$	218,8794179051	$(34\sqrt{3} + 146)$	204,889727457	38	38
106	$(100\sqrt{3})$	173,2050807569	-100	-100	55	55
107	$(100\sqrt{3})$	173,2050807569	-100	-100	-55	-55
108	$(100\sqrt{3} - 100)$	73,2050807569	$-(300 - 100\sqrt{3})$	-126,794919243	$-(80 - 25\sqrt{3})$	-36,698729811
109	$(100\sqrt{3} - 100)$	73,2050807569	$-(300 - 100\sqrt{3})$	-126,794919243	$(80 - 25\sqrt{3})$	36,698729811
110	100	100	$-(300\sqrt{3} - 400)$	-119,615242271	30	30
111	$(82\sqrt{3} - 18)$	124,0281662206	$-(18\sqrt{3} + 82)$	-113,176914536	30	30
112	$(82\sqrt{3} - 18)$	124,0281662206	$-(18\sqrt{3} + 82)$	-113,176914536	64	64
113	100	100	$(300\sqrt{3} - 400)$	119,615242271	-30	-30

№	Coordinates X		Coordinates Y		Coordinates Z	
114	$(82 \cdot \sqrt{3} - 18)$	124,0281662206	$(18 \cdot \sqrt{3} + 82)$	113,176914536	-30	-30
115	$(82 \cdot \sqrt{3} - 18)$	124,0281662206	$(18 \cdot \sqrt{3} + 82)$	113,176914536	-64	-64
116	$(100 \cdot \sqrt{3})$	173,2050807569	100	100	-55	-55
117	$(100 \cdot \sqrt{3})$	173,2050807569	100	100	55	55
118	$(142 \cdot \sqrt{3} + 22)$	267,9512146748	$(142 - 22 \cdot \sqrt{3})$	103,894882234	30	30
119	$(142 \cdot \sqrt{3} + 38)$	283,9512146748	$(142 - 38 \cdot \sqrt{3})$	76,182069312	30	30
120	$(148 \cdot \sqrt{3} - 26)$	230,3435195202	$(26 \cdot \sqrt{3} + 148)$	193,033320997	42	42
121	$(127 \cdot \sqrt{3} - 12)$	207,9704525612	$(12 \cdot \sqrt{3} + 127)$	147,784609691	0	0
122	$(127 \cdot \sqrt{3} + 12)$	231,9704525612	$(127 - 12 \cdot \sqrt{3})$	106,215390309	0	0
123	$(142 \cdot \sqrt{3} + 22)$	267,9512146748	$-(142 - 22 \cdot \sqrt{3})$	-103,894882234	-30	-30
124	$(142 \cdot \sqrt{3} + 38)$	283,9512146748	$-(142 - 38 \cdot \sqrt{3})$	-76,182069312	-30	-30
125	$(180 \cdot \sqrt{3})$	311,7691453624	-180	-180	-106	-106
126	$(146 \cdot \sqrt{3} - 34)$	218,8794179051	$-(34 \cdot \sqrt{3} + 146)$	-204,889727457	-38	-38
127	$(127 \cdot \sqrt{3} + 12)$	231,9704525612	$-(127 - 12 \cdot \sqrt{3})$	-106,215390309	0	0
128	$(148 \cdot \sqrt{3} + 26)$	282,3435195202	$-(148 - 26 \cdot \sqrt{3})$	-102,966679003	42	42
129	$(130 \cdot \sqrt{3} + 50)$	275,1666049840	$(130 - 50 \cdot \sqrt{3})$	43,397459622	-30	-30
130	$(146 \cdot \sqrt{3} + 34)$	286,8794179051	$(146 - 34 \cdot \sqrt{3})$	87,110272543	-38	-38
131	$(360 \cdot \sqrt{3} - 360)$	263,5382907248	0	0	30	30
132	$(360 \cdot \sqrt{3} - 360)$	263,5382907248	0	0	-30	-30
133	$(142 \cdot \sqrt{3} - 38)$	207,9512146748	$-(38 \cdot \sqrt{3} + 142)$	-207,817930688	30	30
134	$(180 \cdot \sqrt{3})$	311,7691453624	-180	-180	106	106
135	$(146 \cdot \sqrt{3} + 34)$	286,8794179051	$-(146 - 34 \cdot \sqrt{3})$	-87,110272543	38	38
136	$(130 \cdot \sqrt{3} + 50)$	275,1666049840	$-(130 - 50 \cdot \sqrt{3})$	-43,397459622	30	30
137	$-(130 \cdot \sqrt{3} - 50)$	-175,1666049840	$(50 \cdot \sqrt{3} + 130)$	216,602540378	-30	-30
138	$-(400 - 200 \cdot \sqrt{3})$	-53,5898384862	$(200 \cdot \sqrt{3} - 200)$	146,410161514	-30	-30
139	0	0	-200	-200	55	55
140	0	0	-200	-200	-55	-55
141	$(400 - 200 \cdot \sqrt{3})$	53,5898384862	$-(200 \cdot \sqrt{3} - 200)$	-146,410161514	-30	-30
142	36	36	-164	-164	-30	-30
143	36	36	-164	-164	-64	-64
144	$-(109 \cdot \sqrt{3})$	-188,7935380250	109	109	30	30
145	-44	-44	284	284	-30	-30
146	0	0	218	218	-30	-30
147	-36	-36	164	164	-30	-30
148	$(148 \cdot \sqrt{3} + 26)$	282,3435195202	$(148 - 26 \cdot \sqrt{3})$	102,966679003	-42	-42
149	$(109 \cdot \sqrt{3})$	188,7935380250	109	109	-30	-30
150	$(109 \cdot \sqrt{3})$	188,7935380250	109	109	30	30
151	$(142 \cdot \sqrt{3} - 22)$	223,9512146748	$(22 \cdot \sqrt{3} + 142)$	180,105117767	-30	-30
152	52	52	296	296	-42	-42
153	$(82 \cdot \sqrt{3} + 18)$	160,0281662206	$(82 - 18 \cdot \sqrt{3})$	50,823085464	64	64
154	$(82 \cdot \sqrt{3} + 18)$	160,0281662206	$(82 - 18 \cdot \sqrt{3})$	50,823085464	30	30
155	-36	-36	164	164	-64	-64
156	$(200 \cdot \sqrt{3} - 200)$	146,4101615138	0	0	$-(80 - 25 \cdot \sqrt{3})$	-36,698729811
157	$(200 \cdot \sqrt{3} - 200)$	146,4101615138	0	0	$(80 - 25 \cdot \sqrt{3})$	36,698729811
158	$(500 - 200 \cdot \sqrt{3})$	153,5898384862	$(200 - 100 \cdot \sqrt{3})$	26,794919243	30	30
159	$(130 \cdot \sqrt{3} - 50)$	175,1666049840	$-(50 \cdot \sqrt{3} + 130)$	-216,602540378	-30	-30
160	$(180 \cdot \sqrt{3} - 180)$	131,7691453624	$-(540 - 180 \cdot \sqrt{3})$	-228,230854638	30	30
161	$(180 \cdot \sqrt{3} - 180)$	131,7691453624	$-(540 - 180 \cdot \sqrt{3})$	-228,230854638	-30	-30
162	100	100	-260	-260	30	30
163	$(500 - 200 \cdot \sqrt{3})$	153,5898384862	$-(200 - 100 \cdot \sqrt{3})$	-26,794919243	-30	-30
164	0	0	218	218	30	30
165	$-(127 \cdot \sqrt{3} - 12)$	-207,9704525612	$-(12 \cdot \sqrt{3} + 127)$	-147,784609691	0	0
166	$-(127 \cdot \sqrt{3} + 12)$	-231,9704525612	$-(127 - 12 \cdot \sqrt{3})$	-106,215390309	0	0
167	$-(180 \cdot \sqrt{3})$	-311,7691453624	-180	-180	106	106
168	$-(180 \cdot \sqrt{3})$	-311,7691453624	-180	-180	-106	-106

Coordinates of face vertexes (7, 3; 7)-V1.

Table 3

Face №	Vertex number from the table 2.							Face №	Vertex number from the table 2.						
	1	2	3	4	5	6	7		1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	37	122	148	115	114	149	150	118
2	8	9	10	11	12	13	14	38	151	103	102	130	148	122	121
3	15	16	17	18	19	20	21	39	145	50	49	76	152	59	58
4	22	23	24	25	26	21	20	40	121	120	153	154	150	149	151
5	27	28	29	11	10	30	31	41	54	53	39	38	89	88	57
6	32	25	24	17	16	33	34	42	76	75	113	91	90	155	152
7	35	36	28	27	37	38	39	43	117	116	156	157	158	154	153
8	40	41	42	43	44	45	46	44	159	126	125	134	133	160	161
9	10	9	33	16	15	47	30	45	81	86	143	142	73	63	62
10	48	49	50	51	52	53	54	46	79	84	83	161	160	162	80
11	55	56	48	54	57	58	59	47	113	75	78	103	151	149	114
12	3	2	60	61	62	63	64	48	105	104	93	92	117	153	120
13	65	66	67	32	34	68	69	49	130	129	163	156	116	115	148
14	70	71	72	64	63	73	74	50	97	96	107	106	157	156	163
15	75	76	49	48	56	77	78	51	163	129	132	124	123	98	97
16	62	61	79	80	71	70	81	52	164	87	93	104	77	56	55
17	4	82	44	43	13	12	5	53	135	136	158	157	106	112	128
18	74	83	84	85	86	81	70	54	126	159	141	108	107	96	95
19	87	88	89	90	91	92	93	55	141	159	161	83	74	73	142
20	94	95	96	97	98	99	100	56	150	154	158	136	131	119	118
21	101	102	103	78	77	104	105	57	65	42	41	165	7	6	66
22	106	107	108	109	110	111	112	58	7	165	166	69	68	167	1
23	92	91	113	114	115	116	117	59	65	69	166	14	13	43	42
24	118	119	101	105	120	121	122	60	14	166	165	41	40	168	8
25	123	124	125	126	95	94	127	61	58	57	88	87	164	146	145
26	127	128	112	111	99	98	123	62	59	152	155	147	146	164	55
27	129	130	102	101	119	131	132	63	21	26	36	35	144	47	15
28	100	133	134	135	128	127	94	64	20	19	31	30	47	144	22
29	134	125	124	132	131	136	135	65	80	162	110	109	139	72	71
30	25	32	67	29	28	36	26	66	133	100	99	111	110	162	160
31	18	137	138	37	27	31	19	67	167	168	40	46	60	2	1
32	45	85	84	79	61	60	46	68	11	29	67	66	6	5	12
33	64	72	139	140	82	4	3	69	38	37	138	147	155	90	89
34	140	139	109	108	141	142	143	70	168	167	68	34	33	9	8
35	23	22	144	35	39	53	52	71	17	24	23	52	51	137	18
36	138	137	51	50	145	146	147	72	85	45	44	82	140	143	86

Polyhedron (7, 3; 7)-V2.

The polyhedron also consists of 72 heptagonal faces and differs from the previous one in that all vertex coordinates are integer, since the angles between the segments are straight. But this reduces the number of symmetries.

The polyhedron has two horizontal and one vertical 2-fold figure axes.

Fig. 6. General views (7, 3; 7)-V2.

Fig. 7. Sections with enlarged scale and transparent faces (7, 3; 7)-V2.

Fig. 8. Layout of faces on the plane (7, 3; 7)-V2.

Fig. 9. Axis of symmetry (7, 3; 7)-V2.

Coordinates of polyhedron vertices (7, 3; 7)-V2.

Table 4

№	X	Y	Z	№	X	Y	Z	№	X	Y	Z	№	X	Y	Z
1	-268	168	-24	43	132	-212	64	85	197	-147	-24	127	-12	-168	-24
2	-280	180	-26	44	268	-348	-24	86	247	-147	-24	128	-12	-348	-24
3	-472	-12	-58	45	280	-360	-26	87	640	0	-196	129	-140	-220	-24
4	-580	42	-76	46	132	-212	24	88	472	-12	24	130	-132	-212	-24
5	-580	60	-76	47	140	-220	24	89	460	0	24	131	-140	-204	-24
6	-280	360	-26	48	140	-220	-24	90	496	0	24	132	148	-212	-64
7	-268	348	-24	49	268	348	24	91	12	348	-24	133	0	-360	-27
8	-554	29	24	50	280	360	26	92	0	360	-27	134	0	-180	-27
9	-554	86	24	51	280	360	-27	93	0	360	27	135	-83	147	24
10	-640	0	196	52	132	212	-64	94	148	212	64	136	-140	204	24
11	-580	-60	76	53	132	212	-24	95	148	212	24	137	-140	40	131
12	-580	-42	76	54	140	220	-24	96	268	168	24	138	-232	132	39
13	-542	-23	0	55	140	220	24	97	140	204	24	139	0	180	27
14	-542	23	0	56	-140	220	-24	98	197	147	24	140	-12	168	24
15	-496	0	-24	57	-148	212	-24	99	247	147	24	141	-33	147	24
16	-472	-12	-24	58	-140	204	-24	100	-64	128	-43	142	140	40	131
17	-460	0	-24	59	-197	147	-24	101	-48	132	-39	143	48	132	39
18	-292	-168	-24	60	-247	147	-24	102	-140	40	-131	144	64	128	43
19	-292	-348	-24	61	-232	-132	-39	103	-173	171	0	145	-216	128	43
20	-554	-86	-24	62	-216	-128	-43	104	-107	171	0	146	-12	348	24
21	-554	-29	-24	63	-132	-212	-64	105	140	40	-131	147	-140	220	24
22	-640	0	-196	64	268	-168	-24	106	33	147	-24	148	-132	212	24
23	-292	348	24	65	280	-180	-26	107	83	147	-24	149	-132	212	64
24	-280	360	27	66	472	12	-58	108	107	171	0	150	-140	-220	24
25	-280	-360	-27	67	580	-42	-76	109	173	171	0	151	-148	-212	24
26	-280	-360	26	68	580	-60	-76	110	216	128	-43	152	-140	-204	24
27	-472	12	24	69	554	-29	24	111	232	132	-39	153	-197	-147	24
28	-460	0	24	70	554	-86	24	112	12	168	-24	154	-247	-147	24
29	-292	168	24	71	640	0	196	113	140	204	-24	155	83	-147	24
30	-496	0	24	72	580	60	76	114	-148	212	-64	156	140	-204	24
31	-280	180	27	73	580	42	76	115	0	180	-27	157	140	-40	131
32	-472	12	58	74	542	23	0	116	280	180	-27	158	0	-180	27
33	-280	-180	26	75	542	-23	0	117	64	-128	-43	159	12	-168	24
34	-280	-180	-27	76	496	0	-24	118	48	-132	-39	160	33	-147	24
35	-268	-168	24	77	472	12	-24	119	140	-40	-131	161	-140	-40	131
36	-268	-348	24	78	460	0	-24	120	173	-171	0	162	-48	-132	39
37	280	-360	27	79	292	168	-24	121	107	-171	0	163	0	-360	27
38	292	-348	24	80	292	348	-24	122	-140	-40	-131	164	-148	-212	64
39	292	-168	24	81	554	86	-24	123	-33	-147	-24	165	-64	-128	43
40	280	-180	27	82	554	29	-24	124	-83	-147	-24	166	12	-348	24
41	232	-132	39	83	148	-212	-24	125	-107	-171	0	167	280	180	26
42	216	-128	43	84	140	-204	-24	126	-173	-171	0	168	472	-12	58

Coordinates of face vertexes (7, 3; 7)-V2.

Table 5

Face №	Vertex number from the table 2.							Face №	Vertex number from the table 2.						
	1	2	3	4	1	2	3		1	2	3	4	1	2	3
1	1	2	3	4	5	6	7	37	118	117	132	133	128	127	134
2	8	9	10	11	12	13	14	38	109	98	97	113	53	52	110
3	15	16	17	18	19	20	21	39	104	135	136	58	57	114	100
4	22	10	9	23	24	6	5	40	102	137	138	31	2	1	60
5	25	26	11	10	22	20	19	41	115	139	140	141	137	102	101
6	27	28	29	23	9	8	30	42	105	142	143	139	115	112	106
7	4	14	13	21	20	22	5	43	144	143	142	99	98	109	108
8	14	4	3	16	15	30	8	44	137	141	135	104	103	145	138
9	29	28	17	16	3	2	31	45	135	141	140	146	147	148	136
10	28	27	32	33	34	18	17	46	56	147	146	93	92	114	57
11	13	12	32	27	30	15	21	47	143	144	94	93	146	140	139
12	35	33	32	12	11	26	36	48	113	97	95	94	144	108	107
13	37	38	39	40	41	42	43	49	58	136	148	149	145	103	59
14	44	45	37	43	46	47	48	50	31	138	145	149	24	23	29
15	49	50	51	52	53	54	55	51	6	24	149	148	147	56	7
16	1	7	56	57	58	59	60	52	35	36	150	151	152	153	154
17	34	61	62	63	25	19	18	53	126	153	152	131	130	63	62
18	64	65	66	67	68	45	44	54	121	155	156	84	83	132	117
19	69	70	71	72	73	74	75	55	39	89	78	77	66	65	40
20	76	77	78	79	80	81	82	56	40	65	64	86	119	157	41
21	64	44	48	83	84	85	86	57	134	158	159	160	157	119	118
22	87	71	70	38	37	45	68	58	122	161	162	158	134	127	123
23	51	50	72	71	87	81	80	59	129	150	36	26	25	63	130
24	88	89	39	38	70	69	90	60	133	163	164	151	150	129	128
25	67	75	74	82	81	87	68	61	165	162	161	154	153	126	125
26	75	67	66	77	76	90	69	62	157	160	155	121	120	42	41
27	55	54	91	92	93	94	95	63	34	33	35	154	161	122	61
28	96	49	55	95	97	98	99	64	155	160	159	166	47	46	156
29	100	101	102	60	59	103	104	65	48	47	166	163	133	132	83
30	105	106	107	108	109	110	111	66	162	165	164	163	166	159	158
31	107	106	112	91	54	53	113	67	131	152	151	164	165	125	124
32	101	100	114	92	91	112	115	68	84	156	46	43	42	120	85
33	51	80	79	116	111	110	52	69	116	167	96	99	142	105	111
34	117	118	119	86	85	120	121	70	89	88	168	167	116	79	78
35	122	123	124	125	126	62	61	71	74	73	168	88	90	76	82
36	124	123	127	128	129	130	131	72	96	167	168	73	72	50	49

Polyhedron (7, 3; 3)-V1.

Characteristics of a polyhedron (7, 3; 3)-V1:

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p	α_1	α_2
3	24	56	84	7	180	90

Fig. 10. General views (7, 3; 3)-V1.

Fig. 11. Sections with enlarged scale and transparent faces (7, 3; 3)-V1.

Fig. 12. Layout of faces on the plane (7, 3; 3)-V1.

Fig. 13. Angles between segments.

Polyhedron (7, 3; 4)-V1.

Characteristics of a polyhedron (7, 3; 4)-V1:

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p	α_1	α_2
4	36	84	126	7	120	60

Fig. 14. General views (7, 3; 4)-V1.

Fig. 15. Sections with enlarged scale and transparent faces (7, 3; 4)-V1.

Fig. 16. Layout of faces on the plane (7, 3; 4)-V1.

Fig. 17. Angles between segments. (7, 3; 4)-V1.

Polyhedron (7, 3; 5)-V1.

Characteristics of a polyhedron (7, 3; 5)-V1:

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p	α_1	α_2
5	48	112	168	7	90	45

Fig. 18. General views (7, 3; 5)-V1.

Fig. 19. Sections with enlarged scale and transparent faces (7, 3; 5)-V1.

Fig. 20. Layout of faces on the plane (7, 3; 5)-V1.

Fig. 21. Angles between segments (7, 3; 5)-V1.

Polyhedron (7, 3; 6)-V1.

Characteristics of a polyhedron (7, 3; 6)-V1:

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p	α_1	α_2
6	60	140	210	7	72	36

Fig. 22. General views (7, 3; 6)-V1.

Fig. 23. Sections with enlarged scale and transparent faces (7, 3; 6)-V1.

Fig. 24. Layout of faces on the plane (7, 3; 6)-V1.

Fig. 25. Angles between segments (7, 3; 6)-V1.

Polyhedron (7, 3; 14)-V1.

Characteristics of a polyhedron (7, 3; 14)-V1:

Genus, g	Faces, f	Vertices, v	Edges, e	Angles, p	α_1	α_2
14	156	364	546	7	360/13	180/13

Fig. 26. General views (7, 3; 14)-V1.

Fig. 27. Sections with enlarged scale and transparent faces (7, 3; 14)-V1.

Fig. 28. Layout of faces on the plane (7, 3; 14)-V1.

Fig. 29. Angles between segments (7, 3; 14)-V1.

3. Conclusions

This work allows us to conclude that locally regular toroidal polyhedra consisting of flat heptagonal faces are geometrically realizable in three-dimensional space. Based on the considered examples, one can notice the possibility to implement higher-genus polyhedra when changing the scale of faces and angles between segments. In this case, we can assume that an infinite realization is possible for this series.

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