An integral representation for the Riemann zeta function on positive integer arguments

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Abstract

In this brief note, we give an integral representation for the Riemann zeta function for positive integer arguments. To the best of our knowledge, the representation is new.

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We prove the following

Theorem 1. *We have, for integers $r \geq 2$,*

$$\zeta(r) = \frac{1}{\pi(2-2^r)} \int_0^\infty \frac{x^{-1/2} \mathbf{Li}_r(-x)}{1+x} dx \quad (1)$$

where $\mathbf{Li}_r(x) = \sum_{k=1}^\infty \frac{x^k}{k^r}$ is the polylogarithm function, and $\zeta(r) = \sum_{n=1}^\infty \frac{1}{n^r}$ is the Riemann zeta function.

The above can be obtained as a direct consequence of the following result

Theorem 2. *For integers $r \geq 2$ and $0 < n < 1$, we have*

$$\int_0^\infty x^{n-1} \frac{\mathbf{Li}_r(-x)}{1+x} dx = \frac{\pi}{\sin n\pi} \left(\zeta(r) + (-1)^{r-1} \frac{\psi_{r-1}(1-n)}{(r-1)!} \right). \quad (2)$$

where $\mathbf{Li}_r(x) = \sum_{k=1}^\infty \frac{x^k}{k^r}$ is the polylogarithm function, $\psi_n(x) = \frac{d^n}{dx^n} \frac{\Gamma'(x)}{\Gamma(x)}$ is the Polygamma function, $\Gamma(x)$ is the Gamma function, and $\zeta(r)$ is the Riemann zeta function.

Proof. Let

$$H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r}$$

be the generalized Harmonic number. Then, we have the following generating function from [2]

$$\frac{\mathbf{Li}_r(x)}{1-x} = \sum_{n=1}^\infty H_n^{(r)} x^n,$$

for $|x| < 1$. We can also write

$$\frac{\mathbf{Li}_r(-x)}{(1+x)} = \sum_{n=1}^\infty H_n^{(r)} (-x)^n. \quad (3)$$

We have for the following explicit form from [3]

$$H_n^{(r)} = \zeta(r) + (-1)^{r-1} \frac{\psi_{r-1}(n+1)}{(r-1)!}. \quad (4)$$

First note in this form, we have, $H_0^{(r)} = \zeta(r) + (-1)^{r-1} \frac{\psi_{r-1}(1)}{(r-1)!} = 0$. Next, we use Ramanujan's Master Theorem (RMT) from [1], which is,

$$\int_0^\infty x^{s-1} \{\phi(0) - x\phi(1) + x^2\phi(2) - \dots\} dx = \frac{\pi}{\sin s\pi} \phi(-s), \quad (5)$$

where the integral is convergent for $0 < \mathbf{Re}(s) < 1$, and after certain conditions are satisfied by ϕ . Now, using RMT with equations (3) and (4) gives us required equation (2). $\square$

Proof of Theorem 1. We get the required equation (1) by plugging in $n = 1/2$ in equation (2), and using the following fact from [3]

$$\psi_{r-1} \left(\frac{1}{2} \right) = (-1)^r \cdot (r-1)! \cdot (2^r - 1) \cdot \zeta(r).$$

$\square$

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