Operator of Sign of Energy

Christian Rakotonirina

Institut Supérieur de Technologie d'Antananarivo, IST-T

School of Civil engineering

Laboratoire de la Dynamique de l'Atmosphère,

du Climat et des Océans, DYACO, University of Antananarivo

October 30, 2018

Abstract

In this paper we have defined what we call sign operator of energy in the Dirac theory. We have shown the analogy between this operator with the helicity operator. That has lead us to a signification of the sign of the energy.

Keywords: Dirac equation, Negative energy, Kronecker product, Tensor product.

Introduction

One of the success of the Dirac equation is the discovery of antiparticle. The hypothesis of the existence of the antiparticle of an electron was forwarded by Dirac for interpreting the solution with negative energy of his equation. This has been the Dirac sea interpretation. This Dirac's interpretation was abandoned. Effectively, why could an electron from the state of negative energy $-E$ excited by a photon take the state of positive energy $+E$ but not $+0,9E$, $+1,1E$ or $+1,11E$, etc?

Another interpretation of the negative energy solution is the Feynman-Stückelberg interpretation (See for example[1]), which is expressed mathematically by $\exp\{-iEt\} \equiv \exp\{-i(-E)(-t)\}$. But in the application of this interpretation we encounter the same problem in the process of e^+e^- . Why does, in this process, the electron emit a photon with energy $2E$ but not $+1,9E$, $+2,1E$ or $+2,11E$, etc?

The resolution of the Dirac equation by using the tensor product or Kronecker product of matrices gives rise an operator [2] whose eigenvalues are the negative energy and the positive energy. We called this operator the "sign operator of energy". We think that a study of this operator could give other solutions at least of the problem above. We start this study by examining the solutions of the Dirac equation in considering this operator.

1 Dirac particle wave functions

The Dirac equation

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

is the quantum relativistic equation for a free spin- $\frac{1}{2}$ fermion, where the γ^μ 's are the gamma matrices. In this equation $\hbar$ is the Planck constant, c the speed of light, m the mass of the spin- $\frac{1}{2}$ fermion and ψ is its wave function.

The gamma matrices may be written as tensor product or Kronecker product of the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and the } 2 \times 2\text{-unit matrix} \\ \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\gamma^0 = \sigma^3 \otimes \sigma^0$, $\gamma^1 = i\sigma^2 \otimes \sigma^1$, $\gamma^2 = i\sigma^2 \otimes \sigma^2$, $\gamma^3 = i\sigma^2 \otimes \sigma^3$ are the gamma matrices in the Dirac representation.

The wave function solution of the Dirac equation may be written as Kronecker product or tensor product

$$\nu(t, \vec{x}) = \xi \otimes s e^{-\frac{i}{\hbar}(\pm Et - \vec{p} \cdot \vec{x})}$$

of the energy state $\xi e^{-\frac{i}{\hbar}(\pm Et - \vec{p} \cdot \vec{x})}$ and helicity state s , where $\xi = |\xi(E, p)\rangle = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} 1 \\ \frac{\epsilon cp}{E+mc^2} \end{pmatrix}$ is the eigenvector associated to the positive energy $E = +\sqrt{c^2p^2 + m^2c^4}$ or $\xi = |\bar{\xi}(E, p)\rangle = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} -\frac{\epsilon cp}{E+mc^2} \\ 1 \end{pmatrix}$, eigenvector associated to the negative energy $-E = -\sqrt{c^2p^2 + m^2c^4}$ of the hamiltonian operator $h_D = \epsilon cp\sigma^1 + mc^2\sigma^3$, and s is the eigenvector of the helicity operator $\frac{\hbar}{2}\vec{\sigma} \cdot \vec{n}$, that is the spin operator in the direction of the momentum $\vec{p}$, with $\vec{n} = \frac{\vec{p}}{\|\vec{p}\|} = \frac{\vec{p}}{p}$.

In spherical coordinates $\rho, \theta, \varphi, \vec{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$, $s = |s(\vec{p})\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$ is the eigenvector of $\frac{\hbar}{2}\vec{\sigma}\cdot\vec{n}$ associated to the helicity $+\frac{\hbar}{2}$, right handed particle or $s = |\bar{s}(\vec{p})\rangle = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix}$ is the eigenvector of $\frac{\hbar}{2}\vec{\sigma}\cdot\vec{n}$ associated to the helicity $-\frac{\hbar}{2}$, left handed neutrino (See for example, [1]).

In all of that ϵ is the sign of the helicity or the handedness.

So, we have four Dirac particle wave functions

$$\begin{aligned} \nu_{pp}(t, \vec{x}) &= |\xi(E, p)\rangle \otimes |s(\vec{p})\rangle e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})} & \nu_{pn}(t, \vec{x}) &= |\xi(E, p)\rangle \otimes |\bar{s}(\vec{p})\rangle e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})} \\ \bar{\nu}_{np}(t, \vec{x}) &= |\bar{\xi}(E, p)\rangle \otimes |s(\vec{p})\rangle e^{\frac{i}{\hbar}(Et + \vec{p}\cdot\vec{x})} & \bar{\nu}_{nn}(t, \vec{x}) &= |\bar{\xi}(E, p)\rangle \otimes |\bar{s}(\vec{p})\rangle e^{\frac{i}{\hbar}(Et + \vec{p}\cdot\vec{x})} \end{aligned}$$

solutions of the Dirac equation, respectively for particle with positive energy/positive helicity, positive energy/negative helicity, negative energy/positive helicity and negative energy/negative helicity.

On the exponential we can notice that only the energy change sign with the change of the first factor of the Kronecker product which is eigenvector of hamiltonian operator h_D .

2 Sign of Energy Operator

We called the operator $h_D = \epsilon cp\sigma^1 + mc^2\sigma^3$ "operator sign of energy" [3]. This operator occurs when solving the Dirac equation by using the tensor product or Kronecker product.

In fact this operator divided by the energy, $\frac{h_D}{E} = \frac{\epsilon cp}{E}\sigma^1 + \frac{mc^2}{E}\sigma^3$ which has -1 and $+1$ as eigenvalues should be called "operator sign of energy".

Let us introduce the following vector $\vec{E} = \begin{pmatrix} \epsilon cp \\ 0 \\ mc^2 \end{pmatrix}$. We call this vector

"energy vector", whose first component is the energy due to the momentum and the third component is the energy due to the mass. The energy $E = +\sqrt{c^2p^2 + m^2c^4}$ is the intensity, module or norm of the energy vector.

Therefore, the operator

$$\frac{\hbar}{2} \frac{h_D}{E} = \frac{\hbar}{2} \frac{\vec{\sigma}\cdot\vec{E}}{E}$$

is the projection in the direction of the energy vector $\vec{E}$ of the spin operator. Let us call the eigenvalues of this operator "enginity" and this operator the "enginity operator".

Therefore there is the probabilities of the particle of having the positive

enginity $+\frac{\hbar}{2}$ or the negative enginity $-\frac{\hbar}{2}$. It is like a spin- $\frac{1}{2}$ particle can have probabilities for having positive helicity or negative helicity .

For seeing that more clearly let us compare the enginity operator with the helicity operator.

$h_D = \epsilon c p \sigma^1 + m c^2 \sigma^3$ hamiltonian operator	$\vec{\sigma} \cdot \vec{p} = p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3$
let $\vec{E} = \begin{pmatrix} \epsilon c p \\ 0 \\ m c^2 \end{pmatrix}$ energy vector	with $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ momentum vector
$E = \ \vec{E}\ = \sqrt{m^2 c^4 + c^2 p^2}$ the energy	$p = \ \vec{p}\ = \sqrt{p_1^2 + p_2^2 + p_3^2}$
$\frac{\hbar}{2E} h_D = \frac{\hbar}{2E} \epsilon c p \sigma^1 + \frac{\hbar}{2E} m c^2 \sigma^3$	$\frac{\hbar}{2p} \vec{\sigma} \cdot \vec{p} = \frac{\hbar}{2p} p_1 \sigma_1 + \frac{\hbar}{2p} p_2 \sigma_2 + \frac{\hbar}{2p} p_3 \sigma_3$
enginity operator	helicity operator or
spin operator in the direction of $\vec{E}$	spin operator in the direction of $\vec{p}$
$\frac{h_D}{E} = \frac{\epsilon c p}{E} \sigma^1 + \frac{m c^2}{E} \sigma^3$ enginity sign operator	$\frac{\vec{\sigma} \cdot \vec{p}}{p} = \frac{p_1}{p} \sigma_1 + \frac{p_2}{p} \sigma_2 + \frac{p_3}{p} \sigma_3$ helicity sign operator
Probability for having positive or negative enginity	Probability for having positive or negative helicity

So, a spin- $\frac{1}{2}$ particle can be in a superposition of a state of positive and a state of negative enginity.

Conclusion

We have introduced the energy vector. Let us call enginity operator the projection of the spin operator in the direction of the energy vector. However, unlike the momentum vector we can not observe the energy vector. We have remarked that the sign of the enginity is the sign of the energy. The comparaison between enginity operator and helicity operator allow us to think that in the Dirac theory a spin- $\frac{1}{2}$ particle is in the superposition of a state of positive and a state of negative enginity.

References

- [1] Thomson A., *Modern Particle Physics*, Cambridge University Press, 105-107, 2013.
- [2] Raelina Andriambololona, Rakotonirina C., *A Study of the Dirac-Sidharth Equation*, EJTP 8, No.25, 177-182, 2011.
- [3] Rakotonirina C., *Produit Tensoriel de Matrices en Théorie de Dirac*, Thèse de Doctorat de Troisième Cycle (Supervised by Raelina Andri-

ambololona), Université d'Antananarivo, Antananarivo, Madagascar, p
61, 2003.