On an integral involving the logarithm function

Sumit Kumar Jha
International Institute of Information Technology
Hyderabad-500 032, India
kumarjha.sumit@research.iiit.ac.in

Abstract

We use the Ramanujan's master theorem to evaluate the integral

$$\int_0^\infty \frac{x^{l-1}}{(1+x)^{m+1}} \log^n(1+x) dx$$

in terms of the digamma function, the gamma function, and the Hurwitz zeta function.

1 Main Result

Theorem 1. *For all non-negative integers n we have*

$$\int_0^\infty \frac{x^{l-1}}{(1+x)^{m+1}} \log^n(1+x) dx = (-1)^n P_n(\psi(m+1-l) - \psi(m+1), \zeta(2, m+1) - \zeta(2, m+1-l), \\ \zeta(3, m+1) - \zeta(3, m+1-l), \dots, \zeta(n, m+1) - \zeta(n, m+1-l)) \cdot \frac{\Gamma(m+1-l) \cdot \Gamma(l)}{\Gamma(m+1)}$$

where m and l are complex numbers for which the integral on the left side converges, $\Gamma(s)$ is the gamma function, $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ is the digamma function, $\zeta(s, n)$ is the Hurwitz zeta function, and the polynomial $P_n(s_1, \dots, s_n)$ is defined by $P_0 = 1$ and

$$P_n(s_1, \dots, s_n) = (-1)^n Y_n(-s_1, -s_2, -2s_3, \dots, -(n-1)s_n)$$

where Y_n is the familiar Bell polynomial. The first few values of the polynomials being:

$$P_1(s_1) = s_1,$$

$$P_2(s_1, s_2) = s_1^2 - s_2,$$

$$P_3(s_1, s_2, s_3) = s_1^3 - 3s_1s_2 + 2s_3,$$

$$P_4(s_1, s_2, s_3, s_4) = s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4.$$

Proof. Zave [1] proved the following series expansion:

$$\frac{\log^n(1+x)}{(1+x)^{m+1}} = (-1)^n \sum_{k=0}^{\infty} P_n(H_{m+k}^{(1)} - H_m^{(1)}, H_{m+k}^{(2)} - H_m^{(2)}, \dots, H_{m+k}^{(n)} - H_m^{(n)}) \binom{m+k}{m} (-x)^k \quad (1)$$

where

$$H_n^{(k)} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots + \frac{1}{n^k}.$$

We first recall that

$$H_n^{(1)} = \psi(n+1) + \gamma$$

γ being the Euler's constant, and

$$H_n^{(s)} = \zeta(s) - \zeta(s, n+1),$$

for $s \geq 2$.

Now recalling the Ramanujan's master theorem [2] which states that

$$\int_0^\infty x^{l-1} \{\phi(0) - x\phi(1) + x^2\phi(2) - \dots\} dx = \frac{\pi}{\sin l\pi} \phi(-l)$$

gives us our result whenever m and l are chosen such that the integral is convergent. $\square$

Substituting $n = 1$ in our result Theorem 1 gives us:

Corollary 2.

$$\int_0^\infty \frac{x^{l-1}}{(1+x)^{m+1}} \log(1+x) dx = (\psi(m+1) - \psi(m+1-l)) \cdot \frac{\Gamma(m+1-l) \cdot \Gamma(l)}{\Gamma(m+1)}.$$

References

- [1] Zave, D. A. (1976). A series expansion involving the harmonic numbers. Information Processing Letters, 5(3), 75-77.
- [2] G. H. Hardy, *Ramanujan. Twelve Lectures on subjects suggested by his life and work (3rd ed.)*, Chelsea, 1978.