An extension of Lagrange interpolation to approximate derivative, integral and bivariate function

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Abstract

Lagrange interpolation is the effective method to approximate an arbitrary function by a polynomial. But there is a need to estimate derivative and integral given a set of points. Although it is possible to make Lagrange interpolation first, which produces Lagrange polynomial; after that we take derivative or integral on such polynomial. However this approach does not result out the best estimation of derivative and integral. This research proposes a different approach that makes approximation of derivative and integral based on point data first, which in turn applies Lagrange interpolation into the approximation. Moreover, the research also proposes an extension of Lagrange interpolation to bivariate function, in which interpolation polynomial is converted as two-variable polynomial.

Keywords: Lagrange interpolation, polynomial, approximation.

1. Lagrange interpolation of derivative and integral

Given univariate function $f: \mathbb{R} \rightarrow \mathbb{R}$ where $\mathbb{R}$ is real number field, let $P(x)$ be interpolation polynomial [Wikipedia 2014] of f . Suppose $D = \{(x_i, f(x_i))\}$ is the experimental data to estimate the polynomial $P(x)$ where all x_i (s) are arranged in ascending order such as $x_0 \leq x_1 \leq \dots \leq x_n$ and so x_i (s) form a partition whose elements $[x_i, x_{i+1}]$ are called volumes. In one-dimension space like $\mathbb{R}$, each volume is real number interval. According to Lagrange interpolation [Ta 2010] [Chapra, Canale 2010], we have:

$$P(x) = \sum_{i=0}^n \left(f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} \right)$$

Let $P_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$ be the indicator function. Obviously, we have

$$P(x) = \sum_{i=0}^n f(x_i) P_i(x)$$

And,

$$P_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

It is necessary to modify Lagrange method for interpolating derivative and integral. Let $\frac{\Delta f}{\Delta x}$ be the approximation of the derivative of function f . The derivative approximation at x_i denoted $\frac{\Delta f}{\Delta x}(x_i)$ is computed as following:

$$\frac{\Delta f}{\Delta x}(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Let $\frac{\Delta P}{\Delta x}$ be the interpolation polynomial of derivative of function f , we have:

$$\frac{\Delta P}{\Delta x} = \sum_{i=0}^{n-1} \frac{\Delta f}{\Delta x}(x_i) P_i(x) = \sum_{i=0}^{n-1} \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \prod_{\substack{j=0 \\ j \neq i}}^{n-1} \frac{x - x_j}{x_i - x_j} \right)$$

The ideology of interpolation of derivative is to make the discrete approximation of derivatives first, and then, to make interpolation on such approximations later. The interpolation of integral is computed by the similar way. Let $s(x)$ be the approximation of the integral of function f . The integral approximation at x_i denoted $s(x_i)$ is computed as follows:

$$s(x_i) = \frac{1}{2} \sum_{j=0}^{i-1} (f(x_j) + f(x_{j+1})) (x_{j+1} - x_j)$$

Where,

$$s(x_0) = 0 \text{ and } 1 \leq i \leq n$$

Essentially, $s(x_i)$ is the sum of areas of all trapeziums on the partition $x_0 \leq x_1 \leq \dots \leq x_i$. We recognize that $s(x_i)$ is an accumulated function and so we should take advantage of its recursion so as to get high performance in iterative algorithm.

$$s(x_i) = s(x_{i-1}) + \frac{1}{2} (f(x_{i-1}) + f(x_i)) (x_i - x_{i-1})$$

Let $S(x)$ be the interpolation polynomial of integral of function f , we have:

$$S(x) = \sum_{i=0}^n s(x_i) P_i(x) = \sum_{i=0}^n \left(s(x_i) \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} \right)$$

The interpolation polynomial is deviated from the real integral of f a constants C with attention that the interpolation operation starts with the value x_1 . The fact $s(x_0) = 0$ implies that $S(x)$ is pulled back to the coordinate origin $(0, 0)$ a constant C according to $f(x)$ -axis. Hence, $S(x)$ needs pushed away from the origin a distance C . Let $A(x)$ be the adjusted interpolation polynomial, we have:

$$A(x) = S(x) + C$$

The constant C is calculated as the area of triangle formed by three points $(0, 0)$, $(x_0, 0)$ and $(x_0, f(x_0))$.

$$C = \frac{1}{2} f(x_0) x_0$$

The adjusted interpolation polynomial $A(x)$ are totally determined as follows:

$$A(x) = S(x) + \frac{1}{2} f(x_0) x_0$$

2. Interpolation of bivariate function

When f becomes bivariate function as $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, it is necessary to extend Lagrange interpolation. In this case, the experimental data becomes multivariate data $D = \{(x_i, y_j, f(x_i, y_j))\}$ whose partition is $\{[x_i, x_{i+1}] \times [y_j, y_{j+1}]\}$ such that $x_0 \leq x_1 \leq \dots \leq x_m$ and $y_0 \leq y_1 \leq \dots \leq y_n$. Note that volumes $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ are rectangles. The indicator function $P_{ij}(x, y)$ is re-defined as follows:

$$P_{ij}(x, y) = \prod_{\substack{k=0 \\ k \neq i}}^m \left(\frac{x - x_k}{x_i - x_k} \prod_{\substack{l=0 \\ l \neq j}}^n \frac{y - y_l}{y_j - y_l} \right) = \prod_{\substack{k=0 \\ k \neq i}}^m \prod_{\substack{l=0 \\ l \neq j}}^n \frac{(x - x_k)(y - y_l)}{(x_i - x_k)(y_j - y_l)}$$

It is easy to infer that $P_{ij}(x, y)$ is multi-target indicator function.

$$P_{ij}(x_k, y_l) = \delta_{ijkl} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \text{ and } f(x_k, y_l) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

The bivariate interpolation polynomial $P(x, y)$ is expended as following:

$$P(x, y) = \sum_{i=0}^m \sum_{j=0}^n f(x_i, y_j) P_{ij}(x, y) = \sum_{i=0}^m \sum_{j=0}^n \left(f(x_i, y_j) \prod_{\substack{k=0 \\ k \neq i}}^m \prod_{\substack{l=0 \\ l \neq j}}^n \frac{(x - x_k)(y - y_l)}{(x_i - x_k)(y_j - y_l)} \right)$$

If experimental data is sparse, in other words, there is lack of some values $f(x_i, y_j)$, then, any expression $f(x_i, y_j)$ not evaluated is removed from the formula.

Now we consider the interpolation of derivative of bivariate function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. The gradient vector of f is:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

Where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial derivatives with regard to x and y , respectively. Note that the gradient ∇f is row vector. What we need to do is to take the interpolation to $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ individually. Let $\frac{\Delta f}{\Delta x}$ and $\frac{\Delta f}{\Delta y}$ be the approximations of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively, we have:

$$\begin{aligned} \frac{\Delta f}{\Delta x}(x_i, y_j) &= \frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{x_{i+1} - x_i} \\ \frac{\Delta f}{\Delta y}(x_i, y_j) &= \frac{f(x_i, y_{j+1}) - f(x_i, y_j)}{y_{j+1} - y_j} \end{aligned}$$

Let $\frac{\Delta P}{\Delta x}$ and $\frac{\Delta P}{\Delta y}$ be the interpolation polynomials of $\frac{\Delta f}{\Delta x}$ and $\frac{\Delta f}{\Delta y}$, respectively, we have:

$$\begin{aligned} \frac{\Delta P}{\Delta x} &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left(\frac{\Delta f}{\Delta x}(x_i, y_j) P_{ij}(x_i, y_j) \right) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left(\frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{x_{i+1} - x_i} \prod_{\substack{k=0 \\ k \neq i}}^{m-1} \prod_{\substack{l=0 \\ l \neq j}}^{n-1} \frac{(x - x_k)(y - y_l)}{(x_i - x_k)(y_j - y_l)} \right) \\ \frac{\Delta P}{\Delta y} &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left(\frac{\Delta f}{\Delta y}(x_i, y_j) P_{ij}(x_i, y_j) \right) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left(\frac{f(x_i, y_{j+1}) - f(x_i, y_j)}{y_{j+1} - y_j} \prod_{\substack{k=0 \\ k \neq i}}^{m-1} \prod_{\substack{l=0 \\ l \neq j}}^{n-1} \frac{(x - x_k)(y - y_l)}{(x_i - x_k)(y_j - y_l)} \right) \end{aligned}$$

Where $P_{ij}(x, y)$ is the indicator function as aforementioned. Please pay attention that any expression $f(x_{i+1}, y_j)$, $f(x_i, y_j)$ or $f(x_i, y_{j+1})$ not evaluated is removed from the formula. Now it is easy to determine the interpolation denoted ΔP of gradient vector ∇f .

$$\Delta P = \left(\frac{\Delta P}{\Delta x}, \frac{\Delta P}{\Delta y} \right)$$

Now we find out the interpolation of multivariate integral. Suppose $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ is a rectangle of given partition $x_0 \leq x_1 \leq \dots \leq x_m$ and $y_0 \leq y_1 \leq \dots \leq y_n$. Such rectangle and function f form two triangular prisms denoted T_1 and T_2 . Prism T_1 is created by five faces:

- Bottom triangular face $F_{11} = \{(x_i, y_j, 0), (x_i, y_{j+1}, 0), (x_{i+1}, y_{j+1}, 0)\}$.
- Top triangular face $F_{12} = \{(x_i, y_j, f(x_i, y_j)), (x_{i+1}, y_{j+1}, f(x_{i+1}, y_{j+1})), (x_i, y_{j+1}, f(x_i, y_{j+1}))\}$.
- Left-side trapezoidal face $F_{13} = \{(x_{i+1}, y_{j+1}, 0), (x_i, y_{j+1}, 0), (x_i, y_{j+1}, f(x_i, y_{j+1})), (x_{i+1}, y_{j+1}, f(x_{i+1}, y_{j+1}))\}$.
- Right-side trapezoidal face $F_{14} = \{(x_i, y_{j+1}, 0), (x_i, y_j, 0), (x_i, y_j, f(x_i, y_j)), (x_i, y_{j+1}, f(x_i, y_{j+1}))\}$.
- Common rectangle face $F_r = \{(x_i, y_j, 0), (x_{i+1}, y_{j+1}, 0), (x_{i+1}, y_{j+1}, f(x_{i+1}, y_{j+1})), (x_i, y_j, f(x_i, y_j))\}$.

Similarly, prism T_2 is created by five faces:

- Bottom triangular face $F_{21} = \{(x_i, y_j, 0), (x_{i+1}, y_{j+1}, 0), (x_{i+1}, y_j, 0)\}$
- Top triangular face $F_{22} = \{(x_i, y_j, f(x_i, y_j)), (x_{i+1}, y_{j+1}, f(x_{i+1}, y_{j+1})), (x_{i+1}, y_j, f(x_{i+1}, y_j))\}$.
- Left-side trapezoidal face $F_{23} = \{(x_i, y_j, 0), (x_{i+1}, y_j, 0), (x_{i+1}, y_j, f(x_{i+1}, y_j)), (x_i, y_j, f(x_i, y_j))\}$.
- Right-side trapezoidal face $F_{24} = \{(x_{i+1}, y_{j+1}, 0), (x_{i+1}, y_j, 0), (x_{i+1}, y_j, f(x_{i+1}, y_j)), (x_{i+1}, y_{j+1}, f(x_{i+1}, y_{j+1}))\}$.
- Common rectangular face $F_r = \{(x_i, y_j, 0), (x_{i+1}, y_{j+1}, 0), (x_{i+1}, y_{j+1}, f(x_{i+1}, y_{j+1})), (x_i, y_j, f(x_i, y_j))\}$.

Both T_1 and T_2 have the same rectangle face F_r . Let $v(F_{13})$, $v(F_{14})$, $v(F_{23})$ and $v(F_{24})$ be areas of trapezoidal faces F_{13} , F_{14} , F_{23} and F_{24} , respectively, we have:

$$v(F_{13}) = \frac{1}{2} (f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1})) (x_{i+1} - x_i)$$

$$v(F_{14}) = \frac{1}{2} (f(x_i, y_j) + f(x_i, y_{j+1})) (y_{j+1} - y_j)$$

$$v(F_{23}) = \frac{1}{2} (f(x_{i+1}, y_j) + f(x_i, y_j)) (x_{i+1} - x_i)$$

$$v(F_{24}) = \frac{1}{2} (f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})) (x_{i+1} - x_i)$$

The volume of a triangular prism is the half of product of two areas of trapezoidal faces because the top and bottom faces of such prism are right triangles. Let $v(T_1)$ and $v(T_2)$ be volumes of prisms T_1 and T_2 , respectively, we have:

$$\begin{aligned} v(T_1) &= \frac{1}{2} * v(F_{13}) * v(F_{14}) \\ &= \frac{1}{8} (f(x_i, y_j) + f(x_i, y_{j+1})) (f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1})) (x_{i+1} - x_i) (y_{j+1} - y_j) \end{aligned}$$

$$\begin{aligned} v(T_2) &= \frac{1}{2} * v(F_{23}) * v(F_{24}) \\ &= \frac{1}{8} (f(x_i, y_j) + f(x_{i+1}, y_j)) (f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})) (x_{i+1} - x_i) (y_{j+1} - y_j) \end{aligned}$$

Let $v(x_i, y_j)$ be the volume of polyhedron formed by the rectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ and function f . It is easy to infer that the polyhedron is instituted of two prisms T_1 and T_2 ; in other words, the volume $v(x_i, y_j)$ is sum of $v(T_1)$ and $v(T_2)$ and so we have:

$$\begin{aligned}
v(x_i, y_j) &= v(T_1) + v(T_2) \\
&= \frac{1}{8} \left[\left(f(x_i, y_j) + f(x_i, y_{j+1}) \right) \left(f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1}) \right) \right. \\
&\quad \left. + \left(f(x_i, y_j) + f(x_{i+1}, y_j) \right) \left(f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1}) \right) \right] (x_{i+1} - x_i)(y_{j+1} - y_j)
\end{aligned}$$

Please pay attention that any expression $f(x_{i+1}, y_j)$, $f(x_i, y_j)$, $f(x_i, y_{j+1})$ or $f(x_{i+1}, y_{j+1})$ not evaluated is removed from the formula. Suppose $s(x_i, y_j)$ be approximation of the integral of function f on the $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we recognize that $s(x_i, y_j)$ is sum of a block of volumes $v(x_k, y_l)$ where $k \leq i$ and $l \leq j$.

$$s(x_i, y_j) = \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} v(x_k, y_l)$$

Where,

$$s(x_0, y_j) = s(x_i, y_0) = 0 \text{ and } 1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

Let $S(x, y)$ be the bivariate interpolation polynomial of integral of function f , we have:

$$S(x, y) = \sum_{i=0}^m \sum_{j=0}^n s(x_i, y_j) P_{ij}(x, y) = \sum_{i=0}^m \sum_{j=0}^n \left(s(x_i, y_j) \prod_{\substack{k=0 \\ k \neq i}}^m \prod_{\substack{l=0 \\ l \neq j}}^n \frac{(x - x_k)(y - y_l)}{(x_i - x_k)(y_j - y_l)} \right)$$

In similar to univariate case, the interpolation polynomial is deviated from the real integral of f a constants C with attention that the interpolation operation starts with the point (x_1, y_1) . The fact $s(x_0, y_0) = 0$ implies that $S(x, y)$ is pulled back to the coordinate origin $(0, 0, 0)$ a constant C according to $f(x, y)$ -axis. Hence, $S(x, y)$ needs pushed away from the origin a distance C .

The constant C is defined as volume of polyhedron formed by the rectangles $[x_0, x_i] \times [y_0, y_j]$ and function f .

$$C = v(x_0, y_0) + \sum_{k=1}^{i-1} v(x_k, y_0) + \sum_{l=1}^{j-1} v(x_0, y_l)$$

When C is calculated, the adjusted interpolation polynomial denoted $A(x)$ are totally determined as following:

$$A(x) = S(x, y) + C$$

3. Conclusion

In general, there are two main aspects of the research:

- Proposing a method to interpolate polynomial of derivative and integral given a set of points. The ideology of such method is to make the discrete approximation of derivatives first, and then, to make interpolation on such approximations later. The method is opposite to normal way in which Lagrange interpolation is the most important procedure which is calculated first.
- Extending Lagrange interpolation to bivariate function f , in which the interpolation polynomial becomes two-variable function. When f is multivariate function, the approach is keep intact but estimation formulas get much more complicated.

Derivative and integral interpolation with regard to two-variable function gets complicated because the main point of proposed method is to make approximations based on given experimental data. Derivative is translated as the speed of partial discrete variations. Integral is translated as a discrete sum of volumes of prisms. The purpose of these approximations is to eliminate the requirement that the derivative or integral of interpolation polynomial should

be close to the derivative or integral of origin function when this requirement is the cause of inaccurate interpolation.

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