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Sign of the Mertens function

Darrell Cox, Sourangshu Ghosh, and Eldar Sultanow

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Abstract

Lehman proved that the sum of certain Mertens function values is 1. Functions involving the sum of the signs of these Mertens function values are considered here. Specifically, upper bounds of these functions involving the number of Mertens function values equal to zero are determined. Franel and Landau derived an arithmetic statement involving the Farey sequence that is equivalent to the Riemann hypothesis. Since there is a relationship between the Mertens function and the Riemann hypothesis, there should be a relationship between the Mertens function and the Farey sequence. Functions of subsets of the fractions in Farey sequences that are analogous to the Mertens function are introduced. Results analogous to Lehman's theorem are the defining property of these functions. A relationship between the Farey sequence and the Riemann hypothesis other than the Franel-Landau theorem is postulated.

1 Introduction

Let $M(x)$ denote the Mertens function. Lehman [1] proved that

Theorem (1) $\sum_{i=1}^x M(\lfloor x/i \rfloor) = 1$

Let $n(j)$ denote $2 + \sum_{i=1}^j |\operatorname{sgn}(M(i))|$ where j is a natural number. Note that there is no contribution to the sum when $M(i) = 0$. Let $t(x)$ denote $-\sum_{i=1}^{\lfloor x/(j+1) \rfloor} \operatorname{sgn}(M(\lfloor x/i \rfloor))$. Based on empirical evidence,

$$(2) \lfloor x/n(j) \rfloor \geq t(x), j=1,2,3,\dots$$

A plot of $t(x)$ and $\lfloor x/n(j) \rfloor$ for $j = 1$ and $x \leq 100$ is

$M(x)$ equals 0 at $x = 2, 40, 41, 58, 65$, and 93. A plot of $t(x)$ and $[x/n(j)]$ for $j = 100$ and $x \leq 2000$ is

There are 92 $M(x)$ zeros for x less than 1000. A plot of $t(x)$ and $\lfloor x/n(j) \rfloor$ for $j = 1000$ and $x \leq 10000$ is

2 The Number of Mertens Function Values Equal to Zero

Let $V(x)$ denote the number of Mertens function zeros less than x . Hurst [2] determined that $3.5\sqrt{x}$ is a good estimate of $V(x)$ and compiled the following table of $V(x)$ values for x equal to powers of 10.

$x = n$ th power of 10	$V(x)$
$n = 1 :$	1
$n = 2 :$	6
$n = 3 :$	92
$n = 4 :$	406
$n = 5 :$	1549
$n = 6 :$	5361
$n = 7 :$	12546
$n = 8 :$	41908
$n = 9 :$	141121
$n = 10 :$	431822
$n = 11 :$	1628048
$n = 12 :$	4657633
$n = 13 :$	12917328
$n = 14 :$	40604969
$n = 15 :$	109205859
$n = 16 :$	366567325

3 A Variant of $t(x)$

Let $s(x)$ denote $-\sum_{i=1}^{\lfloor x/(j+1) \rfloor} \text{sgn}(M(\lfloor x/i \rfloor))i$. Based on empirical evidence,

$$(1) (\lfloor x/n(j) \rfloor (\lfloor x/n(j) \rfloor + 1))/2 \geq s(x), j=1,2,3,\dots$$

A plot of $s(x)$ and $(\lfloor x/n(j) \rfloor (\lfloor x/n(j) \rfloor + 1))/2$ for $j = 100$ and $x \leq 1000$ is

A plot of $s(x)$ and $(\lfloor x/n(j) \rfloor (\lfloor x/n(j) \rfloor + 1))/2$ for $j = 1000$ and $x \leq 10000$ is

4 A Variant of $s(x)$

Let $k = \sum_{i=1}^j |\operatorname{sgn}(M(i))|$ and let $r = (k-1) - \lfloor (k-1)/4 \rfloor$. Let $d=2, 3, 3$, or 4 when $r=0, 1, 2$, or 3 respectively. Let $n(k) = 3\lfloor (k-1)/4 \rfloor + d$ and $\sigma_1(i) = \sum_{d|i} d$. Let $u(x) = -\sum_{i=1}^{\lfloor x/(j+1) \rfloor} \operatorname{sgn}(M(\lfloor x/i \rfloor)) \sigma_1(i)$. Based on empirical evidence,

$$(3) \quad (\lfloor x/n(k) \rfloor (\lfloor x/n(k) \rfloor + 1))/2 + 1 \geq u(x), \quad j=1,2,3,\dots$$

A plot of $u(x)$ and $(\lfloor x/n(k) \rfloor (\lfloor x/n(k) \rfloor + 1))/2 + 1$ for $j = 3$ and $x \leq 200$ is

5 Farey Sequences and the Mertens Function

The Farey sequence F_x of order x is the ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed x . Let $A(x)$ denote the number of terms in the Farey sequence of order x . For $v = 1, 2, 3, \dots, A(x)$ let δ_v denote the amount by which the v th term of the Farey sequence differs from $v/A(x)$. The theorem of Franel and Landau [4] is that the Riemann hypothesis is equivalent to the statement that $|\delta_1| + |\delta_2| + |\delta_3| + \dots + |\delta_{A(x)}| = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$ as $x \rightarrow \infty$.

Let m_x denote the number of fractions before $1/4$ and n_x the number of fractions between $1/4$ and $1/2$ in a Farey sequence of order x . A plot of $n_x - m_x$ for

$x = 2, 3, 4, \dots, 10000$ is

A plot of $M(x)$ for $x = 2, 3, 4, \dots, 10000$ is

$n_x - m_x$ is an analogue of $M(x)$. Let $h(x)$ denote $\sum_{i=1}^x (n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})$. (Although $1/4 < 1/3 < 1/2$, n_3 is set to 0 since $1/4$ is not in a Farey sequence of order 3.) Based on empirical evidence,

(4) $h(2), h(3), h(4), \dots, h(13)$ equal 0, 0, 1, 1, 0, 1, 2, 1, 1, 2, 2, 2 respectively and $h(x+12) = h(x) + 2$.

(5) $\sum_{i=1}^x (m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6})$ equals 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$, or $-\frac{2}{3}$.

Based on empirical evidence,

(6) $\sum_{i=1}^x \text{sgn}(M(\lfloor x/i \rfloor)) \geq h(x)$

A plot of $\sum_{i=1}^x \text{sgn}(M(\lfloor x/i \rfloor)) - h(x)$ for $x = 2, 3, 4, \dots, 1000$ is

$\sum_{i=1}^x \text{sgn}(M(\lfloor x/i \rfloor))$ equals $h(x)$ in 47 instances.

Based on empirical evidence,

$$(7) \quad 1 + \sum_{i=1}^x (n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})^2 \geq \sum_{i=1}^x \text{sgn}(M(\lfloor x/i \rfloor))$$

A plot of $1 + \sum_{i=1}^x (n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})^2 - \sum_{i=1}^x \text{sgn}(M(\lfloor x/i \rfloor))$ for $x = 2, 3, 4, \dots, 1000$ is

$1 + \sum_{i=1}^x (n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})^2$ equals $\sum_{i=1}^x \text{sgn}(M(\lfloor x/i \rfloor))$ in 39 instances.

6 Farey Sequences and the Second Chebyshev Function

Let $\Lambda(i)$ denote the Mangoldt function ($\Lambda(i)$ equals $\log(p)$ if $i = p^m$ for some prime p and some $m \geq 1$ or 0 otherwise). Let $\psi(x)$ denote the second Chebyshev function ($\psi(x) = \sum_{i \leq x} \Lambda(i)$). Mertens [3] proved that

Theorem (2) $\sum_{i=1}^x M(\lfloor x/i \rfloor) \log(i) = \psi(x)$

A plot of $\psi(x)$ and $\sum_{i=1}^x (m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6}) \log(i)$ for $x = 2, 3, 4, \dots, 3000$ is

The slope of $\sum_{i=1}^x (m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6}) \log(i)$ is approximately $\frac{1}{5}$ whereas the slope of $\psi(x)$ is approximately 1.0.

The Riemann hypothesis is equivalent to the arithmetic statement $\psi(x) - x = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$. For a linear least-squares fit of $\psi(x)$ for $x = 2, 3, 4, \dots, 5000$, $p_1 = 0.9997$ with a 95% confidence interval of (0.9995, 0.9999), $p_2 = -0.824$ with a 95% confidence interval of (-1.419, -0.2294), SSE= $5.739 \cdot 10^5$, R-squared=0.9999, and RMSE=10.72. For a linear least-squares fit of $\sum_{i=1}^x (m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6}) \log(i)$ for $x = 2, 3, 4, \dots, 5000$, $p_1 = 0.1992$ with a 95% confidence interval of (0.199, 0.1994), $p_2 = 0.879$ with a 95% confidence interval of (0.2823, 1.476), SSE= $5.781 \cdot 10^5$, R-squared=0.9986, and RMSE=10.76. The sum-squared errors (SSE) and root-mean-squared errors (RMSE) are approximately equal. This is the basis for the following conjecture.

Conjecture (1) The Riemann hypothesis is equivalent to the arithmetic statement $\sum_{i=1}^x (m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6}) \log(i) - x/5 = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$.

7 Farey Sequences and Other Arithmetic Functions

Another relationship (among many) proved by Cox [5] is;

Theorem (3) $\sum_{i=1}^x M(\lfloor x/i \rfloor) \sigma_1(i) = x(x+1)/2$

A plot of $\sum_{i=1}^x (m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6}) \sigma_1(i)$ for $x = 2, 3, 4, \dots, 1000$ is

For a quadratic least-squares fit of the curve, $p_1 = 0.09764$ with a 95% confidence interval of $(0.09757, 0.09771)$, $p_2 = 0.08419$ with a 95% confidence interval of $(0.01015, 0.1582)$, $p_3 = 1.693$ with a 95% confidence interval of $(-14.37, 17.76)$, $SSE = 7.321 \cdot 10^6$, $R\text{-squared} = 1$, and $RMSE = 85.74$. The first and second parameters should be approximately equal.

8 Infinitely Many Analogues of the Mertens Function

Let o_x denote the number of fractions before $1/5$ and p_x the number of fractions between $1/5$ and $2/5$ in a Farey sequence of order x . Let q_x denote the number of fractions before $1/6$ and r_x the number of fractions between $1/6$ and $1/3$ in a Farey sequence of order x . Let s_x denote the number of fractions before $1/7$ and t_x the number of fractions between $1/7$ and $2/7$ in a Farey sequence of order x . Let u_x denote the number of fractions before $1/8$ and v_x the number of fractions between $1/8$ and $1/4$ in a Farey sequence of order x . Empirical results are

(8) $\sum_{i=1}^x (o_{\lfloor x/i \rfloor} - p_{\lfloor x/i \rfloor} + \frac{2}{5})$ equals $\frac{4}{5}$, $\frac{1}{5}$, 0 , $-\frac{1}{5}$, or $-\frac{2}{5}$

(9) $\sum_{i=1}^x (q_{\lfloor x/i \rfloor} - r_{\lfloor x/i \rfloor} + \frac{1}{3})$ equals 1, $\frac{2}{3}$, $\frac{1}{3}$, 0, or $-\frac{1}{3}$

(10) $\sum_{i=1}^x (s_{\lfloor x/i \rfloor} - t_{\lfloor x/i \rfloor} + \frac{3}{7})$ equals $\frac{9}{7}$, $\frac{6}{7}$, $\frac{5}{7}$, $\frac{3}{7}$, $\frac{1}{7}$, 0, or $-\frac{3}{7}$

(11) $\sum_{i=1}^x (u_{\lfloor x/i \rfloor} - v_{\lfloor x/i \rfloor} + \frac{3}{8})$ equals $\frac{9}{8}$, $\frac{3}{2}$, $\frac{7}{8}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{1}{4}$, 0, or $-\frac{3}{8}$

In general, the differences between the number of fractions less than $1/I$ and greater than $1/I$ but less than $2/I$, $I = 5, 6, 7, \dots$ are incremented by $\frac{\lfloor (I+1)/2 \rfloor - 1}{I}$.

Let $\sigma_0(i)$ denote the number of positive divisors function. Another relationship is

Theorem (4) $\sum_{i=1}^x M(\lfloor x/i \rfloor) \log(i) \sigma_0(i) / 2 = \log(x!)$

A plot of $\log(x!)$ and $\sum_{i=1}^x (s_{\lfloor x/i \rfloor} - t_{\lfloor x/i \rfloor} + \frac{3}{7}) \sigma_0(i) / 2$ for $x = 2, 3, 4, \dots, 2000$ is

1.7958 times $\sum_{i=1}^x (s_{\lfloor x/i \rfloor} - t_{\lfloor x/i \rfloor} + \frac{3}{7}) \sigma_0(i) / 2$ is approximately equal to $\log(x!)$. A plot of the relative error is

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