On the hit problem for the algebra $H^*(B\mathbb{F}_2^s, \mathbb{F}_2)$ as a module over the mod two Steenrod algebra and applications

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Abstract: Let $\mathbb{F}_2^s$ be the elementary Abelian 2-group of rank s , which can be considered as a s -dimensional $\mathbb{F}_2$ -vector space. Denote the Steenrod algebra over the prime field $\mathbb{F}_2$ by $\mathcal{A}_2$. We are interested in determining the dimension of the space of the indecomposables $\mathbb{F}_2 \otimes_{\mathcal{A}_2} H^*(B\mathbb{F}_2^s, \mathbb{F}_2)$ in each s and degree $N \geq 0$, where $B\mathbb{F}_2^s$ is the classifying space of $\mathbb{F}_2^s$ and $H^*(B\mathbb{F}_2^s, \mathbb{F}_2)$ is isomorphic to $P_s := \mathbb{F}_2[x_1, \dots, x_s]$, the polynomial algebra on s variables of degree one as a module over $\mathcal{A}_2$. This problem, offered by Frank Peterson [10], is called the "hit problem". The solution of the hit problem is not yet known for $s \geq 5$ and all degrees N . The aim of the current research is to give an explicit description of the dimension of $\mathbb{F}_2 \otimes_{\mathcal{A}_2} H^*(B\mathbb{F}_2^s, \mathbb{F}_2)$ in the case $s = 5$ and generic degree $N_d := N = 5(2^d - 1) + 22 \cdot 2^d$ for d an arbitrary non-negative integer. Applying the results, we obtain the dimension of the space $\mathbb{F}_2 \otimes_{\mathcal{A}_2} H^*(B\mathbb{F}_2^6, \mathbb{F}_2)$ in the degree $5(2^{d+4} - 1) + N_1 \cdot 2^{d+4}$ for all $d > 0$ and the information about the behavior of the fifth Singer cohomological transfer in degree N_d . This work is a continuation of our recent work [14].

Key words: Steenrod algebra; Peterson hit problem; Adams spectral sequences; Primary cohomology operations; Actions of groups on commutative rings.

1. Introduction We write $\mathbb{F}_2$ for the field of characteristic two and $\mathcal{A}_2$ for the (mod two) Steenrod ring. It is known that $\mathcal{A}_2$ acts on mod 2 cohomology, making it indispensable computational tools in homotopy theory at the prime 2. As an algebra over $\mathbb{F}_2$, $\mathcal{A}_2$ is generated by the Steenrod squares Sq^i for $i \geq 0$ which act by post-composition. Let $B\mathbb{F}_2^s$ be the classifying space of the elementary Abelian 2-group $\mathbb{F}_2^s$ of rank s , which considered as a s -dimensional $\mathbb{F}_2$ -vector space. The action of $\mathcal{A}_2$ on the cohomology algebra $P_s := H^*(B\mathbb{F}_2^s, \mathbb{F}_2)$ has been a topic of much study (see also Nam [8], Wood [22]). This algebra could be identified with an unstable $\mathcal{A}_2$ -module $\mathbb{F}_2[x_1, \dots, x_s]$ [20]. One of a central problems of Algebraic topology is to determine a minimal set of generators for the ring of invariants $(P_s)^{G_s}$, in which G_s is a subgroup of the general linear group $GL(s, \mathbb{F}_2)$. Equivalently, we need to find the dimension of the $\mathbb{Z}$ -graded vector space

$$(1) \quad \{(\mathbb{F}_2 \otimes_{\mathcal{A}_2} (P_s)^{G_s})_N\}_{N \geq 0}.$$

The structure of this space has been thoroughly studied by many authors (see Hưng-Peterson [4], Janfada-Wood [5], Pengelley-Williams [9], Peterson [11], Singer [16, 17], and others). In addition, its ap-

plications to homotopy theory are surveyed by Peterson [11] and Singer [16]. For $G_s = \Sigma_s$, the space (1) is known by Janfada-Wood [5] with $s = 3$, and by the work of Singer [17] for the dual of (1) with $s \geq 0$. Here, Σ_s is the symmetric group of rank s . The problem when $G_s = GL(s, \mathbb{F}_2)$ was probed by Singer [16] for $s = 2$, and by Hưng-Peterson [4] for $s = 3, 4$. The case where $G_s = \{0\}$ was explicitly determined by Peterson [10] for $s = 1, 2$, by Kameko [6] for $s = 3$, by Sum [18] for $s = 4$. For $s = 5$, it has been investigated by the author [13, 14, 15], the author and Sum [12], Tín [19] in some certain degrees. But in the general case, the problem has been long open. In this text, we are primarily concerned with investigating $G_s = \{0\}$. For this, the space (1) becomes

$$(2) \quad \{(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_s)_N\}_{N \geq 0},$$

a modular representation of the group $GL(s, \mathbb{F}_2)$. Studying the structure of the space (2) is the content of the classical "hit problem", which is proposed by Franklin F. Peterson in [10]. The Kameko squaring operation [6] is one of the relatively efficient tools to approach the hit problem. It is a homomorphism $\widetilde{Sq}_*^0 : (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_s)_N \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_s)_{(N-s)/2}$,

determined by $\widetilde{Sq}_*^0([x]) = [y]$, if $x = \prod_{1 \leq j \leq s} x_j y^2$, while $\widetilde{Sq}_*^0([x]) = 0$ otherwise. It can be seen that $\widetilde{Sq}_*^0$ is an epimorphism. Specifically, Kameko [6] proves that if $\mu(N) = s$, then $\widetilde{Sq}_*^0$ is an isomorphism of $\mathbb{F}_2 GL(s, \mathbb{F}_2)$ -modules. Here $\mu(N)$ is the smallest number of integers of the form $2^j - 1$ (with repetitions allowed) whose sum is N . Moreover, by Wood [21], $(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_s)_N = 0$ for all N , if $\mu(N) > s$. It is to be noted that this establishes the 1987 conjecture of Peterson which first stimulated interest in the hit problem. As consequences, we only need to compute the spaces (2) in the "generic" degrees N of the "special" form (see also [13]):

$$(3) \quad N = r(2^d - 1) + m \cdot 2^d,$$

where $0 \leq \mu(m) < r \leq s$, and $d \geq 0$. Some results for $s = 5$ and these generic degrees have been investigated by us [12, 13, 14], Tân [19], and others. Up to now, it is still a mystery waiting to be solved in general.

The goal of this article is to continue to investigate and discuss a minimal set of $\mathcal{A}_2$ -generators for the space (2) in the cases $s = 5, 6$, and the generic degrees of the form (3). As an application, we probe the fifth cohomological transfer of W. Singer [16] in internal degrees given. Detailed proofs of these results will be published elsewhere. The following theorem is one of our main results.

Theorem 1. *With the generic degrees of the form (3), when $r = s = 5$ and $m = 22$, we have $N_d := N = 5(2^d - 1) + 22 \cdot 2^d$ and*

$$\dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_d} = \begin{cases} 965 & \text{if } d = 0, \\ 3012 & \text{if } d \geq 1. \end{cases}$$

We should note that the result when $d = 0$ was computed in [14]. We shall give the sketch of proof of the theorem for $d \geq 1$. Before we begin, we will briefly review some of the essential known results; the reader may wish to refer to Kameko [6], Phức-Sum [12], Phức [14] for further background detail.

Remark 2. The set of all the admissible monomials of degree N in P_s is a *minimal set of $\mathcal{A}_2$ -generators for P_s in degree N* . Therefore, the space of indecomposables $(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_s)_N$ has a basis consisting of all the classes represent by the admissible monomials of degree N in P_s .

For a natural number k , writing $\alpha_j(k)$ and $\alpha(k)$ for the j -th coefficients and the number of 1's in dyadic expansion of k , respectively. Thence,

$\alpha(k) = \sum_{j \geq 0} \alpha_j(k)$, and k can be written in the form $\sum_{j \geq 0} \alpha_j(k) 2^j$, where $\alpha_j(k)$ belongs to $\{0, 1\}$, for all $j \geq 0$. For a monomial $x = x_1^{a_1} x_2^{a_2} \dots x_s^{a_s}$ in P_s , we consider a sequence associated with x by $\omega(x) := (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots)$ where $\omega_i(x) = \sum_{1 \leq t \leq s} \alpha_{i-1}(a_t) \leq s$, for all $i \geq 1$. This sequence is called the *weight vector* of x .

Theorem 3 (see Kameko [6]). *Let x and y be two monomials in P_s and let q be an positive integer. Suppose that there is an index $t > q$ such that $\omega_t(x) = 0$, from which if y is inadmissible, then xy^{2^q} is, too.*

Theorem 4 (see Singer [17]). *Suppose that x is a monomial in P_s with $\mu(\deg(x)) \leq s$. Let z be the minimal spike in P_s with $\deg(z) = \deg(x)$. Then, if $\omega(x) < \omega(z)$, then x is $\mathcal{A}_2$ -decomposable.*

Let $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ be a sequence of non-negative integers. Then, the sequence ω are called the weight vector if $\omega_i = 0$ for $i \gg 0$. The set of all the weight vectors is given the left lexicographical order.

Theorem 5 (see Phức-Sum [12], Phức [14]). *All the spikes in P_s are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \dots)$ is weakly decreasing and $\omega_1 \leq s$, then there is a spike z in P_s such that $\omega(z) = \omega$.*

The strategy for the proof of Theorem 1. Noticing that $\mu(N_d) = 5$ for arbitrary $d > 1$. So, it suffices to determine the space of unhit elements $(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_d}$ for the cases $d = 0$ and 1, because the iterated Kameko homomorphism

$$(\widetilde{Sq}_*^0)^{d-1} : (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_d} \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_1}$$

is an isomorphism, for every positive integer d . When $d = 0$, we have $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_0} = 965$ (see [14]). Then, since Kameko's map

$$\widetilde{Sq}_*^0 : (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_1} \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_0}$$

is an epimorphism of $\mathbb{F}_2$ -vector spaces, to determine $(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_d}$ for the case $d = 1$, we need only to compute the kernel of $\widetilde{Sq}_*^0$. For this, first of all, we remark that if x is an admissible monomial of degree N_1 in P_5 such that $[x]$ belongs to $\text{Ker}(\widetilde{Sq}_*^0)$, then $\omega_1(x) = 3$. Indeed, we observe that $z = x_1^{31} x_2^{15} x_3^3$ is minimal spike of degree N_1 in P_5 and $\omega(z) = (3, 3, 2, 2, 1)$. Then, by Theorem 5, z is admissible. Because x is admissible and $\deg(x) = N_1$ is odd, according to Theorem 4, either $\omega_1(x) = 1$

or $\omega_1(x) = 5$. If $\omega_1(x) = 5$, then x is of the form $\prod_{1 \leq j \leq 5} x_j y^2$ with y is a monomial of degree 22 in P_5 . Since $\prod_{1 \leq j \leq 5} x_j y^2$ is admissible, following Theorem 3, y is admissible and so, $\widetilde{Sq}_*^0(\prod_{1 \leq j \leq 5} x_j y^2) = [y] \neq 0$. This contradicts with the fact that $[x] = [\prod_{1 \leq j \leq 5} x_j y^2] \in \text{Ker}(\widetilde{Sq}_*^0)$. Thus, we must have $\omega_1(x) = 3$ and therefore by the above arguments, x can be represented in the form $x_j x_i x_k y^2$, whenever $1 \leq i < j < k \leq 5$ and y is an admissible monomial of degree 23 in P_5 .

Now, combining the above remark and a result in [19], deduce that the weight vector $\omega(x)$ is one of the following sequences: $\omega_{(1)} := (3, 3, 2, 2, 1)$, $\omega_{(2)} := (3, 3, 4, 1, 1)$ and $\omega_{(3)} := (3, 3, 4, 3)$. So, we have an isomorphism $\text{Ker}(\widetilde{Sq}_*^0) \cong \bigoplus_{1 \leq j \leq 3} (\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_1}^{\omega_{(j)}}$, where $(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_1}^{\omega_{(j)}}$ are the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5$ generated by the classes $[x]$ such that $\omega(x) = \omega_{(j)}$. Thus, the theorem follows from the above arguments and the calculations of $(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_1}^{\omega_{(j)}}$.

In view of Theorem 1, the dimension of the indecomposables $\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5$ in degree $N = 49$ is quite large. So, to verify the result, we provide an algorithm in MAGMA [23], which is established as follows.

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N := 49;

R<x1,x2,x3,x4,x5> := PolynomialRing(GF(2),5);
P<t> := PolynomialRing(R);
Sqhom := hom<R->P | [x + t*x^2 : x in [x1,x2,x3,x4,x5]]>;
function Sq(j,x)
  c := Coefficients(Sqhom(x));
  if j+1 gt #c then
    return R!0;
  else
    return c[j+1];
  end if;
end function;
M := MonomialsOfDegree(R,N);
V := VectorSpace(GF(2),#M);
function MtoV(m)
  if IsZero(m) then
    return V!0;
  else
    cv := [Index(M,mm) : mm in Terms(m)];
    return CharacteristicVector(V,cv);
  end if;
end function;
M1 := MonomialsOfDegree(R,N-1);
M2 := MonomialsOfDegree(R,N-2);
M4 := MonomialsOfDegree(R,N-4);
M8 := MonomialsOfDegree(R,N-8);
M16 := MonomialsOfDegree(R,N-16);
M32 := MonomialsOfDegree(R,N-32);
print [#m : m in [M1,M2,M4,M8, M16, M32]];

> output: [270725, 249900, 211876, 148995, 66045, 5985]

S1 := [Sq(1,x) : x in M1];
print "S1";
H1 := sub<V | [MtoV(x) : x in S1]>;
print Dimension(H1);

> output: 140724

S2 := [Sq(2,x) : x in M2];
print "S2";
H2 := sub<V | [MtoV(x) : x in S2]>;
print Dimension(H2);

> output:

H := H1 + H2;
print Dimension(H);

> output: 247349

S4 := [Sq(4,x) : x in M4];
print "S4";
H4 := sub<V | [MtoV(x) : x in S4]>;
print Dimension(H4);

> output: 168824

H := H + H4;
print Dimension(H);

> output: 282594

S8 := [Sq(8,x) : x in M8];
print "S8";
H8 := sub<V | [MtoV(x) : x in S8]>;
print Dimension(H8);

> output: 139019

H := H + H8;
print Dimension(H);

> output: 289589

S16 := [Sq(16,x) : x in M16];
print "S16";
H16 := sub<V | [MtoV(x) : x in S16]>;
print Dimension(H16);

> output: 66020

H := H + H16;
print Dimension(H);

> output: 289813

S32 := [Sq(32,x) : x in M32];
print "S32";
H32 := sub<V | [MtoV(x) : x in S32]>;
print Dimension(H32);

> output: 0

H := H + H32;
print Dimension(H);

> output: 289813

print "\n",Dimension(V);

> output: 292825

print Dimension(H);

> output: 289813

print Dimension(V) - Dimension(H);

```

Thus, the algorithm indicates that

$$\begin{aligned} \dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_N &= \dim(V) - \dim(H) \\ &= 292825 - 289813 = 3012. \end{aligned}$$

Remark 6. In the above algorithm, to compute the hit monomials and the inadmissible monomials, we considered the Steenrod squares, Sq^j , are the homomorphisms $Sq^j : (P_5)_{N-j} \rightarrow (P_5)_N$, where $j = 2^t$, $t \geq 0$, $2j \leq N$, and $(P_5)_N$ denotes the $\mathbb{F}_2$ -subspace of P_5 consisting of all the homogeneous polynomials of degree N in P_5 . In fact, $\mathcal{A}_2$ is generated algebraically by the Steenrod squares Sq^{2^t} for every non-negative integer t and since $Sq^{2^t}(x) = 0$ if $\deg(x) = N - j < 2^t$, we need only to consider effects of Sq^{2^t} with $2^t \leq \deg(x)$. It is remarkable that by using the Kameko squaring operation and changing the number of variables in the polynomial ring " $R < x_1, x_2, \dots, x_5 >$ " and degree " N " in the algorithm, we can verify all the results of Peterson [10], Kameko [6] and Sum [18] on the dimension of the space (2) for $s \leq 4$. So, establishing an algorithm on a computer algebra is very useful for surveying the space (2).

2. Some applications We now mention two applications of the contributions of this paper. In work [18], Sum proved that

Theorem 7. *Let us consider the generic degrees of the form (3). If*

$$1 \leq s - 2 \leq \mu(m) = \alpha(m + \mu(m)) \leq s - 1,$$

then for each $d \geq s$, one has that

$$\dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_{s+1})_N = (2^{s+1} - 1) \dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_s)_m.$$

This result together with the iteration of the Kameko map $\widetilde{Sq}_*^0$ and Theorem 1 show that

Corollary 8. *With the degree N_d as in Theorem 1, for every positive integer d , we have*

$$\begin{aligned} \dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_6)_{5(2^{d+4}-1)+N_1 \cdot 2^{d+4}} \\ &= (2^6 - 1) \dim(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_d} \\ &= 189756. \end{aligned}$$

The next application of Theorem 1 is to investigate the behavior of the fifth Singer cohomological transfer [16] in the internal degree N_d . This transfer is a linear function

$$\varphi_5^{A_2} : [(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)^{GL(5, \mathbb{F}_2)}]^* \rightarrow \text{Ext}_{\mathcal{A}_2}^{5, 5+*}(\mathbb{F}_2, \mathbb{F}_2),$$

from the space of $GL(5, \mathbb{F}_2)$ -coinvariants of $\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5$ to the E_2 -term of the Adams spectral sequence. We also note that the domain of

$\varphi_5^{A_2}$ is isomorphic to $[PH_*(B\mathbb{F}_2^5, \mathbb{F}_2)]_{GL(5, \mathbb{F}_2)}$, where $PH_*(B\mathbb{F}_2^5, \mathbb{F}_2)$ is the subspace of the divided power algebra $H_*(B\mathbb{F}_2^5, \mathbb{F}_2) = \Gamma(a_1^{(1)}, \dots, a_5^{(1)})$ consisting of all elements that are annihilated by all positive degree Steenrod squares. The Singer transfer can be viewed as a useful tool for depicting the structure of the E_2 -terms, $E_2^{5, 5+*} = \text{Ext}_{\mathcal{A}_2}^{5, 5+*}(\mathbb{F}_2, \mathbb{F}_2)$. Besides that, the lambda algebra Λ [1] is also the best way to compute these mysterious Ext groups. This algebra is viewed as the E_1 -term of the classical Adams spectral sequence converging to the 2-component of the stable homotopy groups of spheres. Remarkably, in literature [3], Chơn and Hà constructed an interesting $\mathbb{F}_2$ -linear map $\delta_5 : H_*(B\mathbb{F}_2^5, \mathbb{F}_2) \rightarrow \Lambda^{5, *}$, which is considered as the E_1 -level of $\varphi_5^{A_2}$ and determined by the inductive formula:

$$\begin{aligned} &\delta_5(a_1^{(j_1)} a_2^{(j_2)} \dots a_5^{(j_5)}) \\ &= \sum_{i \geq j_5} \delta_4(\prod_{1 \leq k \leq 4} Sq_*^{i-j_5}(a_k^{(j_k)})) \lambda_i \end{aligned}$$

for any $a_1^{(j_1)} a_2^{(j_2)} \dots a_5^{(j_5)}$ in $H_*(B\mathbb{F}_2^5, \mathbb{F}_2)$. Here $\Lambda^{5, *}$ is an $\mathbb{F}_2$ -vector subspace of Λ spanned by all monomials in λ_i of length 5 and $a_k^{(1)}$ is linear dual to x_k . The authors pointed that if a element θ in $H_*(B\mathbb{F}_2^5, \mathbb{F}_2)$ is $\mathcal{A}_2$ -annihilated, then $\delta_5(\theta)$ is a cycle in Λ and is a representative of $\varphi_5^{A_2}([\theta])$. We shall use these data and the following theorem to compute the image of $\varphi_5^{A_2}$ in $\text{Ext}_{\mathcal{A}_2}^{5, 5+*}(\mathbb{F}_2, \mathbb{F}_2)$ with $* = N_d$.

Theorem 9. *With the generic degree N_d as in Theorem 1, the following assertion is true:*

$$[(\mathbb{F}_2 \otimes_{\mathcal{A}_2} P_5)_{N_d}^{GL(5, \mathbb{F}_2)}]^* = \begin{cases} 0 & \text{if } d = 0, \\ \langle [\zeta_d] \rangle & \text{if } d \geq 1, \end{cases}$$

where ζ_d belongs to $PH_{N_d}(B\mathbb{F}_2^5, \mathbb{F}_2)$ and is the following sum:

$$\begin{aligned} &a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(6 \cdot 2^{d-1}-1)} a_4^{(2 \cdot 2^{d-1}-1)} a_5^{(10 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(6 \cdot 2^{d-1}-1)} a_4^{(3 \cdot 2^{d-1}-1)} a_5^{(9 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(7 \cdot 2^{d-1}-1)} a_4^{(2 \cdot 2^{d-1}-1)} a_5^{(9 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(7 \cdot 2^{d-1}-1)} a_4^{(3 \cdot 2^{d-1}-1)} a_5^{(8 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(6 \cdot 2^{d-1}-1)} a_4^{(5 \cdot 2^{d-1}-1)} a_5^{(7 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(7 \cdot 2^{d-1}-1)} a_4^{(4 \cdot 2^{d-1}-1)} a_5^{(7 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(6 \cdot 2^{d-1}-1)} a_3^{(7 \cdot 2^{d-1}-1)} a_4^{(2 \cdot 2^{d-1}-1)} a_5^{(7 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(6 \cdot 2^{d-1}-1)} a_4^{(6 \cdot 2^{d-1}-1)} a_5^{(6 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(7 \cdot 2^{d-1}-1)} a_4^{(5 \cdot 2^{d-1}-1)} a_5^{(6 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(6 \cdot 2^{d-1}-1)} a_3^{(7 \cdot 2^{d-1}-1)} a_4^{(3 \cdot 2^{d-1}-1)} a_5^{(6 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(4 \cdot 2^{d-1}-1)} a_3^{(10 \cdot 2^{d-1}-1)} a_4^{(2 \cdot 2^{d-1}-1)} a_5^{(6 \cdot 2^{d-1}-1)} \\ &+ a_1^{(2^{d+4}-1)} a_2^{(6 \cdot 2^{d-1}-1)} a_3^{(8 \cdot 2^{d-1}-1)} a_4^{(2 \cdot 2^{d-1}-1)} a_5^{(6 \cdot 2^{d-1}-1)} \end{aligned}$$

$$\begin{aligned}
 & +a_1^{(2^{d+4}-1)}a_2^{(4.2^{d-1}-1)}a_3^{(10.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(5.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(6.2^{d-1}-1)}a_3^{(8.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(5.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(4.2^{d-1}-1)}a_3^{(11.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(5.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(7.2^{d-1}-1)}a_3^{(8.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(5.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(6.2^{d-1}-1)}a_3^{(7.2^{d-1}-1)}a_4^{(5.2^{d-1}-1)}a_5^{(4.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(7.2^{d-1}-1)}a_3^{(8.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(4.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(4.2^{d-1}-1)}a_3^{(11.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(4.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(4.2^{d-1}-1)}a_3^{(12.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(3.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(6.2^{d-1}-1)}a_3^{(10.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(3.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(7.2^{d-1}-1)}a_3^{(11.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(2.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(4.2^{d-1}-1)}a_3^{(6.2^{d-1}-1)}a_4^{(10.2^{d-1}-1)}a_5^{(2.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(4.2^{d-1}-1)}a_3^{(6.2^{d-1}-1)}a_4^{(8.2^{d-1}-1)}a_5^{(4.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(8.2^{d-1}-1)}a_3^{(8.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(3.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(7.2^{d-1}-1)}a_3^{(10.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(3.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(10.2^{d-1}-1)}a_3^{(7.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(3.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(11.2^{d-1}-1)}a_3^{(6.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(3.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(6.2^{d-1}-1)}a_3^{(11.2^{d-1}-1)}a_4^{(3.2^{d-1}-1)}a_5^{(2.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(14.2^{d-1}-1)}a_3^{(4.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(2.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(6.2^{d-1}-1)}a_3^{(12.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(2.2^{d-1}-1)} \\
 & +a_1^{(2^{d+4}-1)}a_2^{(10.2^{d-1}-1)}a_3^{(8.2^{d-1}-1)}a_4^{(2.2^{d-1}-1)}a_5^{(2.2^{d-1}-1)}.
 \end{aligned}$$

The proof of this theorem is based on Theorem 1, but the calculations are quite complicated. Now from the calculations of Lin [7] and Chen [2], it follows that

$$\text{Ext}_{\mathcal{A}_2}^{5,5+N_d}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_2 h_0 c_1 \rangle = 0 & \text{if } d = 0, \\ \langle h_{d+4} f_{d-1} \rangle & \text{if } d \geq 1 \end{cases}$$

with $h_{d+4} f_{d-1} \neq 0$. On the other side, notice that, for each positive integer d , the elements $\lambda_{2^{d+4}-1}$ and

$$\begin{aligned}
 \bar{f}_{d-1} = & \lambda_{4.2^{d-1}-1} \lambda_{6.2^{d-1}-1} \lambda_{7.2^{d-1}-1} \lambda_{5.2^{d-1}-1} \\
 & + \lambda_{4.2^{d-1}-1} \lambda_{8.2^{d-1}-1} \lambda_{6.2^{d-1}-1} \\
 & + \lambda_{8.2^{d-1}-1} \lambda_{6.2^{d-1}-1} \lambda_{4.2^{d-1}-1} \\
 & + \lambda_{8.2^{d-1}-1} \lambda_{6.2^{d-1}-1} \lambda_{5.2^{d-1}-1} \lambda_{3.2^{d-1}-1}
 \end{aligned}$$

are the cycles in $\Lambda^{1,2^{d+4}-1}$ and $\Lambda^{4,22.2^{d-1}-3}$, representatively. So, by direct calculations using the E_1 -level of $\varphi_5^{A_2}$ and Theorem 9, we get

$$\begin{aligned}
 \varphi_5^{A_2}([\zeta_d]) &= [\delta_5(\zeta_d)] \\
 &= [\lambda_{2^{d+4}-1} \bar{f}_{d-1} \\
 &\quad + \partial(\lambda_{2^{d+4}-1} \lambda_{4.2^{d-1}-1} \lambda_{6.2^{d-1}-1} \lambda_{12.2^{d-1}-1})] \\
 &= h_{d+4} f_{d-1},
 \end{aligned}$$

where ∂ denotes the differential $\Lambda \rightarrow \Lambda$, which is determined by $\partial(\lambda_{s-1}) = \sum_{j \geq 1} \binom{s-j-1}{j} \lambda_{s-j-1} \lambda_{j-1}$ for all $s > 0$ and $\partial(\lambda_0) = 0$. The above computation shows that $h_{d+4} f_{d-1}$ is detected by the fifth cohomological transfer for d an arbitrary positive integer. This, together with Theorem 9, yields the following.

Corollary 10. *The rank 5 transfer is an isomorphism in the internal degree N_d for all $d \geq 0$.*

3. Conclusion Solving the classical hit problem (or computing a monomial basis of the space (2)) with various applications of it will continue to be an important topic of Algebraic topology. So we believe that our results in this paper will make an important contribution for developing the hit problem of five variables and its applications.

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