A NEW EXPLICIT FORMULA FOR THE EULER NUMBERS IN TERMS OF THE STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. In this note, we derive a possibly new explicit formula for the Euler numbers in terms of the Stirling numbers of the second kind.

Definition 1. The *Euler numbers* are a sequence of integers, denoted by E_n , which can be defined by the following Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n,$$

where $\cosh t$ is the hyperbolic cosine.

Definition 2. The *Stirling numbers of the second kind*, denoted by $S(n, m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

We prove the following

Theorem 1. *We have*

$$E_r = 2^{2r-1} \sum_{k=1}^r \frac{(-1)^k S(r, k)}{k+1} \left(3 \left(\frac{1}{4} \right)^{(k)} - \left(\frac{3}{4} \right)^{(k)} \right), \quad (1)$$

where $x^{(n)} = (x)(x+1) \cdots (x+n-1)$ denotes the rising factorial.

Proof. We begin with the following result

$$\frac{\sin n\pi}{\pi} \int_0^{\infty} x^{n-1} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \zeta(s) - \zeta(s, 1-n), \quad (2)$$

where $\mathbf{Li}_s(-x)$ denotes the Polylogarithm function, $\zeta(s)$ is the Riemann zeta function, and $\zeta(s, 1-n)$ is the Hurwitz zeta function. The above integral is valid for all $s \in \mathbb{C} \setminus \{1\}$, and $0 < n < 1$. This integral can be obtained from formula 3.2.1.6 in the book [1].

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Plugging in $n = 3/4$, and $n = 1/4$ in the equation (2), we get

$$\int_0^\infty x^{-3/4} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \sqrt{2}\pi(\zeta(s) - \zeta(s, 3/4)) \quad (3)$$

$$\int_0^\infty x^{-1/4} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \sqrt{2}\pi(\zeta(s) - \zeta(s, 1/4)) \quad (4)$$

Subtracting (4) from (3), we get

$$\int_0^\infty \frac{\mathbf{Li}_s(-x)}{1+x} (x^{-3/4} - x^{-1/4}) dx = \sqrt{2}\pi(\zeta(s, 1/4) - \zeta(s, 3/4)) = \sqrt{2}\pi 4^s \beta(s), \quad (5)$$

where $\beta(s)$ is the Dirichlet Beta function.

Now plugging, $s = -r$, a negative integer in the equation (5), we get

$$\sqrt{2}\pi 4^{-r} \beta(-r) = \int_0^\infty \frac{\mathbf{Li}_{-r}(-x)}{1+x} (x^{-3/4} - x^{-1/4}) dx \quad (6)$$

Now, we use the following representation from the note [2]

$$\mathbf{Li}_{-r}(-x) = \sum_{k=1}^r k! S(r, k) \left(\frac{1}{1+x} \right)^{k+1} (-x)^k,$$

which can be easily proved using induction on r .

As a result, we have

$$\begin{aligned} \int_0^\infty x^{-1/4} \frac{\mathbf{Li}_{-r}(-x)}{1+x} dx &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \int_0^\infty \frac{x^{k-1/4}}{(1+x)^{k+2}} dx \\ &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \cdot \frac{\Gamma(k+3/4) \Gamma(5/4)}{\Gamma(k+2)} \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \Gamma(k+3/4) \Gamma(5/4) \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4} \right) \cdot \left(\frac{3}{4} + 1 \right) \cdots \left(\frac{3}{4} + k - 1 \right) \Gamma(3/4) \Gamma(5/4) \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4} \right) \cdot \left(\frac{3}{4} + 1 \right) \cdots \left(\frac{3}{4} + k - 1 \right) \frac{\pi}{2\sqrt{2}} \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4} \right)^{(k)} \frac{\pi}{2\sqrt{2}}, \end{aligned}$$

and

$$\int_0^\infty x^{-3/4} \frac{\mathbf{Li}_{-r}(-x)}{1+x} dx = \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{1}{4}\right)^{(k)} \frac{3\pi}{2\sqrt{2}}.$$

But, from [3], we know that

$$\beta(-r) = \frac{E_r}{2}. \quad (7)$$

Now, we can deduce our main result (1) readily. $\square$

Remark 1. In [4], the author proved the following explicit formula for the *even* Euler numbers

$$E_{2l} = -4^{2l} \sum_{k=1}^{2l} (-1)^k \cdot \frac{S(2l, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}.$$

This along with our result (equation (1)) gives us

$$E_{2l} = 3 \cdot 4^{2l} \sum_{k=1}^{2l} (-1)^k \cdot \frac{S(2l, k)}{k+1} \cdot \left(\frac{1}{4}\right)^{(k)}.$$

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