2-D AXIOMATIC CONFORMAL FIELD THEORY

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Abstract

In this article I have tried to give a self-contained introduction to two-dimensional conformally field theory through some basic concepts and postulates – that is using a system of axioms as presented in [1] and based on the work of Osterwalder and Schrader [2], [3]. I will assume the Euclidean signature $(+,+)$ on $\mathbb{R}^2$ (or on Riemann surfaces), as it is customary because of the close connection of conformal field theory to statistical mechanics ([4] and [5]) and its relation to complex analysis.

Objects of a Two-Dimensional Quantum Field Theory

The basic objects of a two-dimensional quantum field theory are:

1. the fields φ_i , $i \in B_0$, subject, to a number of properties:
 - a. These fields are also called field operators or operators.
 - b. They are defined as maps φ from Riemann surfaces $\mathcal{M}$ to a Hilbert space $\mathbb{H}$, and set of all φ is:

$$\Phi := \{\varphi: \mathcal{M} \rightarrow \mathcal{O}(\mathbb{H}) \mid \mathbb{H} := \mathcal{S}(\mathcal{M})\}$$

Where $\mathcal{S}(\mathcal{M})$ is Schwartz space of $\mathcal{M}$.i.e. set of function f from $\mathcal{M}$ to $\mathbb{C}$ such that $\sup_{z \in \mathbb{C}} |z^{\alpha, \bar{\alpha}} \partial_{z^{\beta, \bar{\beta}}} f| < \infty$ for all $\alpha, \bar{\alpha}, \beta, \bar{\beta} \in \mathbb{N}$, where $z^{\alpha, \bar{\alpha}} := \bar{z}^{\bar{\alpha}} z^{\alpha}$ and $\partial_{z^{\beta, \bar{\beta}}} = (\partial_{\bar{z}})^{\bar{\beta}} (\partial_z)^{\beta}$

2. Green Function or so-called correlation functions:

$$G_{\Xi(N)}(z^{N, \bar{N}}) := \left\langle \Omega, \prod_{i \in I_n} \varphi_{\Xi(i)}(z^i, \bar{z}^i) \Omega \right\rangle$$

Where $I_n := \{1, \dots, n\}$ and B_0 be a countable indexed set, also

$$\Xi \in \Theta := \left\{ \iota: I_n^n \rightarrow B_0^n, \iota(N) := (\iota(i))_{i=1}^n \text{ and } \iota(i) \neq \iota(j) \Leftrightarrow i \neq j \text{ for all } n \right\},$$

such that $N := (i)_{i=1}^n \in I_n^n$ and $\Omega \in \mathbb{H}$ and $\varphi_{\Xi(i)} \in \Phi$ for all $i \in N$. In our special case in which $\mathbb{H} := \mathcal{S}(\mathcal{M})$

$$G_{\Xi(N)}(z^N) := \int_{p \in \mathcal{M}} \Omega(p) \prod_{i \in I_n} \varphi_{\Xi(i)}(z^i, \bar{z}^i) \Omega(p)$$

For all $n \in \mathbb{N}_0$ where,

$$z^N \in M_n := \left\{ z^N := (z^i)_{i=1}^n \in \mathbb{C}^n \mid z^i \neq z^j \Leftrightarrow i \neq j \right\}$$

3. The other objects:

- a. $M_n^+ := \{z^N \in M_n \mid \operatorname{Re}(z^i) > 0 \text{ for all } i \in I\}$
- b. $M_n^\odot := \{z^N \in M_n \mid z^k = e^{\tau + i\alpha}, |z^k| > |z^m| \Leftrightarrow k > m \text{ for all } k, m \in I\}$
- c. $\mathcal{S}_0^+ := \mathbb{C}$
- d. $\mathcal{S}_n^+ := \{f \in \mathcal{S}(\mathbb{C}^n) \mid \operatorname{supp}(f) \subset M_n^+\}$

Where, $\mathcal{S}(\mathbb{C}^n)$ is the Schwartz space of $\mathbb{C}^n$ and

$$\operatorname{supp}(f) := \{z^N \in \mathbb{C}^n \mid f(z^{N, \bar{N}}) \neq 0\}$$

4. Conformal Transformations group is generated by:

- a. The group $E = E_2$ of Euclidean motions, that is the Euclidean group (which corresponds to the Poincare group in this context), is generated by the
 - i. Rotations

$$r_\alpha: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto e^{i\alpha} z, \alpha \in \mathbb{R}$$

- ii. Translations

$$t_a: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto z + a, a \in \mathbb{C}$$

- b. Mobius transformations (Mb) are the
 - i. dilatations

$$d_\tau: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto e^\tau z, \tau \in \mathbb{R}$$

- ii. Inversion

$$\iota: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto z^{-1}$$

- c. time reflection

$$\theta: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto -\bar{z}$$

Osterwalder–Schrader Axioms

First thing first, the correlation functions are defined as:

$$G_{\Xi(N)} : M_n \rightarrow \mathbb{C}$$

Where $G_\emptyset = 1$ For all $n \in \mathbb{N}_0$

- **Axiom 1 | Locality**: Let σ is permutation of N , where $\sigma(N) := (\sigma(i))_{i=1}^n$, then

$$G_{\Xi(N)}(z^{N,\bar{N}}) = G_{\Xi(\sigma(N))}(z^{\sigma(N),\overline{\sigma(N)}})$$

- **Axiom 2 | Covariance**: For every $\Xi(i) \in B_0$ there are conformal weights

$$h_{\Xi(i)}, \bar{h}_{\Xi(i)} \in \mathbb{R}$$

$$G_{\Xi(N)}(z^{N,\bar{N}}) = \prod_{i=1}^n \left(\frac{dw}{dz}(z^i) \right)^{h_{\Xi(i)}} \left(\overline{\frac{dw}{dz}(z^i)} \right)^{\bar{h}_{\Xi(i)}} G_{\Xi(N)}(w(z)^{N,\bar{N}})$$

For all $n \in \mathbb{N}_0$ here, $s_{\Xi(i)} := h_{\Xi(i)} - \bar{h}_{\Xi(i)}$ is called the conformal spin for the index $\Xi(i)$ and $d_{\Xi(i)} := h_{\Xi(i)} + \bar{h}_{\Xi(i)}$ is called the scaling dimension. Furthermore, we assume

$$s_{\Xi(i)}, d_{\Xi(i)} \in \mathbb{Z} \quad \text{for all } \Xi(i) \in B_0$$

Which implies,

$$h_{\Xi(i)}, \bar{h}_{\Xi(i)} \in \frac{1}{2}\mathbb{Z}$$

- **Axiom 3 | Reflection Positivity**: There exist a map $*$: $B_0^n \rightarrow B_0^n$ for all n such that:

a. $* \circ * := id$

b. $G_{\Xi(N)}(z^{N,\bar{N}}) = G_{\Xi(N)*}(\theta(z)^{N,\bar{N}}) = G_{\Xi(N)*}(-z^{\bar{N},N})$ where θ is time reversal

- c. Let $\underline{\mathcal{S}}^+$ be the space of all sequences $\underline{f} := (f_{\Xi(N)})_{|N| \in \mathbb{N}_0}$, where, if $N = (i)_{i=1}^n$ and $|N| := n$ with $f_{\Xi(N)} \in \mathcal{S}_n^+$ for $\Xi(N) \in B_0^n$ and $f_{\Xi(N)} \neq 0$ for only finite n

$$\langle \underline{f}, \underline{f} \rangle \geq 0 \text{ for all } \underline{f} \in \underline{\mathcal{S}}^+$$

Here we define $\langle \underline{f}, \underline{f} \rangle$ as:

$$\sum_{\Xi \in \Theta} \sum_{|N|, |K| \in \mathbb{N}_0} \int_{M_{|N|+|K|}} G_{\Xi(N)^*, \Xi(K)}(\theta(z)^N, v^K) \overline{f_{\Xi(N)}(z^N)} f_{\Xi(K)}(v^K) d^N z d^K v$$

Reconstruction of the Hilbert Space: This axiom yields a positive semi-definite Hermitian form H on $\underline{\mathcal{S}}^+$ and hence the Hilbert space $\mathbb{H}$ as the completion of $\underline{\mathcal{S}}^+ / \ker H$ with the inner product $\langle \cdot, \cdot \rangle$.

Reconstruction of the Field Operators:

In this construction yields a unitary representation U of the group E of Euclidean motions of the plane in $\mathbb{H}$. Moreover, there exists a dense subspace $\mathbb{D} \subset \mathbb{H}$ left invariant by the unitary representation such that the maps $\varphi_j(f) : [\underline{g}] \rightarrow [\underline{g} \times f]$ are defined on $\mathbb{D}$ for all $j \in \mathbb{N}_0$ and $\varphi_{\Xi(j)}(f)(\mathbb{D}) \subset \mathbb{D}$. In addition, with the $\Omega \in \mathbb{H}$.

Where $[\underline{g}] \in \underline{\mathcal{S}}^+ / \ker H$, $f \in \mathcal{S}_\infty^+ \supset \mathcal{S}_n^+$ and

$$\underline{g} \times f := \begin{cases} (g_{\Xi(N)} \otimes f_{\Xi(i)})_{|N|, i \in \mathbb{N}_0} & \Xi(i) \neq \emptyset \\ (g_{\Xi(N)} \otimes f_\emptyset)_{|N| \in \mathbb{N}_0} & \text{otherwise} \end{cases}$$

Also,

$$(g_{\Xi(N)} \otimes f_{\Xi(i)})(z^N, z^{|N|+1}) := g_{\Xi(N)}(z^N) f_{\Xi(i)}(z^{|N|+1})$$

(namely $\Omega = [f]$, with $f_0 = 1$ and $f_i = 1$ for $i = 0$) the following properties are satisfied:

- For all $\Xi(j) \in B_0$ the mapping $\varphi_{\Xi(j)} : \mathcal{S}_1^+ \rightarrow \text{End}(\mathbb{D})$ is linear, and $\varphi_{\Xi(j)}$ is a field operator.
- he fields $\varphi_{\Xi(j)}$ transform covariantly with respect to the representation U .

$$U(w) \circ \varphi_{\Xi(j)}(z^N) \circ U(w)^{-1} = \left(\frac{\partial w}{\partial z} \right)^{h_{\Xi(i)}} \circ \varphi_{\Xi(j)}(w(z^N))$$

- Let $M_n^> := \{z^N \in M_n^+ \mid \text{Re}(z^i) > \text{Re}(z^j) \Leftrightarrow i > j \text{ for all } i, j \in I\}$

$$G_{\Xi(N)} : M_n^> \rightarrow \mathbb{C}$$

$G_{\Xi(N)}$ is holomorphic on $M_n^>$, same is true for

$$G_{\Xi(N)} : M_n^\odot \rightarrow \mathbb{C}$$

The largest domain subset of $\mathbb{C}^n$, for which $G_{\Xi(N)}$ is holomorphic is not known.

- Axiom 4 | Scaling Covariance:** The correlation functions $G_{\Xi(N)}$ satisfy also for the dilatations $w(z) = e^\tau z$, $\tau \in \mathbb{R}$. Hence

$$G_{\Xi(N)}(z^{N,\bar{N}}) = (e^\tau)^{h_N+h_{\bar{N}}} G_{\Xi(N)}(e^\tau z^{N,\bar{N}})$$

Conformal Ward Identities: Under the assumption that the correlation function $G_{\Xi(N)}(z^{N,\bar{N}})$ satisfies the covariance condition for all Mobius transformations the following Ward identities hold:

$$\sum_{i \in I} \partial_{z^i} G_{\Xi(N)}(z^{N,\bar{N}}) = 0$$

$$\sum_{i \in I} (z^i \partial_{z^i} + h_i) G_{\Xi(N)}(z^{N,\bar{N}}) = 0$$

$$\sum_{i \in I} ((z^i)^2 \partial_{z^i} + 2h_i z^i) G_{\Xi(N)}(z^{N,\bar{N}}) = 0$$

- **Axiom 5 | Existence of the Energy–Momentum Tensor:** Among the set of fields $\{\varphi_{\Xi(N)}\}_{|N| \in \mathbb{N}_0}$ there are four fields, $T_{\mu\nu} \in \{\varphi_{\Xi(N)}\}_{|N| \in \mathbb{Z}} \subset \{\varphi_{\Xi(N)}\}_{|N| \in \mathbb{N}_0}$ where $\mu, \nu \in \{\tau, \sigma\}$, also $\llbracket T \rrbracket$ is matrix $[T_{\mu\nu}]_{\mu, \nu \in \{\tau, \sigma\}}$ with the following properties:

- T symmetric i.e. $T_{\mu\nu} = T_{\nu\mu} \overline{T_{\mu\nu}(z)} = T_{\mu\nu}(\theta(z))$
- $\text{tr } \llbracket T \rrbracket = T_{\tau\tau} + T_{\sigma\sigma} = 0$
- $\partial_\tau T_{\mu\tau} + \partial_\sigma T_{\mu\sigma} = 0$
- $d(T_{\mu\nu}) := h_{\mu\nu} + \bar{h}_{\mu\nu} = 2$
- $s(2T) = 2$ and $s(2T) = -2$ where $T := \frac{(T_{\tau\tau} - T_{\sigma\sigma}) - i(T_{\tau\sigma} + T_{\sigma\tau})}{2} = T_{\tau\tau} - iT_{\tau\sigma}$

Luscher–Mack Theorem: The **Axiom 1-5** imply

- T is independent of $\bar{z}$ hence holomorphic, hence $\bar{T}$ is antiholomorphic also, the respective conformal weight h and $\bar{h}$ and $h(T) = 2 = \bar{h}(\bar{T})$ and $h(\bar{T}) = 0 = \bar{h}(T)$.
- For Laurent series

$$T(z) = \sum_{n \in \mathbb{Z}} L_{-n} z^n, \quad \bar{T}(z) = \sum_{n \in \mathbb{Z}} \bar{L}_{-n} \bar{z}^n$$

where

$$L_{-n} := \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{T(\zeta)}{\zeta^{n+1}} d\zeta, \quad \bar{L}_{-n} := \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{T}(\zeta)}{\bar{\zeta}^{n+1}} d\bar{\zeta}$$

The operator $L_{-n}, \bar{L}_{-n}$ satisfy the commutation relations of two commuting Virasoro algebras with the same central charge $c \in \mathbb{C}$.

- $[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}$
- $[\bar{L}_n, \bar{L}_m] = (n - m)\bar{L}_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}$
- $[L_n, \bar{L}_m] = 0$

c. The representations of the Virasoro algebra defined by L_n and $\bar{L}_n$, respectively, are unitary:

- $L_n^* = L_{-n}$
- $\bar{L}_n^* = \bar{L}_{-n}$

Where L_n^* is conjugate of L_n and same for $\bar{L}_n^*$ also it's very obvious by definition that $\bar{L}_n$ not conjugate of L_n as $\bar{L}_{-n}$ is not conjugate of L_{-n} .

The operators $L_n, \bar{L}_n$ define a unitary representation of $Vir \times \overline{Vir}$. In general, this representation decomposes into unitary highest-weight representations as follows:

$$\oplus (W(c, h) \otimes W(c, \bar{h}))$$

where one has to sum over a suitable collection of central charges c and conformal weights $h, \bar{h}$. The theory is called minimal, if this sum is finite.

$$T(z)T(v) \sim \frac{c}{2} \frac{1}{(z-v)^4} + \frac{2T}{(z-v)^2} + \frac{dT}{dz}(v) \frac{1}{(z-v)}$$

Definition | Primary Field: A conformal field $\varphi_{\Xi(i)}$, $i \in I_n$, for all $n \in \mathbb{N}_0$ is called a primary field if:

$$[L_n, \varphi_{\Xi(i)}(z)] = z^{n+1} \partial_z \varphi_{\Xi(i)}(z) + h_{\Xi(i)}(n+1)z^n \varphi_{\Xi(i)}(z)$$

Conformal Ward Identities: For every correlation function $G_{\Xi(N)}(z^{N, \bar{N}})$ where all the fields $\varphi_{\Xi(i)}$ are primary the Ward identities.

$$\sum_{i \in I_n} \left((z^i)^{m+1} \partial_{z^i} + h_{\Xi(i)}(m+1)(z^i)^m \right) G_{\Xi(N)}(z^{N, \bar{N}}) = 0$$

are satisfied for all $m \in \mathbb{Z}$

$$L_0 b_{\Xi(i)} = h_{\Xi(i)} b, \quad L_m v = 0, n > 0$$

asymptotic state $b_{\Xi(i)} = \lim_{z \rightarrow 0} \varphi_{\Xi(i)}(z) \Omega$ exists as a vector in the Hilbert space $\mathbb{H}$, and $h_{\Xi(i)}$ is the conformal weight

In a quantum field theory satisfying **Axioms 1–5**. The conformal family $[\varphi_{\Xi(i)}]$ for $i \in I_n$, for all $n \in \mathbb{N}_0$ is the complex vector space generated by

$$\varphi_{\Xi(i)}^{\alpha(K)}(z) := \prod_{q \in I_k} L_{-\alpha(q)}(z) \varphi_{\Xi(i)}(z)$$

here $\alpha(K) := (\alpha(q))_{i=1}^k \in \mathbb{N}^k$ where $K := (q)_{i=1}^k$, $i < j \Rightarrow \alpha_i \geq \alpha_j$ and $\alpha_i > 0$ for all $j \in I_k$, $k \in \mathbb{N}$

$$L_{-n}(z) := \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{T(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for $z \in \mathbb{C}$. The operators $\varphi_{\Xi(i)}^{\alpha(q)}(z)$ are called **secondary fields** or **descendants**. The operators $L_{-n}(z)$ are in close connection with the Virasoro generators L_n because of

$$L_{-n} := \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{T(\zeta)}{\zeta^{n+1}} d\zeta = L_{-n}(0)$$

The secondary fields $\varphi_{\Xi(i)}^{\alpha(K)}$ can be expressed as integrals as well. For instance, for $\varphi_{\Xi(i)}^{\alpha(q)}$, $q \in \mathbb{N}$

$$\varphi_{\Xi(i)}^{\alpha(q)}(z) := L_{-\alpha(q)}(z) \varphi_{\Xi(i)}(z) := \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{T(\zeta)}{(\zeta - z)^{n+1}} \varphi_{\Xi(i)}(z) d\zeta$$

Similarly

$$\varphi_{\Xi(i)}^{\alpha(K)}(z) = \frac{1}{(2\pi i)^k} \int_{(|\zeta^q|=1)_{q=1}^k} \prod_{q \in I_k} \frac{T(\zeta^q)}{(\zeta^q - z^q)^{n+1}} \varphi_{\Xi(i)}(z^q) d\zeta^K$$

where

$$\int_{(|\zeta^q|=1)_{q=1}^k}^* := \int_{|\zeta^1|=1} \dots \int_{|\zeta^k|=1}^*$$

And

$$d\zeta^K := \prod_{q \in I_k} d\zeta^q$$

State Field Correspondence

Assume that the asymptotic states of the primary fields together with their descendants generate a dense subspace V of $\mathbb{H}$. Then to each state $a \in V$ there corresponds a field $\varphi_{\Xi(i)}$ such that $\lim_{z \rightarrow 0} \varphi_{\Xi(i)}(z)\Omega = a$.

Operator Product Expansion

For the primary fields of a conformal field theory it is postulated that they obey the following operator product expansion (OPE)

$$\varphi_{\Xi(i)}(z)\varphi_{\Xi(j)}(v) \sim \sum_{k \in I_n} C_{\Xi(i), \Xi(j)}^{\Xi(k)}(z-v)^{h_{\Xi(k)}-h_{\Xi(i)}-h_{\Xi(j)}} \varphi_{\Xi(k)}(v)$$

Where $z, v \in \mathbb{C}$

The **central object of conformal field theory** is the determination of

- the scaling dimensions $d_{\Xi(i)} = h_{\Xi(i)} + \bar{h}_{\Xi(i)}$
- the central charge c_i for the family $[\varphi_{\Xi(j)}]$, and
- The coefficients $C_{\Xi(i), \Xi(j)}^{\Xi(k)}$ structures constantes

Axiom 6 | Operator Product Expansion: The primary fields have the OPE and this OPE is associative.

The associativity of the OPE allows to represent G with the aid of so-called holomorphic and antiholomorphic, respectively “conformal blocks” $F_{\Xi(k)}, \bar{F}_{\Xi(k)}$:

$$G_{\Xi(i), \Xi(j)}(z, \bar{z}) = \sum_{k \in \mathbb{N}_0} C_{\Xi(i), \Xi(j)}^{\Xi(k)} C_{\Xi(i), \Xi(j)}^{\Xi(k)} F_{\Xi(k)}(z) \bar{F}_{\Xi(k)}(\bar{z})$$

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