COMPUTING SUMS IN TERMS OF BETA, POLYGAMMA, AND GAUSS HYPERGEOMETRIC FUNCTIONS

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Dedicated to people facing and battling COVID-19

ABSTRACT. In the paper, by virtue of the binomial inversion formula, a general formula of higher order derivatives for a ratio of two differentiable function, and other techniques, the authors compute several sums in terms of the beta function and its partial derivatives, polygamma functions, the Gauss hypergeometric function, and a determinant. These results generalize known ones in combinatorics.

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1. BACKGROUND AND MOTIVATIONS

In [14, Theorem 1] and [23, p. 80, Eq. (7.5)], it was obtained that

$$\sum_{q=0}^n \binom{n}{q} \frac{(-1)^q}{q+k} = \frac{1}{k \binom{k+n}{k}} \quad (1.1)$$

for $k \geq 1$ and $n \geq 0$. The binomial inversion formula in [4, p. 192, (5.48)] reads that

$$g(n) = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell f(\ell) \iff f(n) = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell g(\ell). \quad (1.2)$$

Applying (1.2) into (1.1) yields

$$\sum_{q=0}^n \binom{n}{q} \frac{(-1)^q}{\binom{q+k}{k}} = \frac{k}{k+n} \quad (1.3)$$

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for $k, n \geq 0$. This can be rewritten as

$$\sum_{q=0}^n \binom{n}{q} [(-1)^q B(k, q+1)] = \frac{1}{k+n} \quad (1.4)$$

for $k \geq 1$ and $n \geq 0$, where

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt, \quad \Re(z), \Re(w) > 0$$

denotes the classical Euler beta function. The beta function $B(z, w)$ and the classical Euler gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

have the relation

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \Re(x), \Re(y) > 0.$$

The logarithmic derivative $[\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ is denoted by $\psi(z)$ and the derivatives $\psi^{(k)}(z)$ for $k \geq 0$ are called polygamma functions. For very recent results on the beta, gamma, and polygamma functions, please refer to the papers [22, 25, 26, 27, 28, 29] and closely related references therein.

In this paper, we will extend the identities (1.1), (1.3), and (1.4), compute generalized sums in terms of the beta function $B(z, w)$ and its partial derivatives, polygamma functions $\psi^{(k)}(z)$ for $k = 0, 1$, the Gauss hypergeometric function ${}_2F_1(-n, z; 1+z; -1)$, a determinant, and an identity.

2. COMPUTING SUMS IN TERMS OF BETA AND POLYGAMMA FUNCTIONS

In this section, via replacing corresponding integers k by a complex variable $\Re(z)$, we generalize identities (1.1), (1.3), and (1.4) in terms of the beta function and its partial derivatives, polygamma functions, and integrals as follows.

Theorem 2.1. *Let $n \geq 0$ be an integer and $\Re(z) > 0$. Then*

$$\sum_{q=0}^n \binom{n}{q} \frac{(-1)^q}{q+z} = B(z, n+1) \quad (2.1)$$

and

$$\sum_{q=0}^n \binom{n}{q} [(-1)^q B(z, q+1)] = \frac{1}{z+n}. \quad (2.2)$$

Proof. Let

$$f(t) = \sum_{q=0}^n \frac{(-1)^q}{q+z} \binom{n}{q} t^{q+z}, \quad t \geq 0.$$

It is obvious that $f(0) = 0$ and

$$f'(t) = \sum_{q=0}^n (-1)^q \binom{n}{q} t^{q+z-1} = t^{z-1} \sum_{q=0}^n (-1)^q \binom{n}{q} t^q = t^{z-1} (1-t)^n.$$

Integrating on both sides of the above equality with respect to t over $[0, 1]$ gives $f(1) = B(z, n+1)$. The formula (2.1) is thus proved.

The identity (2.2) follows from applying the binomial inversion formula (1.2) to (2.1). The proof of Theorem 2.1 is complete. $\square$

Remark 2.1. The identity (2.1) recovers [23, p. 82, Eq. (7.7)].

Theorem 2.2. *Let $m, n \geq 0$ be an integer and $\Re(z) > 0$. Then*

$$\sum_{q=0}^n \binom{n}{q} \frac{(-1)^q}{(q+z)^{m+1}} = \frac{(-1)^m}{m!} \frac{\partial^m B(z, n+1)}{\partial z^m}$$

and

$$\sum_{q=0}^n (-1)^q \binom{n}{q} \frac{\partial^m B(z, n+1)}{\partial z^m} = \frac{(-1)^m m!}{(z+n)^{m+1}}.$$

In particular, we have

$$\begin{aligned} \sum_{q=0}^n \binom{n}{q} \frac{(-1)^q}{(q+z)^2} &= [\psi(z+n+1) - \psi(z)] B(z, n+1), \\ \sum_{q=0}^n \binom{n}{q} \frac{(-1)^q}{(q+z)^3} &= \frac{\{[\psi(z) - \psi(z+n+1)]^2 + \psi'(z) - \psi'(z+n+1)\} B(z, n+1)}{2}, \\ \sum_{q=0}^n \binom{n}{q} (-1)^q [\psi(q+z+1) - \psi(z)] B(z, q+1) &= \frac{1}{(z+n)^2}, \end{aligned}$$

and

$$\begin{aligned} \sum_{q=0}^n \binom{n}{q} (-1)^q \{[\psi(z) - \psi(q+z+1)]^2 + \psi'(z) - \psi'(q+z+1)\} B(z, q+1) \\ = \frac{2}{(z+n)^3}. \end{aligned}$$

Proof. These identities follow from differentiating with respect to z on both sides of (2.1) and (2.2) and employing partial derivatives

$$\frac{\partial B(z, w)}{\partial z} = B(z, w) [\psi(z) - \psi(z+w)]$$

and

$$\frac{\partial^2 B(z, w)}{\partial z^2} = B(z, w) \{[\psi(z) - \psi(z+w)]^2 + \psi'(z) - \psi'(z+w)\}.$$

The proof of Theorem 2.2 is complete. $\square$

Theorem 2.3. *Let $m, n \geq 0$ be an integer and $\Re(z) > 0$. Then*

$$\sum_{q=0}^n \binom{n}{q} \frac{1}{(q+z)^{m+1}} = \frac{(-1)^m}{m!} \int_0^1 t^{z-1} (1+t)^n (\ln t)^m dt \quad (2.3)$$

and

$$\sum_{q=0}^n \binom{n}{q} \left[(-1)^q \int_0^1 t^{z-1} (1+t)^q (\ln t)^m dt \right] = \frac{(-1)^{m+n} m!}{(z+n)^{m+1}}. \quad (2.4)$$

Proof. Let

$$h(t) = \sum_{q=0}^n \frac{1}{q+z} \binom{n}{q} t^{q+z}, \quad t \geq 0.$$

It is apparent that $h(0) = 0$ and

$$h'(t) = \sum_{q=0}^n \binom{n}{q} t^{q+z-1} = t^{z-1} \sum_{q=0}^n \binom{n}{q} t^q = t^{z-1} (1+t)^n.$$

Integrating the above equation with respect to t over $[0, 1]$ arrives at

$$h(1) = \int_0^1 t^{z-1} (1+t)^n dt.$$

Consequently, it follows that

$$\sum_{q=0}^n \binom{n}{q} \frac{1}{q+z} = \int_0^1 t^{z-1} (1+t)^n dt. \quad (2.5)$$

Further differentiating m times with respect to z on both sides of (2.5) leads to the formula (2.3).

The identity (2.4) follows from applying the binomial inversion formula (1.2) to (2.3). The proof of Theorem 2.3 is complete. $\square$

3. COMPUTING A SUM IN TERMS OF THE GAUSS HYPERGEOMETRIC FUNCTION AND A DETERMINANT

The Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ are defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}$$

for $|z| < 1$, for complex numbers $\alpha, \beta \in \mathbb{C}$, for $\gamma \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, and for

$$(c)_k = \prod_{\ell=0}^{k-1} (c + \ell) = \begin{cases} c(c+1) \cdots (c+k-1), & k \geq 1 \\ 1, & k = 0 \end{cases}$$

which is called the rising factorial of $c \in \mathbb{C}$.

In the above sections, we mainly consider summability of

$$\sum_{q=0}^n \binom{n}{q} \frac{(\pm 1)^q}{(q+z)^{m+1}}, \quad m, n \geq 0, \quad \Re(z) > 0$$

and their inversion formulas in terms of beta functions and integrals. In this section, we compute the sum

$$\sum_{q=0}^n \binom{n}{q} \frac{1}{q+z}, \quad n \geq 0, \quad \Re(z) > 0$$

in terms of the Gauss hypergeometric function ${}_2F_1$, a determinant, and an identity.

Theorem 3.1. *Let $n \geq 0$ be an integer and $\Re(z) > 0$. Then*

$$\sum_{q=0}^n \binom{n}{q} \frac{1}{q+z} = \frac{{}_2F_1(-n, z; 1+z; -1)}{z}. \quad (3.1)$$

Proof. In [3, p. 320, Item 8], it is listed that

$$\int_0^u t^{z-1}(t+a)^\lambda(u-t)^{\mu-1}dt = a^\lambda u^{\mu+z-1} B(\mu, z) {}_2F_1\left(-\lambda, z; \mu+z; -\frac{u}{a}\right) \quad (3.2)$$

for $\Re(\mu), \Re(z) > 0$ and $|\arg \frac{u}{a}| < \pi$. Taking $u = \mu = a = 1$ and $\lambda = n$ in (3.2) leads to

$$\int_0^1 t^{z-1}(1+t)^n dt = \frac{{}_2F_1(-n, z; 1+z; -1)}{z}, \quad \Re(z) > 0.$$

Hence, it follows that

$$\begin{aligned} {}_2F_1(-n, z; 1+z; -1) &= z \int_0^1 t^{z-1}(1+t)^n dt = z \int_0^1 t^{z-1} \sum_{q=0}^n \binom{n}{q} t^q dt \\ &= z \sum_{q=0}^n \binom{n}{q} \int_0^1 t^{q+z-1} dt = z \sum_{q=0}^n \binom{n}{q} \frac{1}{q+z}. \end{aligned}$$

The proof of Theorem 3.1 is complete. $\square$

Theorem 3.2. *Let $n \geq 0$ be an integer and $\Re(z) > 0$. Then*

$$\sum_{q=0}^n \binom{n}{q} \frac{1}{q+z} = \frac{1}{z} \begin{vmatrix} \frac{\langle n+z \rangle_0}{2^0} & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\langle n+z \rangle_1}{2^1} & 1 & 4 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\langle n+z \rangle_2}{2^2} & 0 & 2 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\langle n+z \rangle_{n-2}}{2^{n-2}} & 0 & 0 & 0 & \cdots & n-2 & 4 & 0 \\ \frac{\langle n+z \rangle_{n-1}}{2^{n-1}} & 0 & 0 & 0 & \cdots & 0 & n-1 & 4 \\ \frac{\langle n+z \rangle_n}{2^n} & 0 & 0 & 0 & \cdots & 0 & 0 & n \end{vmatrix}, \quad (3.3)$$

where

$$\langle c \rangle_k = \prod_{\ell=0}^{k-1} (c - \ell) = \begin{cases} c(c-1) \cdots (c-k+1), & k \geq 1 \\ 1, & k = 0 \end{cases}$$

is the falling factorial of $c \in \mathbb{C}$.

Proof. By virtue of the formulas

$${}_2F_1(-n, \alpha+1+\beta+n; \alpha+1; x) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha, \beta)}(1-2x)$$

and

$$P_n^{(\alpha, \beta)}(w) = \frac{(-1)^n}{2^n n!} \frac{1}{(1-w)^\alpha (1+w)^\beta} \frac{d^n}{dw^n} [(1-w)^{n+\alpha} (1+w)^{n+\beta}]$$

in [1, p. 561, Item 15.4.6] or [5, p. 442, Item 18.5.7], we obtain

$$\begin{aligned} {}_2F_1(-n, z; 1+z; -1) &= \frac{n!}{(z+1)_n} P_n^{(z, -(n+1))}(3) \\ &= \frac{n!}{(z+1)_n} \frac{(-1)^n}{2^n n!} \frac{4^{n+1}}{(-2)^z} \lim_{w \rightarrow 3} \frac{d^n}{dw^n} \left[\frac{(1-w)^z (1-w^2)^n}{(1+w)^{n+1}} \right] \\ &= \frac{(-1)^{n-z}}{(z+1)_n} 2^{n-z+2} \lim_{w \rightarrow 3} \frac{d^n}{dw^n} \left[\frac{(1-w)^{n+z}}{1+w} \right]. \end{aligned} \quad (3.4)$$

In [2, p. 40, Exercise 5], it is stated that the n -th derivative of the ratio $\frac{u(t)}{v(t)}$ can be computed by

$$\frac{d^n}{dt^n} \left[\frac{u(t)}{v(t)} \right] = (-1)^n \frac{|W_{(n+1) \times (n+1)}(t)|}{v^{n+1}(t)}, \quad (3.5)$$

where $u(t)$ and $v(t) \neq 0$ are differentiable functions, $|W_{(n+1) \times (n+1)}(t)|$ denotes the determinant of the $(n+1) \times (n+1)$ matrix

$$W_{(n+1) \times (n+1)}(t) = \begin{pmatrix} U_{(n+1) \times 1}(t) & V_{(n+1) \times n}(t) \end{pmatrix},$$

the elements of the $(n+1) \times 1$ matrix $U_{(n+1) \times 1}(t)$ are $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \leq k \leq n+1$, and the elements of the $(n+1) \times n$ matrix $V_{(n+1) \times n}(t)$ are

$$v_{ij}(t) = \begin{cases} \binom{i-1}{j-1} v^{(i-j)}(t), & i-j \geq 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \leq i \leq n+1$ and $1 \leq j \leq n$. This conclusion has been employed in the papers [6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 20, 21, 24] and closely related references. Applying the formula (3.5) to $u(w) = (w-1)^{n+z}$ and $v(w) = w+1$ results in

$$\begin{aligned} \frac{d^n}{dw^n} \left[\frac{(1-w)^{n+z}}{1+w} \right] &= (-1)^{n+z} \frac{d^n}{dw^n} \left[\frac{(w-1)^{n+z}}{w+1} \right] = \frac{(-1)^z}{(w+1)^{n+1}} \\ &\times \begin{vmatrix} \langle n+z \rangle_0 (w-1)^{n+z} & 1+w & 0 & 0 & \cdots & 0 & 0 & 0 \\ \langle n+z \rangle_1 (w-1)^{n+z-1} & 1 & 1+w & 0 & \cdots & 0 & 0 & 0 \\ \langle n+z \rangle_2 (w-1)^{n+z-2} & 0 & 2 & 1+w & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \langle n+z \rangle_{n-2} (w-1)^{2+z} & 0 & 0 & 0 & \cdots & n-2 & 1+w & 0 \\ \langle n+z \rangle_{n-1} (w-1)^{1+z} & 0 & 0 & 0 & \cdots & 0 & n-1 & 1+w \\ \langle n+z \rangle_n (w-1)^{0+z} & 0 & 0 & 0 & \cdots & 0 & 0 & n \end{vmatrix} \\ &\rightarrow \frac{(-1)^z}{4^{n+1}} \begin{vmatrix} \langle n+z \rangle_0 2^{n+z} & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \langle n+z \rangle_1 2^{n-1+z} & 1 & 4 & 0 & \cdots & 0 & 0 & 0 \\ \langle n+z \rangle_2 2^{n-2+z} & 0 & 2 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \langle n+z \rangle_{n-2} 2^{2+z} & 0 & 0 & 0 & \cdots & n-2 & 4 & 0 \\ \langle n+z \rangle_{n-1} 2^{1+z} & 0 & 0 & 0 & \cdots & 0 & n-1 & 4 \\ \langle n+z \rangle_n 2^{0+z} & 0 & 0 & 0 & \cdots & 0 & 0 & n \end{vmatrix} \end{aligned} \quad (3.6)$$

as $w \rightarrow 3$. Combining equalities (3.4) and (3.6) with the identity (3.1) in Theorem 3.1 yields the determinantal expression (3.3). The proof of Theorem 3.2 is complete. $\square$

Theorem 3.3. *Let $n \geq 0$ be an integer and $\Re(z) > 0$. Then*

$${}_2F_1(-n, z; 1+z; -1) = (-1)^n \frac{n!}{(z+1)_n} \left[1 + \sum_{k=1}^n (-1)^k \frac{2^k}{k!} \langle n+z \rangle_k \right]. \quad (3.7)$$

Proof. Let $n \geq 2$ and

$$P_n = \begin{pmatrix} \alpha_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & 0 \\ \alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & 0 \\ \alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \gamma_{n-1} \\ \alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n \end{pmatrix} = (p_{i,j})_{1 \leq i,j \leq n},$$

where

$$p_{i,j} = \begin{cases} \alpha_i, & 1 \leq i \leq n, j = 1; \\ \beta_i, & 2 \leq i = j \leq n; \\ \gamma_i, & 1 \leq i = j - 1 \leq n - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.2 in [19] reads that the determinant $|P_n|$ of the tridiagonal matrix P_n for $n \geq 2$ can be computed explicitly by

$$|P_n| = \alpha_1 \prod_{k=2}^n \beta_k - \sum_{k=2}^n (-1)^k \left(\prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^n \beta_m \right) \alpha_k. \quad (3.8)$$

Setting $\alpha_k = \frac{\langle n+z \rangle_{k-1}}{2^{k-1}}$ for $1 \leq k \leq n+1$, $\beta_k = k-1$ for $2 \leq k \leq n+1$, and $\gamma_k = 4$ for $1 \leq k \leq n$ in (3.8) arrives at

$$\begin{aligned} |P_{n+1}| &= \alpha_1 \prod_{k=2}^{n+1} \beta_k - \sum_{k=2}^{n+1} (-1)^k \left(\prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n+1} \beta_m \right) \alpha_k \\ &= n! - \sum_{k=2}^{n+1} (-1)^k 4^{k-1} \frac{n!}{(k-1)!} \frac{\langle n+z \rangle_{k-1}}{2^{k-1}} \\ &= n! \left[1 + \sum_{k=1}^n (-1)^k \frac{2^k}{k!} \langle n+z \rangle_k \right]. \end{aligned}$$

Substituting this into (3.3) leads to (3.7). The proof of Theorem 3.3 is complete. $\square$

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