HARD TO SELECT POINTS IN A PLANE

Thinh D. Nguyen

Email: kosmofarmer@yandex.com

Abstract. *We show that selecting point in a 2D plane subject to some requirement is a difficult task.*

I. PROBLEM STATEMENT

We are interested in the following optimization problem which is related, in a certain way, to the representation of a set of points using the least number of points among them. More formally, we have a ground set $\mathbf{U}$ of points with coordinates (x_i, y_i) for $i = 1, \dots, n$. Given an integer $k \leq n$, the decision problem consists in finding a subset of points $\mathbf{I}$ of cardinality k such that every point in $\mathbf{U}$ can be written as the convex combination of *at most two points of* $\mathbf{I}$.

For example, let $\mathbf{U} = \{(0,0); (1,3); (2,6); (3,0); (2,3/2); (4,3); (6,0)\}$. There is a solution with $k = 4$ which is $\mathbf{I} = \{(0,0); (2,6); (4,3); (6,0)\}$. Indeed, we have $(1,3) = \frac{(0,0) + (2,6)}{2}$; $(2,3/2) = \frac{(0,0) + (4,3)}{2}$ and $(3,0) = \frac{(0,0) + (6,0)}{2}$.

II. PROOF OF HARDNESS

This problem is reducible from VERTEX COVER. Let the input graph to VERTEX COVER be $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$. Choose integer $n \ll t = \mathbf{O}(n^c)$, and create a convex polygon $\mathbf{P}$ on $1 + n + 2t$ vertices $\mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_n, \mathbf{D}_1, \dots, \mathbf{D}_t, \mathbf{E}_1, \dots, \mathbf{E}_t$. Choose vertices $\mathbf{V}_1, \dots, \mathbf{V}_n$ on sides $\mathbf{C}_0\mathbf{C}_1, \dots, \mathbf{C}_{n-1}\mathbf{C}_n$. Now, for each edge $ij \in \mathbf{E}$ with $i, j \in \mathbf{V}$ and $i < j$, add all points on intersections of $\mathbf{V}_i\mathbf{D}_k$ and $\mathbf{V}_j\mathbf{E}_l$ with $1 \leq k, l \leq t$. We choose coordinates in such a way that no point of the plane has more than two diagonals of $\mathbf{P}$ passing through it, and no three intersection points share non-trivial common lines together with vertices $\mathbf{P}$ (other than diagonals of $\mathbf{P}$ when intended).

Obviously, we have to include all vertices of $\mathbf{P}$ into $\mathbf{I}$. Vertices $\mathbf{V}_i$ are already covered by sides of $\mathbf{P}$, so we have freedom of including any subset of them. For an edge $ij \in \mathbf{E}$, we have to cover all intersections of $\mathbf{V}_i\mathbf{D}_k$ and $\mathbf{V}_j\mathbf{E}_l$ either by including one of $\{\mathbf{V}_i, \mathbf{V}_j\}$ into $\mathbf{I}$, or take some of the intersections into $\mathbf{I}$; in the latter case, $\Omega(t)$ intersections need to be included, which effectively prohibits us from doing this. We can now see that the minimal size of $\mathbf{I}$ is exactly $1 + n + 2t + \mathbf{VC}(\mathbf{G})$, where $\mathbf{VC}(\mathbf{G})$ is the size of minimal vertex cover of $\mathbf{G}$.

The last remark is that we have to make sure that it is possible to choose all coordinates so that length of coordinates of each point is polynomial of n , which is technical.

III. CONCLUSION

Garey and Johnson [1] shape their theory based on previous primal works of Cook, Levin and Karp. Johnson [2] moves on with the guide to this theory. As long as we study a mathematical conjecture, we should encourage ourselves to have moderate amount of labor hours on popular mathematics books like these. Then, reading some articles on theory of computing like [3] is a good practice. Having all those done, we could think of the ultimate final for all mathematics sciences.

REFERENCE

1. Michael R. Garey, David S. Johnson, **Computers and Intractability: A Guide to the Theory of NP-Completeness**
2. David S. Johnson, **The NP-Completeness Column: An Ongoing Guide**
3. Phan Dinh Dieu, Le Cong Thanh, Le Tuan Hoa, **Average Polynomial Time Complexity of Some NP-Complete Problems**, Theor. Comput. Sci. 46(3): 219-237 (1986)