AN EXPLICIT FORMULA FOR THE EULER ZIGZAG NUMBER IN TERMS OF STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. In this note, we derive an explicit formula for the Euler zigzag number in terms of Stirling numbers of the second kind.

Definition 1. The n th Euler zigzag number, denoted by A_n , is the number of alternating permutations of the set $\{1, 2, \dots, n\}$.

Definition 2. The *Stirling numbers of the second kind*, denoted by $S(n, m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

We prove the following

Theorem 1. *We have*

$$A_r = -\frac{4^r}{a_r} \sum_{k=1}^r \frac{(-1)^k S(r, k)}{k+1} \left(\frac{3}{4}\right)^{(k)}, \quad (1)$$

where

$$a_r = \begin{cases} (-1)^{\frac{r-1}{2}} (1 + 2^{-r}) & \text{if } r \text{ is odd} \\ (-1)^{\frac{r}{2}} & \text{if } r \text{ is even} \end{cases},$$

and $(x)^{(n)} = (x)(x+1)\cdots(x+n-1)$ denotes the rising factorial.

Proof. In the article [1], the author proved the following

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)})(1-2^{-r})} \left(\sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right), \quad (2)$$

where B_r and E_r denote the Bernoulli numbers, and the Euler numbers, respectively.

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If we let r to be odd in equation (2), then considering odd indexed Euler numbers are zero, we have

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}.$$

We know that, from [2], we have

$$A_r = (-1)^{\frac{r-1}{2}} \frac{2^{r+1} (2^{r+1} - 1) B_{r+1}}{r+1},$$

for all odd r . As a result, we can conclude that

$$\begin{aligned} \frac{4(-1)^{\frac{r-1}{2}} A_r}{2^{r+1}(2^{r+1} - 1)} (1 + 2^{-(r+1)}(1 - 2^{-r})) &= \frac{(-1)^{\frac{r-1}{2}} A_r}{4^r} (1 + 2^{-r}) \\ &= -\sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}. \end{aligned}$$

which proves our result when r is odd.

Now, if we let r be an even in equation (2), then considering the fact that odd indexed Bernoulli numbers are zero (except 1), we have

$$E_r = -4^r \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}.$$

We know that, from [2], we have

$$A_r = (-1)^{\frac{r}{2}} E_r,$$

for all even r . This proves our result when r is even. □

REFERENCES

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