GRAPH ORIENTATION SATISFYING THAT ALL NODES IN A GIVEN SUBSET HAVE IN- OR OUT-DEGREE 0, AND EVERY OTHER NODE POSITIVE INDEGREE

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ABSTRACT. We show that it is hard to orient a digraph such that all nodes in a given subset have in- or out-degree 0, and every other node positive indegree.

1. DEFINITION AND CLAIM

In a directed graph, the indegree of a node is the number of incoming edges and the outdegree is the number of outgoing edges. Consider the following problem. Given an undirected graph $G = (V, E)$ and a designated subset $C \subseteq V(G)$, decide whether it is possible to orient all edges of G so that every node in C has indegree 0 or outdegree 0, and every other node in G has indegree at least 1?

Our problem is now formally defined:

Definition 1. INOUTDEGORIENTATION:

Input: An undirected graph $G = (V, E)$ and a set $C \subseteq V(G)$

Output: YES if G has an orientation according to which every node in C has either indegree or outdegree equals to 0 and every node in $V \setminus C$ has positive indegree, otherwise NO

Claim 2. We have that $3SAT \leq_p$ INOUTDEGORIENTATION

[1] demonstrated the hardness of 3SAT. So, our claim above implies hardness for INOUTDEGORIENTATION.

2. PROOF OF THE CLAIM 2

Proof. Our problem is NP-complete by a reduction from 3SAT.

For each literal either positive or negative, create a vertex for it.

Set C to be the set of literals themselves.

For each pair of literals of the same variable x_i , namely x_i and $\neg x_i$, connect them by an edge. This ensures that whenever x_i is assigned to TRUE (all outgoing arcs) then $\neg x_i$ is assigned to FALSE (all incoming arcs), and vice versa whenever $\neg x_i$ is assigned to TRUE (all outgoing) then x_i is assigned to FALSE (all incoming). Hence, this makes sure the assignment is consistent.

Now, we take care of the clauses.

For each clause, create for it a new vertex. Connect this new clause-vertex to its 3 literal-vertices in C . And so, we are done.

A satisfying assignment exists if and only if the produced undirected graph can be oriented such that each vertex in C is either a source vertex, i.e. TRUE with all

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outgoing arcs, or a sink vertex, i.e. FALSE with all incoming arcs. A clause-vertex needs to have at least one satisfying literal which is witnessed by the direction of an incoming arc. The claim follows. $\square$

3. CONCLUSION

Garey and Johnson [1] shape their theory based on previous works of Cook, Levin and Karp. Johnson [2] moves on with the guide to this theory. As long as we study a mathematical conjecture, we should encourage ourselves to have enough labour hours on popular mathematics books like these. Then, reading some articles on theory of computing like [3] is a good practice. Only after that, could we think of the ultimate final for all mathematical sciences.

REFERENCES

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