Ministry of Higher Education and
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MODULE : NUMERICAL METHODS

REPORT

Numerical Resolution of Lorenz and Lotka-Volterra predation differential equations

Using Matlab

1st Work Group

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I. INTRODUCTION

In **Mathematics**, a **differential equation** is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Such relations are common; therefore, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

Mainly the study of differential equations consists of the study of their solutions, and of the properties of their solutions.

Only the simplest differential equations are solvable by explicit formulas; however, many properties of solutions of a given differential equation may be determined without computing them exactly.

In our report, we are interested by the resolution of two differential equations, the famous **Lorenz dynamic system** which was developped to understand the chaotic character of **meteorology**, and the **Predator prey model**.

In the folowwing report, we are going to resolve these equations in order to understand their meanings using Matlab, but first of all, we should introduce each problem, then develop and explain both mathematical and numerical issues, our main goal is to resolve these systems with an adequate Matlab formulation, using the ode45 function and finally we should discuss the results.

II. LORENZ MODEL

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1) Introduction to the problem

The Lorenz system is a system of ordinary differential equations first studied by **Edward Lorenz**. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, **the Lorenz attractor** is a set of chaotic solutions of the Lorenz system. In popular media the 'butterfly effect' stems from the real-world implications of the Lorenz attractor, i.e. that in any physical system, in the absence of perfect knowledge of the initial conditions, our ability to predict its future course will always fail. This underscores that physical systems can be completely deterministic and yet still be inherently unpredictable even in the absence of quantum effects. The shape of the Lorenz attractor itself, when plotted graphically, may also be seen to resemble a **butterfly**.

Figure 1 Dynamic Systems of Lorenz, 'Butterfly' shape

2) Mathematical formulation

The Lorenz equations, a system having a number of important applications including weather forecasting and for fluid convection in a two-dimensional layer heated from below are:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = -xz + rx - y$$

$$\frac{dz}{dt} = xy - bz$$

Here **x** denotes the rate of convective overturning, **y** the horizontal temperature difference, and **z** the departure from a linear vertical temperature gradient .

For the Parameters **$\sigma=10$, $b=8/3$ and $r=28$** , Lorenz suggested that trajectories in a bounded region converge to an **attractor**(see picture in page 2) that is a fractal, with dimension about 2.06, as estimated by Liapunov exponents.

The Lorenz equations are very suggestive but do not represent weather systems very well.

3) Numerical formulation

We wrote a simulator in order to model the pattern-formation equation, here is the numerical formulation of Lorenz equations:

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \gamma u(1 - u)$$

4) Matlab resolution

As an example of a system of three simultaneous equations, we consider the **Lorenz system**.

This system, a subject to appropriate initial conditions. As the parameters s , r , and b are varied through various ranges of values, the solutions of this system of differential equations vary in form. In particular, for certain values of the parameters the system exhibits chaotic behavior.

To provide **more accuracy** in the computation process we use the **Matlab function ode45**.

Below our Matlab Script **e4s507.m** for solving this problem:

```
% e4s507.m Solution of the Lorenz equations
r = input('enter a value for the constant r ');
simtime = input('enter runtime ');
acc = input('enter accuracy value ');
xprime = @(t,x) [10*(x(2)-x(1)); r*x(1)-x(2)-x(1)*x(3); ...
    x(1)*x(2)-8*x(3)/3];
initx = [-7.69 -15.61 90.39]';
tspan = [0 simtime];
options = odeset('RelTol',acc);
[t x] = ode45(xprime,tspan,initx,options);
% Plot results against time
figure(1), plot(t,x,'k')
xlabel('Time'), ylabel('x')
figure(2), plot(x(:,1),x(:,3),'k')
xlabel('x'), ylabel('z')
```

After plotting, we get this Figure

Figure 2 Solution of Lorenz Equations for $r=126.52$, $s=10$ and $b=8/3$ using accuracy of 0.000005 and terminating at $t=8$

5) Results discussion

The results of running script e4s507.m are given in Fig 2 which is characteristic of the Lorenz equations

shows the **complexity of the relationship** between x and z

From a technical standpoint, the Lorenz system is nonlinear, non-periodic, three-dimensional and deterministic.

For almost all initial conditions (different from fixed points), the orbit of the system walks over the attractor, the trajectory starting by wrapping around one wing, then jumping from wing to wing to begin, to curl up on the other wing, and so on, apparently erratically.

Supervisor's Remarks:

III. PREDATOR-PREY MODEL

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1) Introduction to the problem

a) General problem definition

In mathematics, the **Lotka-Volterra predation equations**, also referred to as the "**prey-predator model**", are a pair of first-order nonlinear differential equations, and are commonly used to describe the dynamics of biological systems in which a **predator** and its **prey** interact.

They were independently proposed by Alfred James Lotka in 1925 and Vito Volterra in 1926.

This system of equations is conventionally used as a model for the dynamics of lynx and hare snow.

Figure 3 Fox and Rabbit: Predator and Prey

b) Our Assigned problem definition

Our assignment is based on the predator-prey model, we are going to develop step by step the resolution of a case study based on two species. The first species consists of rabbits with a specified population quantity, the second species consists of foxes with a specified population quantity as well. Our intention is to measure the rabbit population in units of 1000 in order to obtain a balanced scale between the two population quantities.

2) Mathematical Formulation

The birth rate of rabbits is considered to be constant ($=a$), the death rate is therefore dependant on the size of the fox population ($=c \times y^2$, where c is a constant). The death rate of the foxes is constant ($=b$). The birth rate, however, depends on the quantity of rabbits that the foxes have at their disposal ($=d \times y^1$, where d is a constant).

We have the following mathematical model:

$$\left\{ \begin{array}{l} \frac{dy_1}{dt} = ay_1 - cy_1y_2 \quad \text{Rabbits (Prey)} \\ \frac{dy_2}{dt} = -by_2 + dy_1y_2 \quad \text{Foxes (Predator)} \end{array} \right.$$

As for initial conditions, the initial populations $y_1(0)$ and $y_2(0)$ are assumed to be known, note that $a=2$, $b=3$, $c=0.001$ and $d=0.006$.

3) Numerical Formulation

To resolve this problem, we intend to use the **set of ordinary differential equations ODEs** describing the predator-prey (or pray) system.

Will proceed to an integration to the system described below with **ODE** from the initial position at $t=0$ to the next step at $t=0.05$ until reaching $t=1$.

Will use a single and constant step size integration method ($h=0.05$)

Here is a short breif to understand the mathematical and numericla symbols:

“The variables y_1 and y_2 measure the sizes of the prey and predator populations, respectively. The quadratic cross term accounts for the interactions between the species. The prey population increases when no predators are present, and the predator population decreases when prey are scarce.”

4) Matlab Resolution

After reading the concerned documentation and doing our researches, we wrote this matlab script **“Lotka_Volterra_Rabbits_and_Foxes.m”** (which was verified by our supervisor) and we was satisfied with the plotted curve result we had after exeuting the code.

```
clear all
close all
hold off
f = @(t,y) [(2-0.001*y(2))*y(1); (-3+0.006*y(1))*y(2)];
[ts,ys] = ode45(f,[0:0.05:1],[1000;500]);
plot(ts,ys,'o-') %plot rabbit and fox population vs. t
hold on
legend('rabbits','foxes'); xlabel('time t')
```

Here we used $[ts,ys] = \text{ode45}(f,[t0,T],y0)$.

Where:

-f is the ODE function

-T0 is the initial time

-T is the final time

-y0 is the initial y-vector at time t0

This returns the arrays ts, ys:

Ts is the vector of time values

Ys is an array containing the values of y: row contains the values of y1 (rabbits) and y2 (foxes)

[ts,ys] gives a table with t in column 1, y1 (rabbits) in column 2, y2 (foxes) in column 3.

After Plotting we had the Figure below:

Figure 4 Foxes and Rabbits Populations Progress throw time -Predator-Pray Model-

6) Results discussion

The results we had match with the literature we have consulted before initiating the problem resolution. To a first approximation, there was apparently nothing keeping the rabbit population in check other than predation by fox, and the fox depended entirely on rabbits for food.

This is so clear by just observing the figure above (Fig-4-).

The remarkable property of the **Lotka-Volterra** model is that the solutions are always **periodic** **(In our case, we was limited to $[0, 1]$ time, but by extending it, we will see the periodicity of this model) .**

The populations always return to their initial values and repeat the cycle. This property is not obvious and not easy to prove. It is rare for nonlinear models to have periodic solutions. The difficult aspect of computing the solution to the predator-prey equations is determining the length of the period.

But if an other variable is established by humans ? like pesticides ? Shall we remodel the problem in order to get a good prediction ?

IV. CONCLUSION

Throw These two very interesting problems: The famous **Lorenz** weather model to determine the chaotic weather related variables in an atmosphere heated from below: convection, horizontal and vertical temperature variations and the **biological/Natural** study model of the two researches **Lotka and Volterra (Predator-pray Model)**, we can really conclude the importance of numerical models to predict valuables and precious information in order to take the right decisions and so the right executed tasks to prevent and/or establish and action in the future.

The Lorenz model is used to forecast the weather and is a good model to study in the meteorology discipline, I am, the writer, a student pilot at the virtual World Simualtion Network (IVAO and VATSIM) studied Meteorology and I know the importance of such modeling for pilots and Aircraft tower controllers in the airport.

Also, the Lotka-Volterra Model is so important for the study of animal extinction.

These analyses are very important for us (students and researches) and this was the fruit of a sophisticated software for numerical modeling: **Matlab**.

V. References

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