Localizing predicate activity in resolution proofs

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1 Introduction

It is known that resolution-derivation of statements in conjunctive-normal-form is both sound and refutation-complete for Classical First Order Logic [1]. This provides a different angle for proving things about derivability in that language. This paper makes an approach along this line. A theorem is (constructively) proven regarding the existence of resolution-derivations satisfying certain structural properties, and this theorem is then used to offer a new constructive proof of an important result regarding Classical First Order Logic.

More specifically, a theorem is shown which implies that given any resolution-derivation D_1 and set of predicate-terms SP_1 , a resolution-derivation D_2 can be constructed whose axiomatic basis is a subset of that of D_1 , whose result-clause is the same as that of D_1 , and whose every step involving an elimination of a literal with a term in SP_1 is such that every step leading to it also involves an elimination of a literal with a term in SP_1 . This implication of the theorem is then used to provide a new constructive proof of the Craig Interpolation Theorem [2] for Classical First Order Logic.

I'll begin by laying out the relevant terminology, and then establishing a set of lemmas. Following this is the statement and proof of the central theorem of the paper, called a *localization* theorem because of the way in which it allows us to localize the activity of a given set of predicates. Finally, the new constructive proof of Craig's Interpolation Theorem is provided.

2 Terminology

2.1 Concepts in the field

Definition 1. *resolution-proof:* By this we will mean a conjunctive-normal-form resolution proof in Classical First Order Logic. Such a proof involves a set of steps, where the input to each step is either a clausal representation of an axiom of some theory, or the output of some other step. If we say that a step involves the elimination of a term t , we mean by this that a literal eliminated at that step includes t .

Definition 2. *variable-substitution*: In resolution-proofs, it is valid under certain circumstances to, for a certain step, transform the literals from an input so that certain variables are replaced with certain other terms (these other terms being variables, constants, or skolem-functions). We will refer to the function specifying such replacements for a given step as a *variable-substitution*.

Definition 3. *valid-resolution-step*: A step in a resolution-proof involves the elimination of a set of literals. Let us define here the conditions under which that elimination is valid, so we can reference them later in proofs. It will suffice for our purposes to only define validity for the case that a step involves no variable-substitutions. Let O_1 be the output clause for a step S_1 that involves no variable-substitutions, and I_1, I_2 its two input clauses. Then S_1 is valid iff there exists a set E such that :

- (i) E is a subset of $I_1 \cup I_2$.
- (ii) E includes at most one element from each of I_1, I_2
- (iii) If E includes two elements, they are complements
- (iv) $O_1 = \{I_1 \cup I_2\} - E$

Definition 4. *tree*: This refers to the usual graph-theoretic concept - a Directed Acyclic Graph where every node is pointed to by at most one other. If a node n_1 points to n_2 , we say that n_1 is n_2 's *parent*, and n_2 is n_1 's *child*. A node's *descendants* are all those nodes reachable by paths pointing away from it, and a node's *ascendants* are all those nodes for which it is an descendant. The *inclusive-descendants* of a node is the set including that node and its descendants, and *inclusive-ascendants* is defined analogously. A *root* is a node with no parents, and a *leaf* is a node with no children. If, in the context of a tree t_1 , we speak of that sub-tree t_2 of it which is rooted by a node n_1 , we mean that sub-tree of t_1 consisting of n_1 's inclusive-descendants, and all of the edges which point among them.

Definition 5. *resolution-tree*: There is a natural way to determine a tree based on a resolution-proof, and we will call such a tree a resolution-tree. First, create a single node, and associate it to the final-clause of the proof. If the final step of the proof had two (as opposed to zero) inputs, then add two nodes to the tree, associate each node to each input, respectively, and establish edges making the original node their parent. Continue doing this iteratively until the only leaf-nodes in the tree are those associated to axiomatic inputs (inputs that are not the output of any step in the proof). Note that a given clause may come to be associated to more than one node. Also note that the tree will have a single root.

2.2 Concepts specific to this paper

Definition 6. *abstractly-equivalent*: Two resolution-trees t_1 and t_2 are abstractly-equivalent if a graph-isomorphism f_1 exists from t_1 to t_2 such

that, for any node n_1 in t_1 , the clause c_1 associated to n_1 and the clause c_2 associated to $f_1(n_1)$ are such that they differ from each other only in regards their variables and constants.

Definition 7. *complete-concretization:* The *complete-concretization* t_2 of a resolution-tree t_1 is the tree produced by altering the clause c_1 associated to each node n_1 in the following way : send c_1 through the composition of variable-substitutions which exist between n_1 and t_1 's root. It can be seen that t_2 will be a valid resolution-tree that involves no variable-substitutions, and which has the same result-clause as t_1 .

Definition 8. *abstract-rebasis:* Given a set of clauses X_1 , and a tree t_1 such that every clause c_1 associated to a leaf-node n_1 in t_1 results from a (valid) variable-substitution operation on some clause c_2 in X_1 , let the *abstract-rebasis* of t_1 be the tree resulting from taking t_1 , and changing each leaf node so that instead of being associated to its original clause, it is now associated to the relevant matching clause in X_1 . Clearly, the resulting tree has the same result-clause as the original, is a valid resolution-tree, and is abstractly-equivalent to the original.

Definition 9. *organizing-condition:* A condition C_{org} on nodes of a resolution-tree such that :

- (i) C_{org} is defined relative to a given set SP_1 of predicate terms
- (ii) C_{org} is satisfied by every leaf-node whose clause includes a term in SP_1
- (iii) For any three nodes n_1 , n_2 , n_3 , where n_3 is parent to n_1 and n_2 , and n_3 involves an elimination in SP_1 : n_1 and n_2 satisfy C_{org} iff n_3 does.
- (iv) If a node does not satisfy C_{org} , then neither does its parent
- (v) For any two nodes n_1 and n_2 , if the sub-trees rooted by n_1 and n_2 , respectively, are abstractly-equivalent, then n_1 satisfies C_{loc} iff n_2 does

Two examples of this kind of condition are :

- For a given set of predicate-terms SP_1 , the condition that every leaf-node in one's inclusive-descendants includes a term in SP_1 .
- For a given set of predicate-terms SP_1 , the condition that every leaf-node in one's inclusive-descendants includes a term in SP_1 , and every non-leaf-node in one's inclusive-descendants involves the elimination of a literal that includes a term in SP_1 .

Definition 10. *localizing-condition:* A condition C_{loc} on nodes of a resolution-tree such that :

- (i) C_{loc} is defined relative to a given organizing-condition C_{org}

- (ii) if SP_1 is the set of predicate-terms that C_{org} is defined in relation to, then a node satisfies C_{loc} iff it (one or both of the following) satisfies C_{org} , or is such that its elimination involves no term in SP_1 .

Definition 11. *localized-form*: For a given localizing-condition C_{loc} , a *localized-form* of a resolution-tree t_x is any resolution-tree whose nodes all satisfy C_{loc} , whose root node is associated to the same clause as that of t_x , and whose leaf nodes are associated to a set of clauses which is a subset of the set of clauses associated to the leaf nodes of t_x .

2.3 Algorithms specific to this paper

Note : the algorithms below produce trees whose nodes are associated to clauses. They may not always produce trees that represent valid resolution-proofs. However, our later work will rely only on the fact that they produce a tree representing a valid resolution-proof when called on particular inputs. What these inputs are will be specified later, and the fact of the output being valid in that case will be proven later. For now we will still refer to their outputs as resolution-trees, keeping in mind that the validity of these trees is to be shown later for particular cases.

Definition 12. `tree_producer_1`(t_x , t_y , n_x , n_y , LS_x) : For n_x , n_y nodes in resolution-trees t_x , t_y , and LS_x a set of literals, return a new resolution-tree constructed by :

1. creating a node and assigning it the clause consisting in the union of the literals in n_x and n_y 's clauses, minus the literals in LS_x
2. taking the sub-trees rooted by n_x and n_y in t_x and t_y , respectively, and adding them to this new tree, and making the roots of these sub-trees children to the node created in part 1.

Definition 13. `tree_producer_2`(t_x , t_y , n_x , N_y , LS_x) : For n_x a node in a resolution-tree t_x , N_y a set of nodes in resolution-tree t_y , none of which are ascendant of any other, and LS_x a set of literals : return the tree which results from taking t_y , replacing every sub-tree rooted by a node n_y in N_y with `tree_producer_1`(t_x , t_y , n_x , n_y , LS_x), and altering the clause-assignment for each ascendant n_a of such a replacement so that, if it was c_1 beforehand, it is now the union of the literals in c_1 and in the clause assigned to n_x , minus LS_x .

Definition 14. `tree_producer_3`(t_x , t_y , N_x , N_y , LS_x) : For sets N_x , N_y of nodes from resolution-trees t_x , t_y , respectively, where no node in the set $\{N_x \cup N_y\}$ is ascendant of another, and a set of literals LS_x , return the tree constructed in the following way : begin with t_x , and then replace each sub-tree rooted by a node n_x in N_x with `tree_producer_2`(n_x , N_y , LS_x). Then, for each ascendant n_a of a replacement, alter its clause assignment so that if it was c_1 beforehand, it is now the union of the literals in c_1 and in the clause assigned to t_y 's root, minus LS_x .

3 On localizing-conditions

Lemma 3.1. *If resolution-trees t_1 and t_2 are abstractly-equivalent, then t_1 's root satisfies a localizing-condition C_{loc} iff t_2 's root does.*

Proof. Let C_{org} be the organizing-condition in relation to which C_{loc} is defined, and SP_1 the set of predicate-terms in relation to which C_{org} is defined. Suppose t_1 's root satisfies C_{loc} . Then either (one or both of the following) the elimination at t_1 's root does not involve a term in SP_1 , or t_1 's root satisfies C_{org} . In the former case, given the nature of abstract-equivalence, we know that the elimination at t_2 's root will likewise not involve a term in SP_1 . In the latter case, given part (v) of Definition 9 of organizing-condition, we know that t_2 's root will also satisfy C_{org} . The same reasoning works for showing that if t_2 's root satisfies C_{loc} , then so will t_1 's root. $\square$

Lemma 3.2. *For any localizing-condition C_{loc} and resolution-tree t_1 , if a sub-tree st_1 of t_1 whose nodes do not all satisfy C_{loc} is replaced by a sub-tree whose root is associated to the same clause as st_1 's root, then any non-replaced node in t_1 that satisfied C_{loc} beforehand will do so afterwards as well.*

Proof. Let C_{org} be the organizing-condition that C_{loc} was defined in relation to, and SP_1 the set of predicate-terms that C_{org} was defined in relation to. Given Definition 10 of localizing-condition, the fact that some nodes in the replaced sub-tree did not satisfy C_{loc} implies that some nodes in the replaced sub-tree did not satisfy C_{org} .

Given that, by part (iv) of Definition 9 of organizing-condition, a parent of a node that doesn't satisfy such a condition cannot itself satisfy such a condition, we know that all the ascendants of the replaced sub-tree could not have satisfied C_{org} beforehand.

Thus, any ascendant of the replaced sub-tree which satisfied C_{loc} beforehand must have been such that the step it represents does not involve the elimination of a literal with a term in SP_1 . The replacement does not effect which steps these nodes represent, so any of these nodes which satisfied C_{loc} beforehand will continue to do so afterwards.

As far as the nodes which were neither replaced nor ascendant to a replacement - that C_{loc} satisfaction is preserved among such nodes follows from Lemma 3.1, and the fact that each is the root of a sub-tree which is abstractly-equivalent to itself. $\square$

Lemma 3.3. *For every localizing-condition C_{loc} , defined in relation to an organizing-condition C_{org} and set of predicate-terms SP_1 , and every pair of nodes n_1 , n_2 in a resolution-tree that involves no variable-substitutions :
If*

- n_2 is an ascendant of n_1
- n_2 's child that is not an ascendant of n_1 has a clause that includes a literal lit_1 with a term in SP_1

- n_2 's clause does not include lit_1
- For N_x the set of nodes between (non-inclusive) n_1 and n_2 , none of the nodes in N_x satisfy C_{org} , but they all satisfy C_{loc}

Then n_1 's clause does not include lit_1 .

Proof. Since the nodes in N_x satisfy C_{loc} but not C_{org} , it must be that none of them involve an elimination with a term in SP_1 . This entails that lit_1 cannot be eliminated in one of those nodes. Suppose that n_1 includes lit_1 . This would then imply that both of n_2 's children include lit_1 , while n_2 does not. This would render the resolution-step that n_2 represents invalid, contradicting the assumption that it comes from a resolution-tree. $\square$

Lemma 3.4. *For every localizing-condition C_{loc} , defined in relation to an organizing-condition C_{org} and set of predicate-terms SP_1 , and every node n_1 in a resolution-tree that involves no variable-substitutions, and every literal lit_1 which includes a term in SP_1 :*

If

- n_1 's parent does not satisfy C_{org}
- no member n_d of n_1 's inclusive-descendants is such that all the following are true of it: n_d satisfies C_{org} , it's parent does not, and n_d 's clause includes lit_1

Then n_1 's clause does not include lit_1 .

Proof. Suppose lit_1 is in n_1 's clause. Then there must be some leaf node n_2 which is in the inclusive-descendants of n_1 , whose clause includes lit_1 , and where lit_1 is not eliminated between n_2 and n_1 . Given part (ii) of Definition 9 of organizing-condition, we know that n_2 satisfies C_{org} . And so there will be some node n_x , which is either n_2 , n_1 , or some node in between, which includes lit_1 , satisfies C_{org} , and is such that its parent does not satisfy C_{org} . Since n_x is in n_1 's inclusive-descendants, this contradicts the second antecedent. $\square$

4 On localized-forms

Lemma 4.1. *For any resolution-tree t_1 and localizing-condition C_{loc} : if a sub-tree st_1 of t_1 whose only C_{loc} -violating node is its root is replaced by a localized-form of st_1 , this produces a tree t_2 where:*

- t_2 's root node is associated to the same clause as that of t_1
- the set of clauses associated to t_2 's leaf-nodes is a subset of the set of clauses associated to t_1 's leaf-nodes
- t_2 has strictly fewer nodes than t_1 that do not satisfy C_{loc}

Proof. The first two points should be clear simply from the nature of trees and Definition 11 of localized-form. The last point follows from Lemma 3.1 and 3.2. $\square$

Lemma 4.2. *For every localizing-condition C_{loc} , and every resolution-tree t_1 whose every node satisfies C_{loc} except for its root, if t_1 involves no variable-substitutions, then we can construct a localized-form t_2 of t_1 which also involves no variable-substitutions.*

Proof. Let C_{org} be the organizing-condition that C_{loc} is defined in relation to, and SP_1 the set of predicate terms that C_{org} is defined in relation to. Since t_1 's root does not satisfy C_{loc} , it must involve the elimination of a literal lit_1 whose predicate term is in SP_1 . Let n_{main} be the child of t_1 whose clause includes lit_1 , and n_{other} its other child. Let LS_1 be the set of literals eliminated at t_1 's root (must consist of lit_1 alone or lit_1 and its complement, lit_1^c). Let N_1 be the subset of n_{main} 's inclusive-descendants such that, for each n_1 in N_1 :

- n_1 's clause includes lit_1
- lit_1 does not get eliminated between n_1 and n_{main}
- n_1 satisfies C_{org}
- n_1 's parent does not satisfy C_{org}

Define N_2 analogously for n_{other} and lit_1^c . Let st_{main} and st_{other} be the sub-trees rooted by n_{main} and n_{other} , respectively. Let $t_3 = \text{tree_producer_3}(st_{main}, st_{other}, N_1, N_2, LS_1)$. Note that the inputs to this call to the function satisfy the function's input-requirements (for example, that $\{N_1 \cup N_2\}$ should include no member which is ascendant of another.) The rest of our proof will involve showing that t_3 is the t_2 whose existence is claimed by the lemma.

Basis Preservation That the set of clauses associated to t_3 's leaf nodes is a subset of that for t_1 follows from the fact that the leaf nodes produced in the output of the function are all copied from its input trees, which are both sub-trees of t_1 .

Result Preservation By definition, the clause associated to t_1 's root is the union of the literals in n_{main} 's clause and the literals in n_{other} 's clause, minus LS_1 . So let us show this will also be true for t_3 's root-node. Consider the following two cases:

1. N_1 includes n_{main} Given how N_1 is defined, this implies that N_1 is simply $\{n_{main}\}$. This then implies that N_2 cannot include n_{other} . For if it did, then both children of t_1 's root would satisfy C_{org} , making it so that t_1 's root, which involves an elimination in SP_1 , would satisfy C_{org} (given part

(iii) of Definition 9 of organizing-condition) and thus C_{loc} . This contradicts the premise that t_1 's root does not satisfy C_{loc} . It can be seen that since N_2 does not include n_{other} , it must include at least one of its descendants.

Considering the definition of **tree_producer_3**, t_3 will be the output of the call **tree_producer_2**(st_{main} , st_{other} , n_{main} , N_2 , LS_1). By definition of **tree_producer_2**, the root node of the output to this call will be n_{other} , except with its clause assignment altered. The new clause to which it will be assigned can be seen from the definition of the algorithm to be precisely the union of the literals in its former clause and the literals in n_{main} 's clause, minus LS_1 .

2. N_1 does not include n_{main} Given how N_1 is defined, this implies N_1 will include some node n_1 which is a descendant of n_{main} . This means the root node of t_3 will be n_{main} , except with its clause-assignment altered. The alteration can be seen, from the definition of the algorithm to produce precisely the clause needed.

Validity It is clear that the functions we defined produced trees where every node is assigned to a clause, but it may not be clear that these trees are resolution-trees - as in, that they represent valid resolution-proofs. We will now show that, when one provides the kind of inputs that we provide to the algorithms in our construction of t_3 , they produce resolution-trees that represent valid resolution-proofs.

The call we make to **tree_producer_3** in turn involves certain calls to **tree_producer_2**. In particular, these are calls of the kind **tree_producer_2**(st_{main} , st_{other} , n_1 , N_2 , LS_1), where n_1 is some node in N_1 . A call of this kind in turn involves calls to **tree_producer_1**, of the kind **tree_producer_1**(st_{main} , st_{other} , n_1 , n_2), where n_1 and n_2 belong to N_1 , N_2 , respectively. So we begin by showing the validity of these 'sub-calls'. Nodes in sub-trees that are copied unaltered from t_1 will be valid given the assumed validity of t_1 , so in the foregoing we will only concern ourselves with showing the validity of nodes that are not of that kind.

Validity of output of call to tree_producer_1 : There is only one node in this output which is not copied unaltered from t_1 - the root node n_r . Its two children, n_1 and n_2 , which come from N_1 and N_2 respectively, have been copied unaltered (with the same clause assignments) from t_1 . n_r 's clause is assigned to be the union of the literals in n_1 and n_2 , minus LS_1 . Recall that LS_1 consists of either lit_1 or lit_1 and its complement lit_1^c . Given how N_1 and N_2 were defined, n_1 's clause contains lit_1 , and n_2 's clause contains lit_1^c . What remains to be shown are two things. First, that n_2 's clause does not contain lit_1 . This can be seen to follow from Lemma 3.3. Second, that if LS_1 includes lit_1^c , then n_1 's clause does not include lit_1^c . This can also be seen to follow from Lemma 3.3.

Validity of output of call to tree_producer_2 : The only nodes in this output which are not copied unaltered from t_1 are those produced by calls to

`tree_producer_1` of the kind whose validity we just showed, and those copied from t_1 which are ascendant to nodes replaced by such calls, and have had their clauses altered. Let us consider an arbitrary node n_3 of the latter kind.

At least one of its children, n_3c_1 , will either be the root of the output to the relevant kind of call to `tree_producer_1`, or it will be ascendant to such a node. In the former case, let on_3c_1 be n_3 's child in t_1 that was replaced by the call to `tree_producer_1`. In the latter case, let on_3c_1 simply be n_3c_1 (except with its original clause assignment).

n_3 's other child, n_3c_2 , will either be of one of the two kinds that we mentioned n_3c_1 may be, or the root of a sub-tree copied unaltered from t_1 . In the former case, define on_3c_2 similarly to how we did on_3c_1 . In the latter case, let on_3c_2 simply be n_3c_2 . Note that we have designed the criteria for on_3c_1 and on_3c_2 so that, in any case, both were children of n_3 in t_1 .

Let's introduce some more terminology to simplify our reasoning. We will write $f_c(n_x)$ for a given node n_x to represent the clause assigned to it. In the case that the clause assigned to a node is different in t_1 versus the output tree of the call to `tree_producer_2`, we will write $f_c(n_x, before)$ vs. $f_c(n_x, after)$ to distinguish.

Given our assumption that t_1 is valid and involves no variable-substitutions, we know that :

$$f_c(n_3, before) = \{f_c(on_3c_1) \cup f_c(on_3c_2)\} - E_1 \quad (1)$$

where E_1 is some set of literals satisfying the conditions (i)–(iv) mentioned in Definition 3 of valid-resolution-step.

Given the definition of `tree_producer_2` :

$$f_c(n_3, after) = \{f_c(n_3, before) \cup f_c(n_1)\} - LS_1 \quad (2)$$

Recall that n_1 is one of the inputs to the kind of call to `tree_producer_2` with which we are presently concerned, and that n_1 comes from N_1 . If we combine the information from equation 1 and 2 :

$$f_c(n_3, after) = \left\{ \left\{ f_c(on_3c_1) \cup f_c(on_3c_2) \right\} - E_1 \right\} \cup f_c(n_1) \right\} - LS_1 \quad (3)$$

Given some simple reasoning regarding the nature of sets, this implies :

$$f_c(n_3, after) = \left\{ \left\{ f_c(on_3c_1) \cup f_c(on_3c_2) \cup f_c(n_1) \right\} - LS_1 \right\} - \{E_1 - f_c(n_1)\} \quad (4)$$

From the definition of `tree_producer_2` and `tree_producer_1`:

$$f_c(n_3c_1) = \{f_c(on_3c_1) \cup f_c(n_1)\} - LS_1 \quad (5)$$

When n_3c_2 is copied unaltered from t_1 , clearly :

$$f_c(n_3c_2) = f_c(on_3c_2) \quad (6)$$

and when this is not the case, given the definition of `tree_producer_2` and `tree_producer_1` :

$$f_c(n_3c_2) = \{f_c(on_3c_2) \cup f_c(n_1)\} - LS_1 \quad (7)$$

What we will try to show now (for reasons that will be clear later) is that none of the literals in LS_1 will be in $f_c(n_3c_2)$. Given equations 6 and 7, this amounts to showing that, in the case that n_3c_2 is copied unaltered from t_1 , none of the literals in LS_1 are in $f_c(on_3c_2)$. Recall that LS_1 consists either only of lit_1 or of lit_1 and lit_1^c .

Since n_3 is ascendant of a node which gets replaced by a call to `tree_producer_1`, we know (given the definition of `tree_producer_2`) that n_3 was ascendant to a member of N_2 in the original tree t_1 . Given the definition of N_2 , and part (iv) of Definition 9 of organizing-condition, this implies that n_3 does not satisfy C_{org} . If n_3c_2 is copied unaltered from t_1 , we know from this that (given the definition of `tree_producer_2`) no member of N_2 is in the inclusive-descendants of on_3c_2 . It can be seen that these two facts allow us to apply Lemma 3.4 to show that, in this case, the potential member lit_1^c of LS_1 will not be in $f_c(on_3c_2)$.

To show that the member lit_1 of LS_1 will not be in $f_c(on_3c_2)$ (in the case that n_3c_2 is copied unaltered from t_1), consider the following :

- Given that n_3 was an ascendant of a node in N_2 in t_1 , and n_3 was on_3c_2 's parent, we know that none of the nodes between on_3c_2 and n_r in t_1 satisfied C_{org} .
- Given that n_3 was, in t_1 , within a sub-tree rooted by a child of t_1 's root n_r , and that t_1 's only C_{loc} violating node was n_r , we know that all the nodes between on_3c_2 and n_r in t_1 satisfied C_{loc} .
- n_r , an ascendant of on_3c_2 , does not include lit_1 (lit_1 was originally defined as a literal which is eliminated at n_r).
- n_r 's child that is not an ascendant of on_3c_2 is n_{main} , whose clause includes lit_1 (this is precisely how n_{main} was defined).
- The above points allow us to apply Lemma 3.3 to show that on_3c_2 's clause will not include lit_1 .

Now that we have shown that none of the literals in LS_1 will be in $f_c(n_3c_2)$, we can combine equations 5, 6 and 7 in the following way:

$$f_c(n_3c_1) \cup f_c(n_3c_2) = \{f_c(on_3c_1) \cup f_c(on_3c_2) \cup f_c(n_1)\} - LS_1 \quad (8)$$

And now we can combine 4 and 8 to yield :

$$f_c(n_3, after) = \{f_c(n_3c_1) \cup f_c(n_3c_2)\} - \{E_1 - f_c(n_1)\} \quad (9)$$

Let E_2 stand for $\{E_1 - f_c(n_1)\}$. What remains to be shown is that E_2 satisfies the conditions (i)–(iv) mentioned in Definition 3 of valid-resolution-step. We prove each of these in turn below. Before we begin, note the following:

Note 1. E_1 does not intersect LS_1 . To see this, consider the following. Given the definition of `tree_producer_2`, we know that n_3 was ascendant, in t_1 , to some node in N_2 . Given the definition of N_2 , this implies that n_3 did not satisfy C_{org} in t_1 . Given that t_1 's only C_{loc} violating node is its root, and that n_3 comes from a sub-tree rooted by a child of t_1 's root, we know that n_3 must have satisfied C_{loc} in t_1 . Since n_3 satisfied C_{loc} but not C_{org} , it cannot have involved an elimination of a term in SP_1 , and thus cannot have involved the elimination of any literal in LS_1 .

And now for demonstrating the satisfaction of each part of the definition of valid-resolution-step (each demonstration is labelled by the index of the part of the definition it relates to) :

- (i) By definition, E_1 is a subset of $\{f_c(on_3c_1) \cup f_c(on_3c_2)\}$. From equation 8 and Note 1, we then know that E_1 is a subset of $\{f_c(n_3c_1) \cup f_c(n_3c_2)\}$. Since E_2 defined as a subset of E_1 , we are done.
- (ii) Suppose lit_x is an element in E_2 . It is then in E_1 , and since t_1 is valid, this means it is in one of $f_c(on_3c_1)$ or $f_c(on_3c_2)$, and not the other. Suppose it is in the former (and thus not in the latter). Given equation 5 and note 1, this means it is also in $f_c(n_3c_1)$. Given equation 6 and 7, it is not in $f_c(n_3c_2)$ as long as it is not in $f_c(n_1)$. That it is not in $f_c(n_1)$ follows simply from the fact that the formula we used to define E_2 to begin with was $\{E_1 - f_c(n_1)\}$. Similar reasoning works for the case where lit_x is in $f_c(on_3c_2)$.
- (iii) Follows from fact that E_2 is a subset of E_1 , and t_1 is valid.
- (iv) This is precisely the content of equation 9

Validity of output of call to `tree_producer_3` : The only nodes in the output of this call that aren't copied unaltered from t_1 are outputs from calls to `tree_producer_2` of the kind whose validity we just showed, and nodes which are ascendant of nodes that have been thus replaced - these nodes being copied over from t_1 but having their clause assignment altered.

Showing the validity of these latter kinds of nodes can be done in a way very similar to the way we showed validity for the analogous nodes (nodes ascendant of replacements) in the output of the relevant kind of call to `tree_producer_2`.

For each such node n_4 , its two children n_4c_1 and n_4c_2 in the output will stand in an analogous relation towards two nodes on_4c_1 and on_4c_2 as n_3c_1 and n_3c_2 did towards the nodes on_3c_1 and on_3c_2 we defined in the earlier proof.

The proof then runs analogously to the way it did for the `tree_producer_2` case, with every string “ n_3 ” replaced with “ n_4 ”, and “ n_1 ” with “ n_{other} ”.

Satisfaction of C_{loc} We can divide the nodes of t_3 into three kinds of nodes - those which are the roots of sub-trees simply copied from t_1 , those which are the roots of trees produced by the relevant call to `tree_producer_1`, and those which are ascendant to nodes of the second kind and have had their clause-assignments changed.

We know all the nodes of the first kind will satisfy C_{loc} because they were descendants of t_1 -root, and since t_1 's only C_{loc} violating node is its root, this means they must have satisfied C_{loc} beforehand. From Lemma 3.1, they will continue to do so in t_3 .

That a node n_x of the second kind will satisfy C_{loc} follows from the following considerations. Its children come (unaltered) from $\{N_1 \cup N_2\}$, which all satisfied C_{org} in t_1 . Given part (v) of Definition 9 of organizing-condition, those children will continue to satisfy C_{org} in t_3 . Finally, n_x by definition involves the elimination of literals in the non-empty set LS_1 , which include a predicate-term from SP_1 . Given part (iii) of the definition of organizing-condition, these facts together imply that n_x will satisfy C_{org} , and thus C_{loc} .

That a node of the third kind will satisfy C_{loc} follows from the following considerations. In the work showing the validity of t_3 , it became apparent that the elimination E_2 which occurs at every such node in t_3 is a subset of the elimination E_1 that used to occur in it in t_1 (this was shown directly for nodes within the output to the relevant kind of calls to `tree_producer_2`, and would be seen to be true for all other nodes ascendant of replacements in the proof for the validity of the output of the relevant call to `tree_producer_3` which we described would run in a manner quite analogous to that for the call to `tree_producer_2`). This means that if E_2 includes a term in SP_1 , then so would E_1 have. We also know that since such a node used to be ascendant of a node either in N_1 or N_2 , it must have not originally satisfied C_{org} in t_1 . Thus if E_2 includes a term in SP_1 , the node would not have satisfied C_{loc} in t_1 . However, it can be seen that the node must have satisfied C_{loc} in t_1 . For consider that none of these nodes were the root of t_1 , and that t_1 's only C_{loc} -violating node is its root. Since the supposition of E_2 including a term in SP_1 leads to a contradiction, we know that it can't have included such a term. From Definition 10 of localizing-condition, this implies that this node satisfies C_{loc} in t_3 .

No Variable-Substitutions That t_3 involves no variable-substitutions is demonstrated by the fact that in proving the validity of nodes of t_3 which were not copied unaltered from t_1 , we relied only upon a definition of valid-resolution-step (Definition 3) pertinent to resolution-proofs that involve no variable-

substitutions. As for the nodes which were copied unaltered from t_1 , we know they involve no variable-substitutions because, by premise, t_1 did not. $\square$

Lemma 4.3. *For any localizing-condition C_{loc} and resolution-tree t_1 that involves no variable-substitutions, we can construct a localized-form of t_1 .*

Proof. If t_1 has one node which does not satisfy C_{loc} , then it must have some sub-tree (proper or not) st_1 whose only C_{loc} -violating node is its root. From Lemma 4.2, we know that we can construct a localized-form st_2 of st_1 that has no variable-substitutions. Given Lemma 4.1, if we replace st_2 with st_1 , we will end up with a tree t_2 whose root is associated to the same clause as that of t_1 , whose leaf-nodes are associated to a set of clauses that is a subset of that of t_1 , and that has strictly fewer nodes that do not satisfy C_{loc} . Clearly, t_2 will also involve no variable-substitutions. We can continue iteratively doing this, which will eventually arrive us at a localized-form of t_1 . $\square$

5 A localization theorem

Theorem 1. (localization) *We can construct a localized-form for every resolution-tree and localizing-condition.*

Proof. Consider an arbitrary resolution-tree t_1 and localizing-condition C_{loc} . Let t_2 be the complete-concretization of t_1 . Given Lemma 4.3, we can construct a localized-form t_3 of t_2 . We can then form an abstract-rebasis t_4 of t_3 using the set of clauses associated to the leaf-nodes of t_1 . Clearly, t_4 's root node and leaf-nodes will bear the needed relation to those of t_1 . What remains to be shown is that t_4 's nodes all satisfy C_{loc} . Given that t_4 is abstractly-equivalent to t_3 , and all of t_3 's nodes satisfy C_{loc} , Lemma 3.1 implies that all of t_4 's nodes will satisfy C_{loc} as well. $\square$

6 New proof of Craig's Interpolation Theorem

There are variations in the literature on how to formulate the theorem originally stated and proven by William Craig [2], but the below version will best suit our purposes.

Theorem 2. (Craig Interpolation) *If a statement A entails a statement C , then there is some statement B such that A entails B , B entails C , and every predicate and constant term appearing in B appears in both A and C .*

Proof. If A entails C , this implies that $(A \ \& \ \neg C)$ is not satisfiable. Given that resolution derivation is refutation-complete [1], this means that there exists a resolution derivation whose basis (set of axiomatic clauses) CL_1 is a conjunctive-normal-form representation of $(A \ \& \ \neg C)$, and which ends in the empty clause. Let t_1 be a resolution-tree representing this derivation.

For each constant that appears in A but not C , replace it in the tree with a unique skolem-constant (as though the constant was an existentially quantified

variable). It can be seen that this will produce a valid resolution-tree t_2 ending in the empty clause where :

- The axiomatic basis of t_2 is a clausal representation of $(A_1 \ \& \ \neg C)$, where
 - A_1 is an entailment of A
 - all of A_1 's terms appear in A
 - all constants in t_2 appear in C

Let SP_1 be the set of predicate-terms that are in A but not in C . Let C_{org} be the condition that is satisfied by any leaf-node including a term in SP_1 , or any non-leaf node which is such that it involves an elimination in SP_1 , and so do all of its non-leaf-node descendants. It can be seen that C_{org} is an organizing-condition. Let C_{loc} be the relevant localizing-condition defined in terms of C_{org} . Given Theorem 1 (localization), we can construct a localized-form t_3 of t_2 for C_{loc} . Consider the set N_1 of nodes in t_3 that involve an elimination in SP_1 but whose parents do not. It can be seen that :

- none of the clauses associated to nodes in N_1 include predicate-terms in SP_1
- the set of leaf-nodes with terms in SP_1 is exactly the set of leaf nodes descendant of nodes in N_1
- the set of leaf nodes descendant of nodes in N_1 together represent a statement A_2 where
 - A_1 entails A_2
 - A_2 's terms are all in A

If necessary, modify the clauses in the non-leaf inclusive-descendants of N_1 so that each step is the most general unifier of its children. It can be seen that this preserves the validity and result of the tree, and that now :

- The constant terms in the set of inclusive-descendants of members of N_1 all appear in A .
- And so the set of clauses assigned to N_1 corresponds to a statement A_3 such that :
 - A_2 implies A_3
 - A_3 's terms are all in both A and C

Let N_2 be the set of leaf-nodes that are not descendant of members of N_1 , and yet which correspond to part of the clausal representation of A . Clearly their associated terms are all in A . Since the tree as a whole only includes constant terms from C , they will not include any constant terms not in C . And since all the leaf nodes with terms in SP_1 are descendant of N_1 , they will not

include any predicate terms not in C . And so the set of clauses assigned to N_2 will only include terms which appear in both A and C .

If we then consider the set of nodes $\{N_1 \cup N_2\}$, we see that the set of clauses assigned to them corresponds to the representation of a statement A_4 which is an implication of A , and whose terms all appear in both A and C . Remove all the nodes that are descendant of N_1 , and we are left with a resolution-tree operating over a basis that is a clausal representation of $(A_4 \ \& \ \neg C)$, and which ends in the empty-clause. This shows that A_4 entails C . And so A_4 is our desired B . $\square$

References

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- [2] William Craig. “Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory”. In: *The Journal of Symbolic Logic* 22.3 (1957), pp. 269–285. DOI: 10.2307/2963594.