Polyharmonic Problems with Simply Supported Polynomial Data in the Unit Ball in R^n

Agah D. Garnadi

Abstract

We studied simply supported polynomial data of boundary value problem of Polyharmonic equation. The problem is reformulated as a systems of Laplace-Poisson equation with Polynomial Dirichlet problems. We utilize an exact algorithms for solving Laplace equations with Dirichlet conditions with polynomial data. The algorithm requires differentiation of the boundary function, but no integration.

Keywords : Polyharmonic, Harmonic Polynomials, Simply Supported, Dirichlet Problem

Mathematics Subject Classification : 31B05, 31B20

1 Introduction

In this paper, we studied Polyharmonic equation which have Dirichlet type boundary value problem with polynomials data in R^n . We approached the solution by Lemma that has been formulated by [2] and Lemma by [3].

Let $x = (x_1, x_2, x_3, \dots, x_n) \in R^n$, thus for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ of non-negative integers we say x^α as monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots x_n^{\alpha_n}$. The degree of x^α is $\alpha_1 + \alpha_2 + \alpha_3$. A polynomial is said to be homogeneous of degree m if it is a finite linear combination of monomial x^α of degree m ; $m = 0, 1, 2, \dots$. Let $\mathcal{P}_m$ denotes the vector space of polynomials in R^n , homogeneous of degree m , and $\mathcal{H}_m$ is the subspace of harmonic polynomials of degree m , then we have property by Axler and Ramey following

$$\mathcal{P}_m = \mathcal{H}_m + |x|^2 \mathcal{P}_{m-2}.$$

Theorem 1.1 *if $p \in \mathcal{P}_m$, then $p = \Lambda_m(p) + |x|^2 q$, for some $q \in \mathcal{P}_{m-2}$ and $\Lambda_m(p) \in \mathcal{H}_m$.*

Theorem 1.1 leads to the following corollary, which gives the well-known decomposition (2) and an explicit formula for p_m .

Corollary 1.2 *Every $p \in \mathcal{P}_m$ can be uniquely written in the form*

$$p = p_m + |x|^2 p_{m-2} + \dots + |x|^{2k} p_{m-2k}, \quad (1)$$

where $k = \lfloor \frac{m}{2} \rfloor$ and $p_j \in \mathcal{H}_j$ for each j . Furthermore, $p_m = \Lambda_m(0)$.

Let Ω is a unit sphere in R^n , with the boundary $\Gamma = \partial\Omega = \{x \in R^n \mid x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 = 1\}$. The Polyharmonic problem is as following:

$$\Delta^k u = 0 \text{ in } \Omega, k = 1, 2, \dots, k > 0 \quad (2)$$

with simply supported like boundary value problems

$$u|_{\Gamma} = q_k, \Delta u|_{\Gamma} = q_{k-1}, \dots, \text{ and } \Delta^{(k-1)} u|_{\Gamma} = q_0,$$

where, q_i , are polynomials function.

The main result of this paper is to state the Theorem for the solution of problem in (2). The solution of this problem is a polynomial function that given in Theorem 3.1.

2 Technical Lemmas

This following Lemmas is needed to solve Polyharmonic problem:

Lemma 2.1 *If $p \in \mathcal{P}_m$, then the solution to the Dirichlet problem, $u \in \mathcal{P}_m$, with boundary data $u|_{\Gamma} = p$ is*

$$p_m + p_{m-2} + \dots + p_{m-2k} \quad (3)$$

where $k = \lfloor \frac{m}{2} \rfloor$ and $p_m, p_{m-2}, \dots, p_{m-2k}$ are the harmonic polynomials.

Proof:

Suppose $p \in \mathcal{P}_m$, because $|x| = 1$ in boundary, by using (1) we have (3) equals to p on Γ . Obviously (3) is harmonic on R^n , and hence its restriction to Ω is the solution to the Dirichlet problem.

Lemma 2.1 said that if we have polynomial data for dirichlet problem then we can obtain the solution of the Laplace equation with this boundary value problem. The solution is a linear combination of harmonic functions. We recommend you to study the algorithm to obtain this harmonic polynomials in [2].

By setting $p_{m-1}, p_{m-3}, \dots$ are equal to 0, the solution u can be written as

$$u = \sum_{j=0}^m p_j$$

where $\sum_{j=0}^m p_j|_{\Gamma} = p$.

Lemma 2.2 [Herzog, 2000] *Given $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_{m+2}$ then there exists $u \in \mathcal{P}_{m+2}$ such that*

$$\Delta u(x) = q(x) \quad x \in \Omega \quad \text{and} \quad u|_{\Gamma} = p(x).$$

3 Main Result

Here we provide the Theorem about the solution of Polyharmonic type problem. The Existence of this solution can be presented in the following Theorem:

Theorem 3.1 *Given $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_{m+2}$ then there exists $u_h \in \mathcal{P}_m$ and $u_p \in \mathcal{P}_{m+2}$ such that $u = u_h + u_p$ is the solution of (2).*

Proof: The Polyharmonic type problem was presented in (2) can be written as following:

$$\Delta^{k-1}u = v \quad ; u|_{\Gamma} = q_k, \Delta u|_{\Gamma} = q_{k-1}, \dots, \Delta^{k-1}u|_{\Gamma} = q_1 \quad (4)$$

and

$$\Delta v = f \quad ; v|_{\Gamma} = q_0. \quad (5)$$

The solution of problem (4) can be formulated if we had solved the problem (5). The solution of problem (5), $v \in \mathcal{P}_{m+2}$, can be obtained by using Lemma 2.1 which is a harmonic polynomials function.

Suppose that the solution of (5) is $v = v_h + v_p$, where v_h is the solution of this problem:

$$\Delta v_h = 0 \quad ; v_h|_{\Gamma} = q_0, \quad (6)$$

and u_p is the solution of this problem:

$$\Delta v_p = f \quad ; v_p|_{\Gamma} = 0. \quad (7)$$

The solution v_h can be obtained by using Lemma 2.1 which is the harmonic polynomials function. Next, by solving problem (7) when the solution of this problem is guaranteed by Lemma 2.2, so we have v_p .

While equations (4) can be solved inductively using sequence of Poisson Equations with polynomial data following the same reason as in solving Triharmonic and Biharmonic equation in the unit ball in R^n with simply supported polynomial data, see ([4],[5]).

References

- [1] Sheldon Axler, Paul Bourdon, and Wade Ramey, Harmonic Function Theory, Graduate Text in Mathematics, Springer-Verlag, 1992.
- [2] S. Axler and W. Ramey, Harmonic polynomials and Dirichlet type problems, *proc. Amer. Math. Soc.* 123(12) (1995), 3765-3773.
- [3] G. Herzog, Polynomials solving Dirichlet boundary value problems, *Amer. Math. Monthly* 107(10) (2000), 934-936.
- [4] Garnadi, Agah D, 2017, Simply Supported Biharmonic Equations with Polynomial Data in The Unit Ball of R_n , **INARXiv**, DOI:10.17605/OSF.IO/VR689
- [5] Garnadi, Agah D, 2017, Tri-Dirichlet Type Problem with Polynomial Data in The Unit Sphere of R_n , **INARXiv**, DOI:10.17605/OSF.IO/ADU5X