Proving the Goldbach's conjecture

Ninh Khac Son

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Abstract

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

"Every even integer greater than 2 can be expressed as the sum of two primes".

Manuscript content: Prove that Goldbach's conjecture is correct.

Key words: Prime numbers, Goldbach's conjecture, number theory.

1. Notation system

We briefly mention the symbols and theorems in number theory to apply to this manuscript.

1.1. Notation

- Symbol of positive natural number: $\mathbf{N}^*$
- Symbol of prime number greater than 2: $\mathbf{P}^*$
- Symbol of odd-number greater than 2: $\mathbf{O}^*$

1.2. The operations express odd and prime numbers

- For every odd natural number $\mathbf{O}$ greater than 2, it can always be expressed as:

$$O = 2n + 1 \text{ (With : } O \in O^*, n \in N^*) \quad (1)$$

This deduces the result: For every odd natural number $\mathbf{O}'$ greater than 5, it can always be expressed as:

$$O' = 2n' + P \text{ (With : } n' \in N^*, P \in P^*, P < O') \quad (2)$$

This also deduces the result: For every prime number P greater than 5, it can always be expressed as:

$$P = 2m + P' \text{ (With : } m \in N^*, P' \in P^*, P' < P) \quad (3)$$

1.3. Bertrand's postulate

Bertrand's postulate is a theorem stating that for any integer $n > 3$, there always exists at least one prime number p with

$$n < p < 2n - 2$$

2. Goldbach's conjecture

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

"Every even integer greater than 2 can be expressed as the sum of two primes".

3. Proving the Goldbach's conjecture

3.1. Consider even integer numbers $2 < N \leq 10$

- For $N = 4$, represent: $N = 2 + 2$
- For $N = 6$, represent: $N = 3 + 3$
- For $N = 8$, represent: $N = 3 + 5$
- For $N = 10$, represent: $N = 3 + 7$

3.2. Consider even integer numbers $N > 10$

Because N is an even integer greater than 10, N can always be expressed as the sum of two odd numbers:

$$N = O_1 + O_2 \text{ (With : } O_1, O_2 \in O^*, O_1 < O_2) \quad (4)$$

Because the prime number P_1 is greater than 2 in the set P^* , it also belongs to the set O^* . Therefore, N can always be expressed as the sum of a prime number P_1 and an odd number O :

$$N = P_x + O \text{ (With : } \forall P_x \in P^*, O \in O^*, P_x < N/2) \quad (5)$$

Based on the result of (2), we can express the odd number O to:

$$O = P_x + 2n \text{ (With : } n \in N^*, P_x \in P^*, P_x < O) \quad (6)$$

Finally, this is stated as follows: With every even natural number $N > 10$, there is always at least a prime number P_1 in the set P^* such that

$$N = P_x + (P_x + 2n) \text{ (With : } P_x \in P^*, n \in N^*, P_x, n < N/2) \quad (7)$$

Example: Any even natural number N greater than 10, it can be expressed by $\forall P_x < N/2$ or $N = P_{x=const} + (P_{x=const} + 2n)$.

From expression (7), we transform to produce the result:

$$n = \frac{N}{2} - P_x \text{ (With : } P_x \in P^*, n \in N^*, P_x < N/2) \quad (8)$$

In particular, N has a given value, so the value of n will vary with P_x . While $P_x \in P^*$, we convert P_x into the set N^* to construct a function f :

$$f = \frac{N}{2} - x \text{ (With : } x \in N^*, x < N/2) \quad (9)$$

Thus, the value of f contains the value of n and the value of x contains the value of P_x . This means that the values of n and P_x always belong the graph of the function f .

On the other hand, based on the expressions (5) (6), since $N/2 < P_x + 2n < N$, this results in the value of n also in the graph of the function g :

$$g = x - \frac{N}{2} \text{ (With : } x \in N^*, N/2 < x < N) \quad (10)$$

Thus, the value of g contains the value of n and the value of x contains the value of $2n + P_x$. This means that the values of n and $2n + P_x$ always belong the graph of the function g .

Purpose of functions: We construct the functions f and g to apply the Bertrand's theorem to the x value of those two functions to find the prime numbers in the set $(N/2, N - 1)$. In which, the x value of the function f belongs to the set $x \in (0, N/2)$ and the x value of the function g belongs to the set

$x \in (N/2, N - 1)$ when these two functions are on the same coordinate system.

Graphing two functions f and g on the same coordinate system, we have:

(Two graphs f and g are on the same coordinate system)

Important argument:

We rewrite the expression (7): $N = P_x + (P_x + 2n)$.

Where $O = (P_x + 2n)$ with $O \in O^*$, and $O \in (N/2, N - 1)$. And we only focus on the value of $(P_x + 2n)$, because the value of $(P_x + 2n)$ is equivalent to the function $g = x - \frac{N}{2}$.

Applying Bertrand's theorem to the value of x of the function $g = x - \frac{N}{2}$, we have: There is always at least a prime number $x = P_y$ such that $N/2 < x = P_y < N - 1$.

- Assuming $P_y = P_x + 2m$ with $\forall P_x < N/2$, and $N = P_{x=const} + P_{y=const}$. This satisfies the problem.

- Assuming $P_y = P_x + 2m$, and $N \neq P_x + P_{y=const}$ with $\forall P_x < N/2$. This means:

$$N = O' + P_{y=max} \text{ (With : } O' \in O^*)$$

$$\implies N = P_{x=max} + (2n' + P_{y=max}) \text{ (With : } n' \in N^*)$$

- In which, $P_{y=max}$ is the largest prime number of the set $(N/2, N - 1)$; and $P_{x=max}$ is the largest prime number of the set $[3, N - P_{y=max}]$

To be satisfied with the expression (7), the following system of equations always has solutions with $n, n' \in N^*$:

$$\begin{cases} N = P_{x=max} + (P_{y=max} + 2n') \\ N = P_{x=max} + (P_{x=max} + 2n) \end{cases}$$

$$\implies P_{y=max} + 2n' = P_{x=max} + 2n \text{ (With : } n, n' \in N^*) \quad (11)$$

Consider function $K(n) = P_{x=max} + 2n$ with $n \in N^*, K(n) \in (N/2, N - 1)$:

- We continue to apply Bertrand's theorem to the function $K(n)$, a consequence of it has been proven as follows: "If $N > 10$, there are at least 2 primes $P_y, P_z \in (N/2, N)$ ". Therefore, in addition to the prime number P_y defined in the function $K(n)$, $K(n)$ has at least one more prime P_z if $P_z \in (N/2, N - 1)$; or $K(n)$ only exists one prime number P_y if $P_z = N - 1$.

- In the case of $K(n) = P_z \in (N/2, N - 1)$, this leads to a contradiction with expression (11). Because if $K(n) = P_z = P_{y=max} + 2n'$ then $n' < 0 \neq N^*$. Therefore to satisfy the problem, if there is more than one prime P_y in the set $(N/2, N-1)$ then N can always be expressed as the sum of two primes $N = P_x + P_y$

- In the case of $P_z = N - 1$: From the Bertrand's theorem, there always exist at least 2 primes in the set $(\frac{N-1}{2}, N - 1)$. But because the set $(N/2, N - 1)$ contains only one prime P_y , so the remaining prime will be equal to $N/2$ and N can be expressed as the sum of two primes $N = 2 * N/2$. (If $N/2 \notin P^*$ then there will be at least two primes in the set $(N/2, N - 1)$, this case has been analyzed before).

- Summary of analyzed cases: If $N \neq P_x + P_{y=const}$ with $\forall P_x < N/2$ then $N = P_x + P_{y=max}$; or $N = 2 * P$ with $P = N/2$.

Conclusion: For every even natural number $N > 10$, it can always be expressed as the sum of two primes, with $P_x, P_y \in P^*$, and $P_y = P_x + 2n$.

Combining with even natural numbers $2 < N \leq 10$ has been expressed as the sum of the two primes in section 3.1, leading us to prove that the Goldbach's conjecture is correct.

Proving end.

References

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