A Uniformly Distributed Congruence

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Abstract

An elementary congruence appears to be uniformly distributed. The Weyl criterion is used to confirm this.

1 Introduction

The congruence is $d(n - x) - (l - d)x \equiv -1 \pmod{l}$ where l is a prime, n is a natural number less than l , and d is a natural number less than l that satisfies the congruence $l|dn + 1$. For $d = 1, 2, 3, \dots, l - 1$, the solutions of $l|dn + 1$ are a permutation of the natural numbers $(1, 2, 3, \dots, l - 1)$. When a solution is found, the d and n values are used to solve the first congruence. This is used by Cox et al.[1][2] to derive optimal bounds for the length of rational Collatz and $3n+c$ cycles.

2 $(n - x, x)$ Values and Uniform Distributions

For a real number x , let $[x]$ denote the integral part of x and $\{x\}$ the fractional part. The sequence $\omega = \{x_n\}$, $n=1,2,3,\dots$ of real numbers is said to be uniformly distributed modulo 1 (abbreviated u.d. mod 1) if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ we have $\lim_{N \rightarrow \infty} \frac{A([a,b];N;\omega)}{N} = b - a$. The formal definition of u.d. mod 1 was given by Weyl [3] [4]. Let $\Delta : 0 = z_0 < z_1 < z_2 < \dots$ be a subdivision of the interval $[0, \infty)$ with $\lim_{k \rightarrow \infty} z_k = \infty$. For $z_{k-1} \leq x < z_k$ put $[x]_\Delta = z_{k-1}$ and $\{x\}_\Delta = \frac{x - z_{k-1}}{z_k - z_{k-1}}$ so that $0 \leq \{x\}_\Delta < 1$. The sequence of (x_n) , $n = 1, 2, 3, \dots$ of non-negative real numbers is said to be uniformly distributed modulo Δ (abbreviated u.d. mod Δ) if the sequence $(\{x_n\}_\Delta)$, $n = 1, 2, 3, \dots$ is u.d. mod 1. The notion of u.d. mod Δ was introduced by Levaque [5].

If f is a function having a Riemann integral in the interval $[a, b]$, then its integral is the limit of Riemann sums taken by sampling the function f in a set of points chosen from a fine partition of the interval. This is then a criterion for determining if a sequence is uniformly distributed. A sequence of real numbers is uniformly distributed (mod 1) if and only if for every Riemann-integrable function f on $[0, 1]$ one has $\lim_{N \rightarrow \infty} 1/N \sum_{n \leq N} f(\{x_n\}) = \int_0^1 f(x)dx$. In the

following, evidence that $n - x$ and x are u. d. mod Δ is presented using this criterion and Weyl's criterion [3] [4].

Weyl's criterion is that (γ_n) is u.d. mod 1 if and only if $\lim_{N \rightarrow \infty} 1/N \sum_{n=1}^N e^{2\pi i m \gamma_n} = 0$ for every integer $m \neq 0$. In the following, the z increments in the subdivision are set to $\sqrt{2}$ to avoid any aliasing with the integer $n - x$ and x values. A plot of the resulting sequence generated from the sorted $n - x$ values for $n = 1, 2, 3, \dots, l - 1$ and $l = 997$ is

Setting (γ_n) to such sequences in Weyl's formula and computing the moduli of the complex-valued results for $l = 1999$, $n - x$, and $m = 1$ gives

A cubic least-squares fit of the curve (where R-squared=0.9999) is included. The moduli for $l = 1999$, x , and $m = 1$ (excluding 16 values of zero in the input sequence) are

The moduli for $l = 3989$, $n - x$, and $m = 2$ are

The moduli for $l = 3989$, x , and $m = 2$ (excluding 6 values of zero in the input sequence) are

The moduli for $l = 5987$, $n - x$, and $m = 3$ are

The moduli for $l = 5987$, x , and $m = 3$ (excluding 8 values of zero in the input sequence) are

The moduli for $l = 7993$, $n - x$, and $m = 4$ are

The moduli for $l = 7993$, x , and $m = 4$ (excluding 32 values of zero in the input sequence) are

The moduli for $l = 9973$, $n - x$, and $m = 10$ are

The moduli for $l = 9973$, x , and $m = 10$ (excluding 18 values of zero in the input sequence) are

The functions $f(x)$ to be considered are x , x^2 , x^3 , x^4 , $\sqrt{x}$, $\sqrt{\sqrt{x}}$, $\log(x)$, e^x , $\sin(x)$, $\cos(x)$, $\tan(x)$, and $\frac{1}{a^2+x^2}$. The values of $\int_0^1 f(x)dx$ are $1/2$, $1/3$, $1/4$, $1/5$, $2/3$, $4/5$, -1 , 2.72 , 0.84 , 0.54 , 1.56 , and $\frac{1}{a} \tan^{-1} \frac{x}{a}$ (equal to 0.23 for $a = 2$ and 0.11 for $a = 3$) respectively. For $l = 997$ and the sequence generated from x , the results are 0.49 , 0.31 , 0.22 , 0.17 , 0.66 , 0.80 , -1.00 , 2.69 , 0.84 , 0.56 , 1.45 , 0.23 (for $a = 2$), and 0.11 (for $a = 3$) respectively. For $l = 9973$ and the sequence generated from $n - x$, the results are 0.50 , 0.34 , 0.25 , 0.20 , 0.67 , 0.80 , -0.97 , 2.74 , $.84$, $.54$, 1.55 , 0.23 (for $a = 2$), and 0.11 (for $a = 3$) respectively.

A plot of the integral for $f(x) = x^2$ is

A plot of the integral for $f(x) = e^x$ is

A plot of the integral for $f(x) = \sin(x)$ is

A plot of the integral for $f(x) = \frac{1}{a^2+x^2}$, $a = 3$ is

The trigonometric functions require a fixed amount to be added to the sequence values (apparently to change the phase). The exponential function also requires a fixed amount to be added to the sequence values - the same as for the cosine function. Apparently, this is due to Euler's formula $e^{ix} = \cos(x) + i \cdot \sin(x)$. Denote the amounts for sine and cosine by j and k respectively. These values satisfy the equation $j^2 + k^2 = \cos(1)$, similar to the formula $\sin(x)^2 + \cos(x)^2 = 1$. They also satisfy the equation $j/k = \sqrt{\tan(1)}$, similar to the formula $\sin(x)/\cos(x) = \tan(x)$. The amount required for the sine function is $\sin^{-1}(\cos(1))$. The amount required for the cosine function can be determined by using the formula $j^2 + k^2 = \cos(1)$. The amount required for the tangent function is $1/e$ ($e = \tan(1)/j$).

References

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