Hardness of modular factorial

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July 29, 2018

Abstract

In this note, we show that computing modular factorial (MODFACT) is a very hard task.

I. DEFINITION

We give the definition of what is the modular factorial function problem.

DEFINITION:

MODFACT: *Given n, m such that $m \leq n$, compute $(m! \bmod n)$*

In the following sections, we will give the application of MODFACT in some branches of mathematics - in particular number theory.

II. PRIMALITY TESTING

This section only defends the following observation.

OBSERVATION

If one can compute MODFACT, one can easily do primality testing. Indeed, by using Wilson's theorem, primality testing is a direct application of MODFACT.

WILSON'S THEOREM: *For $p \geq 2$, we have $(p-1)! \equiv -1 \bmod p$*

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PROOF: Let p be a prime; the claim is trivial if $p = 2$, so assume that p is odd. The idea is to look at pairs of the elements of $(\mathbb{Z}/p\mathbb{Z})^\times$. In fact, for every $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ there is an element $a^{-1} \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $a \cdot a^{-1} \equiv 1 \pmod{p}$. In general, $[a]$ and $[a^{-1}]$ are distinct: $[a] = [a^{-1}]$ implies $[a^2] = [1]$, so this can only happen (and does in fact happen) if $[a] = [1]$ or $[a] = [-1] = [p-1]$ (here we use that $(\mathbb{Z}/p\mathbb{Z})$ is a field; in fields, polynomials of degree 2 such as $x^2 - 1$ have at most 2 roots).

Thus, $(\mathbb{Z}/p\mathbb{Z})^\times \setminus \{[-1], [+1]\}$ is the union of pairs $\{[a], [a^{-1}]\}$ with $[a] \neq [a^{-1}]$, hence the product over all elements of $(\mathbb{Z}/p\mathbb{Z})^\times \setminus \{[-1], [+1]\}$ must be $[1]$. We can get $[(p-1)!]$ by multiplying this product with the two missing classes $[1]$ and $[-1]$, and this gives the claimed result $[(p-1)!] = [-1]$.

We still have to prove the converse: assume that $(n-1)! \equiv -1 \pmod{n}$; if p is a prime divisor of n , this congruence implies $(n-1)! \equiv -1 \pmod{p}$. But $p \leq n-1$ also implies that p occurs as factor of $(n-1)!$ on the left hand side, hence we would have $0 \equiv (n-1)! \pmod{p}$. But then $0 \equiv -1 \pmod{p}$, a contradiction. **Q.E.D**

III. SQUARE ROOTS OF -1

First, we give the following observation.

OBSERVATION

Let p be an odd prime and put $a = \left(\frac{p-1}{2}\right)!$; then we have that $a^2 \equiv (-1)^{(p-1)/2} \pmod{p}$. In particular, $a \equiv \pm 1 \pmod{p}$ if $p \equiv 1 \pmod{4}$, and $a^2 \equiv -1 \pmod{p}$ if $p \equiv 3 \pmod{4}$.

PROOF: We start with Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$; if, in the product $(p-1)!$, we replace the elements $\frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1$ by their negatives $-\frac{p+1}{2} \equiv \frac{p-1}{2}, -\frac{p+3}{2} \equiv \frac{p-3}{2}, \dots, -(p-1) \equiv 1 \pmod{p}$, then we have introduced exactly $\frac{p-1}{2}$ factors -1 ; thus $(p-1)! \equiv (-1)^{(p-1)/2} a^2 \pmod{p}$ with $a = \left(\frac{p-1}{2}\right)!$. This proves the claim. **Q.E.D**

Using this observations, it is fairly straightforward to compute a square root of -1 modulo a prime p congruent to 3 modulo 4 . But in the opposite direction, if we somehow compute both square roots (if one can compute one, then he can compute both) of -1 efficiently, we do not know which one among the two returned values is the modular factorial $a = \left(\frac{p-1}{2}\right)! \pmod{p}$. In a sense, computing square roots of -1 is easier than computing modular factorial.

IV. SUM OF TWO SQUARES

The following claim, together with the previous observation, will be necessary tools to shine lights on the observation of this section.

CLAIM: *Given an integer a not divisible by p , there exists $x, y \in \mathbb{Z}$ with $1 \leq |x|, |y| \leq \sqrt{p} - 1$ such that $ay \equiv x \pmod{p}$.*

PROOF: Let f be the smallest integer greater than $\sqrt{p}$, and consider the residue classes $\{[u + av] : 0 \leq u, v \leq f - 1\}$ modulo p . There are $f^2 \geq p + 1$ such expressions, but only p different residue classes, hence there must exist u, u', v, v' such that $u + av \equiv u' + av' \pmod{p}$. Put $x = u - u'$ and $y = v - v'$; then $x \equiv ay \pmod{p}$, and moreover $-f < x, y < f$. **Q.E.D**

Now, we are ready to give the observation of this section.

OBSERVATION

Every prime $p \equiv 1 \pmod{4}$ is a sum of two integral squares.

PROOF: Since $p \equiv 1 \pmod{4}$, there is an $a \in \mathbb{Z}$ such that $a^2 \equiv -1 \pmod{p}$. Using the previous claim, we imply that there are integers x and y such that $ay \equiv x \pmod{p}$ and $0 < x, y < \sqrt{p}$. Squaring gives $-y^2 \equiv x^2 \pmod{p}$, that is $x^2 + y^2 \equiv 0 \pmod{p}$. Since $0 < x^2, y^2 < p$, we find that $0 < x^2 + y^2 < 2p$; since $x^2 + y^2$ is divisible by p , we must have $x^2 + y^2 = p$. **Q.E.D**

That the solution to $p = x^2 + y^2$ in this case is unique is also known since Gauss. It is also worth noting that computing modular factorial does not help much to find a representation of a prime as sum of two squares. But in this case, another hard problem, namely USAT, can really help.

DEFINITION

USAT = $\{\Phi \text{ --- the Boolean formula } \Phi \text{ has exactly one satisfying variable assignment}\}$

Clearly, given the uniqueness of solution to $p = x^2 + y^2$ when $p \equiv 1 \pmod{4}$, the problem of finding such a representation is reducible to USAT. So, we see here that hardness of problems does not come alone but comes in many.

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