Lattice Model for Valuation

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ABSTRACT

This article proposes a lattice approach for valuing interest rate products within the LIBOR Market Model by introducing a shifted forward measure and several novel fast drift approximation methods. The model has similar accuracy to the current pricing models in the market, but is much faster. Some other merits of the model are that calibration is almost automatic and the approach is less complex and easier to implement than other current approaches.

Key Words: LIBOR Market Model, lattice model, drift approximation, risk management.

The LIBOR Market Model (LMM) is an interest rate model based on evolving LIBOR market forward rates under a risk-neutral forward probability measure. In contrast to models that evolve the instantaneous short rates (e.g., Hull-White, Black-Karasinski models) or instantaneous forward rates (e.g., Heath-Jarrow-Morton (HJM) model), which are not directly observable in the market, the objects modeled using the LMM are market observable quantities. The explicit modeling of market forward rates allows for a natural formula for interest rate option volatility that is consistent with the market practice of using the formula of Black for caps. It is generally considered to have more desirable theoretical calibration properties than short rate or instantaneous forward rate models.

In general, it is believed that Monte Carlo simulation is the only viable numerical method available for the LMM (see Piterbarg [2003]). The Monte Carlo simulation is computationally expensive, slowly converging, and notoriously difficult to use for calculating sensitivities and hedges. Another notable weakness is its inability to determine how far the solution is from optimality in any given problem.

I. LIBOR MARKET MODEL

The simply compounded forward rate reset at t for forward period (T_{k-1}, T_k) is defined by

$$F_k(t) := F(t; T_{k-1}, T_k) = \frac{1}{\delta_k} \left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1 \right) \quad (1)$$

where $P(t, T)$ denotes the time t price of a zero-coupon bond maturing at time T and $\delta_k := \delta(T_{k-1}, T_k)$ is the accrual factor or day count fraction for period (T_{k-1}, T_k) .

Inverting this relationship (1), we can express a zero coupon bond price in terms of forward rates as:

$$P(t, T_k) = P(t, T_{n(t)}) \prod_{j=n(t)}^k \frac{1}{1 + \delta_j F_j(t)} \quad (2)$$

LIBOR Market Model Dynamics

Consider a zero coupon bond numeraire $P(\bullet, T_i)$ whose maturity coincides with the maturity of the forward rate. The measure Q^i associated with $P(\bullet, T_i)$ is called T_i forward measure. Terminal measure Q^N is a forward measure where the maturity of the bond numeraire $P(\bullet, T_N)$ matches the terminal date T_N .

For brevity, we discuss the one-factor LMM only. The one-factor LMM (Brace et al. [1997]) under forward measure Q^i can be expressed as

$$\text{If } i < k, t \leq T_i, \quad dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=i+1}^k \frac{\delta_j \sigma_j(t) F_j(t)}{1 + \delta_j F_j(t)} dt + \sigma_k(t) F_k(t) dX_t \quad (3a)$$

$$\text{If } i = k, t \leq T_{k-1}, \quad dF_k(t) = \sigma_k(t) F_k(t) dX_t \quad (3b)$$

$$\text{If } i > k, t \leq T_{k-1}, \quad dF_k(t) = -\sigma_k(t) F_k(t) \sum_{j=k+1}^i \frac{\delta_j \sigma_j(t) F_j(t)}{1 + \delta_j F_j(t)} dt + \sigma_k(t) F_k(t) dX_t \quad (3c)$$

where X_t is a Brownian motion.

There is no requirement for what kind of instantaneous volatility structure should be chosen during the life of the caplet. All that is required is (see Hull-White [2000]):

$$(\hat{\sigma}_k)^2 := (\hat{\sigma}_k(T_{k-1}, K))^2 = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k^2(u) du \quad (4)$$

where $\hat{\sigma}_k$ denotes the market Black caplet volatility and K denotes the strike. Given this equation, it is obviously not possible to uniquely pin down the instantaneous volatility function. In fact, this specification allows an infinite number of choices. People often assume that a forward rate has a piecewise constant instantaneous volatility.

Here we choose the forward rate $F_k(t)$ has constant instantaneous volatility regardless of t (see Brigo-Mercurio [2006]).

Shifted Forward Measure

The $F_k(t)$ is a Martingale or driftless under its own measure Q^k . The solution to equation (3b) can be expressed as

$$F_k(t) = F_k(0) \exp\left(-\frac{1}{2} \int_0^t \sigma_k(s)^2 ds + \int_0^t \sigma_k(s) dX_s\right) \quad (5)$$

where $F_k(0) = F(0; T_{k-1}, T_k)$ is the current (spot) forward rate. Under the volatility assumption described above, equation (5) can be further expressed as

$$F_k(t) = F_k(0) \exp\left(-\frac{\sigma_k^2}{2} t + \sigma_k X_t\right) \quad (6)$$

Alternatively, we can reach the same Martingale conclusion by directly deriving the expectation of the forward rate (6); that is

$$\begin{aligned} E_0(F_k(t)) &= F_k(0) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_k^2}{2} t + \sigma_k X_t\right) \exp\left(-\frac{X_t^2}{2t}\right) dX_t \\ &= F_k(0) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(X_t - t\sigma_k)^2}{2t}\right) dX_t = F_k(0) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{Y_t^2}{2t}\right) dY_t = F_k(0) \end{aligned} \quad (7)$$

where X_t, Y_t are both Brownian motions with a normal distribution $(0, t)$ at time t , $E_t(\bullet) \equiv E(\bullet | \mathcal{F}_t)$ is the expectation conditional on the $\mathcal{F}_t$, and the variable substitution used for derivation is

$$Y_t = X_t - t\sigma_k \quad (8)$$

After applying this variable substitution (8), equation (6) can be expressed as

$$F_k(t) = F_k(0) \exp\left(-\frac{\sigma_k^2}{2} t + \sigma_k X_t\right) = F_k(0) \exp\left(\frac{\sigma_k^2}{2} t + \sigma_k Y_t\right) \quad (9)$$

Since the LMM models the complete forward curve directly, it is essential to bring everything under a common measure. The terminal measure is a good choice for this purpose, although this is by no means the only choice. The forward rate dynamic under terminal measure Q^N is given by

$$dF_k(t) = -\sigma_k F_k(t) \sum_{j=k+1}^N \frac{\delta_j \sigma_j F_j(t)}{1 + \delta_j F_j(t)} dt + \sigma_k F_k(t) dX_t \quad (10)$$

The solution to equation (10) can be expressed as

$$F_k(t) = F_k(0) \exp \left(\mu_k(t) - \int_0^t \frac{\sigma_k^2}{2} ds + \int_0^t \sigma_k dX_s \right) = F_k(0) \exp \left(\mu_k(t) - \frac{\sigma_k^2}{2} t + \sigma_k X_t \right) \quad (11a)$$

where the drift is given by

$$\mu_k(t) = -\int_0^t \sum_{j=k+1}^N \psi_j(s) \sigma_k \sigma_j ds = -\int_0^t \sum_{j=k+1}^N \frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \sigma_k \sigma_j ds \quad (11b)$$

where $\psi_j(s) = \delta_j F_j(s) / [1 + \delta_j F_j(s)]$ is the drift term.

Applying (8) to (11a), we have the forward rate dynamic under the shifted terminal measure as

$$F_k(t) = F_k(0) \exp \left(\mu_k(t) + \frac{\sigma_k^2}{2} t + \sigma_k Y_t \right) \quad (12)$$

Drift Approximation

Our model works backwards recursively from forward rate N down to forward rate k . The N -th forward rate $F_N(t)$ without drift can be determined exactly. By the time it takes to calculate the k -th forward rate $F_k(t)$, all forward rates from $F_{k+1}(t)$ to $F_N(t)$ at time t are already known. Therefore, the drift calculation (11b) is to estimate the integrals containing forward rate dynamics $F_j(s)$, for $j=k+1, \dots, N$, with known

beginning and end points given by $F_j(0)$ and $F_j(t)$. For completeness, we list all possible solutions below.

Frozen Drift (FD). Replace the random forward rates in the drift by their deterministic initial values, i.e.,

$$\mu_k(t) = -\int_0^t \sum_{j=k+1}^N \frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \sigma_k \sigma_j ds \approx -\sum_{j=k+1}^N \frac{\delta_j F_j(0)}{1 + \delta_j F_j(0)} \sigma_k \sigma_j t \quad (13)$$

Arithmetic Average of the Forward Rates (AAFR). Apply the midpoint rule (rectangle rule) to the random forward rates in the drift, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^N \frac{\delta_j \frac{1}{2}(F_j(0) + F_j(t))}{1 + \delta_j \frac{1}{2}(F_j(0) + F_j(t))} \sigma_k \sigma_j t \quad (14)$$

Arithmetic Average of the Drift Terms (AADT). Apply the midpoint rule to the random drift terms, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^N \frac{1}{2} \left(\frac{\delta_j F_j(0)}{1 + \delta_j F_j(0)} + \frac{\delta_j F_j(t)}{1 + \delta_j F_j(t)} \right) \sigma_j \sigma_k t \quad (15)$$

Geometric Average of the Forward Rates (GAFR). Replace the random forward rates in the drift by their geometric averages, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^N \frac{\delta_j \sqrt{F_j(0) \times F_j(t)}}{1 + \delta_j \sqrt{F_j(0) \times F_j(t)}} \sigma_j \sigma_k t \quad (16)$$

Geometric Average of the Drift Terms (GADT). Replace the random drift terms by their geometric averages, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^N \sqrt{\frac{\delta_j F_j(0)}{1 + \delta_j F_j(0)} \times \frac{\delta_j F_j(t)}{1 + \delta_j F_j(t)}} \sigma_j \sigma_k t \quad (17)$$

Conditional Expectation of the Forward Rate (CEFR). In addition to the two endpoints, we can further enhance our estimate based on the dynamics of the forward rates. The forward rate $F_j(s)$ follows the dynamic (9) (The drift term is ignored). We can derive the expectation of the forward rate conditional on the two

endpoints and replace the random forward rate in the drift by the conditional expectation of the forward rate.

Proposition 1. Assume the forward rate $F_j(s)$ follows the dynamic (9), with the two known endpoints given by $F_j(0)$ and $F_j(t)$. Based on the conditional expectation of the forward rate $F_j(s)$, the drift of $F_k(t)$ can be expressed as

$$\mu_k(t) \approx -\sum_{j=k+1}^N \int_0^t \frac{\delta_j E_0[F_j(s) | F_j(0), F_j(t)]}{1 + \delta_j E_0[F_j(s) | F_j(0), F_j(t)]} \sigma_j \sigma_k ds \quad (18a)$$

where the conditional expectation of the forward rate is given by

$$E_0[F_j(s) | F_j(0), F_j(t)] = F_j(0) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t} \right) \quad (18b)$$

Proof. See Appendix A.

Conditional Expectation of the Drift Term (CEDT). Similarly, we can calculate the conditional expectation of the drift term and replace the random drift term by the conditional expectation.

Proposition 2. Assume the forward rate $F_j(s)$ follows the dynamic (9), with the two known endpoints given by $F_j(0)$ and $F_j(t)$. Based on the conditional expectation of the drift term ψ_j , the drift of $F_k(t)$ can be expressed as

$$\mu_k(t) \approx -\sum_{j=k+1}^N \int_0^t E_0 \left(\frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \middle| F_j(0), F_j(t) \right) \sigma_j \sigma_k ds \quad (19a)$$

where the conditional expectation of the drift term is given by

$$E_0 \left(\psi_j(s) | F_j(0), F_j(t) \right) = E_0 \left(\frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \middle| F_j(0), F_j(t) \right) = 1 - \frac{1 + \nu_{Cj}(s) / \mu_{Cj}^2(s)}{\mu_{Cj}(s)} \quad (19b)$$

$$\mu_{Cj}(s) = 1 + \delta_j F_j(0) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t} \right) \quad (19c)$$

$$v_{C_j}(s) = \delta_j^2 F_j^2(0) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{2s}{t}} \left(\exp \left(\frac{\sigma_j^2 s(t-s)}{t} \right) - 1 \right) \exp \left(\frac{\sigma_j^2 s(t-s)}{t} \right) \quad (19d)$$

Proof. See Appendix A.

The accuracy and performance of these drift approximation methods are discussed in section IV.

II. THE LATTICE PROCEDURE IN THE LMM

The “lattice” is the generic term for any graph we build for the pricing of financial products. Each lattice is a layered graph that attempts to transform a continuous-time and continuous-space underlying process into a discrete-time and discrete-space process, where the nodes at each level represent the possible values of the underlying process in that period.

This article presents a grid lattice model for the LMM. To illustrate the lattice algorithm, we use a callable exotic as an example. Callable exotics are a class of interest rate derivatives that have Bermudan style provisions that allow for early exercise into various underlying interest rate products. In general, a callable exotic can be decomposed into an underlying instrument and an embedded Bermudan option.

We will simplify some of the definitions of the universe of instruments we will be dealing with for brevity. Assume the payoff of a generic underlying instrument is a stream of payments $Z_i = \delta_i [F_i(T_{i-1}) - C_i]$ for $i=1, \dots, N$, where C_i is the structured coupon. The callable exotic is a Bermudan style option to enter the underlying instrument on any of a sequence of notification dates $t_1^{ex}, t_2^{ex}, \dots, t_M^{ex}$. For any notification date $t = t_j^{ex}$, we define a right-continuous mapping function $n(t)$ by $T_{n(t)-1} \leq t < T_{n(t)}$. If the option is exercised at t , the reduced price of the underlying instrument, from the structured coupon payer’s perspective, is given by

$$\tilde{I}(t) := \frac{I(t)}{P(t, T_N)} = \sum_{i=n(t)}^N E_t \left(\frac{Z_i}{P(T_i, T_N)} \right) = \sum_{i=n(t)}^N E_t \left(\frac{\delta_i (F_i(T_{i-1}) - C_i)}{P(T_i, T_N)} \right) \quad (20)$$

where the ratio $\tilde{I}(t)$ is usually called the reduced value of the underlying instrument or the reduced exercise value or the reduced intrinsic value.

Lattice approaches are ideal for pricing early exercise products, given their “backward-in-time” nature. Bermudan pricing is usually done by building a lattice to carry out a dynamic programming calculation via backward induction and is standard. The lattice model described below also uses backward induction but exploits the Gaussian structure to gain extra efficiencies.

First we need to create the lattice. The random process we are going to model in the lattice is the LMM (12). Unlike traditional trees, we only position nodes at the determination dates (the payment and exercise dates). At each determination date, the continuous-time stochastic equation (12) shall be discretized into a discrete-time scheme. Such discretized schemes basically convert the Brownian motion into discrete variables. There is no restriction on discretization schemes. At any determination date t , for instance, we discretize the Brownian motion to be equally spaced as a grid of nodes $y_{i,t}$, for $i = 1, \dots, S_t$. The number of nodes S_t and the space between nodes $\varphi_t = y_{i,t} - y_{i-1,t}$ at each determination date can vary depending on the length of time and the accuracy requirement. The nodes should cover a certain number of standard deviations of the Gaussian distribution to guarantee a certain level of accuracy. We have the discrete form of the forward rate as

$$F_k(t; y_{i,t}) = F_k(0) \exp \left(\mu_k(t, y_{i,t}) + \frac{\sigma_k^2}{2} t + \sigma_k y_{i,t} \right) \quad (21)$$

The zero-coupon bond (2) can be expressed in discrete form as

$$P(t, T_k; y_{i,t}) = P(t, T_{n(t)}; y_{i,t}) \prod_{j=n(t)}^k \frac{1}{1 + \delta_j F_j(t; y_{i,t})} \quad (22)$$

We now have expressions for the forward rate (21) and discount bond (22), conditional on being in the state $y_{i,t}$ at time t , and from these we can perform valuation for the underlying instrument.

At the maturity date, the value of the underlying instrument is equal to the payoff, i.e.,

$$I(T_N, y_{i,T_N}) = Z_N(y_{i,T_N}) \quad (23)$$

The underlying state process X_t in the LMM (11) is a Brownian motion. The transition probability density from state $(x_{i,t}, t)$ to state $(x_{j,T}, T)$ is given by

$$p(x_{i,t}, t; x_{j,T}, T) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(x_{j,T} - x_{i,t})^2}{2(T-t)}\right] \quad (24)$$

Applying the variable substitution (8), equation (24) can be expressed as

$$p(y_{i,t}, t; y_{j,T}, T) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(y_{j,T} - y_{i,t} + \sigma_T T - \sigma_t t)^2}{2(T-t)}\right] \quad (25)$$

Equation (20) can be further expressed as a conditional value on any state $(y_{i,t}, t)$ as:

$$\frac{I(t; y_{i,t})}{P(t, T_N; y_{i,t})} = \sum_{j=n(t)}^N \frac{1}{\sqrt{2\pi(T_j - t)}} \int \frac{Z_j(y_{T_j})}{P(T_j, T_N; y_{T_j})} \exp\left[-\frac{(y_{T_j} - y_{i,t} + \sigma_{T_j} T_j - \sigma_t t)^2}{2(T_j - t)}\right] dy_{T_j} \quad (26)$$

This is a convolution integral. Some fast integration algorithms, e.g., Cubic Spline Integration, Fast Fourier Transform (FFT), etc., can be used for optimization. We use the Trapezoidal Rule Integration in this paper for ease of illustration.

Next, we determine the option values in each final notification node. On the last exercise date, if we have not already exercised, the reduced option value in any state $y_{i,M}$ is given by

$$\frac{V(t_M^{ex}, y_{i,M})}{P(t_M^{ex}, T_N; y_{i,M})} = \max\left(\frac{I(t_M^{ex}; y_{i,M})}{P(t_M^{ex}, T_N; y_{i,M})}, 0\right) \quad (27)$$

Then, we conduct the backward induction process that is performed by iteratively rolling back a series of long jumps from the final exercise date t_M^{ex} across notification dates and exercise opportunities until we reach the valuation date. Assume that in the previous rollback step t_j^{ex} , we calculated the reduced option value: $V(t_j^{ex}, y_{i,j}) / P(t_j^{ex}, T_N; y_{i,j})$. Now, we go to t_{j-1}^{ex} . The reduced option value at t_{j-1}^{ex} is

$$\frac{V(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} = \max \left\{ \frac{I(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})}, \frac{V^c(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} \right\} \quad (28a)$$

where the reduced continuation value is given by

$$\frac{V^c(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} = \frac{1}{\sqrt{2\pi(t_j^{ex} - t_{j-1}^{ex})}} \int \frac{V(t_j^{ex}, y_j)}{P(t_j^{ex}, T_N; y_j)} \exp \left[-\frac{(y_j - y_{i,j-1} + \sigma_j t_j^{ex} - \sigma_{j-1} t_{j-1}^{ex})^2}{2(t_j^{ex} - t_{j-1}^{ex})} \right] dy_j \quad (28b)$$

We repeat the rollback procedure and eventually work our way through the first exercise date. Then the present value of the Bermudan option is found by a final integration given by

$$pv_{Bermudan}(0) = P(0, T_N) \frac{1}{\sqrt{2\pi t_1^{ex}}} \int \frac{V(t_1^{ex}, y_1)}{P(t_1^{ex}, T_N; y_1)} \exp \left[-\frac{(y_1 + \sigma_1 t_1^{ex})^2}{2t_1^{ex}} \right] dy_1 \quad (29)$$

The present value or the price of the callable exotic from the coupon payer's perspective is:

$$pv_{payer}(0) = pv_{Bermudan}(0) - pv_{underly_instrument}(0) \quad (30)$$

This framework can be used to price any interest rate products in the LMM setting and can be easily extended to the Swap Market Model (SMM).

III. Calibration

First, if we choose the LMM as the central model, we need to price interest rate derivatives that depend on either or both of cap and swaption markets. Second, we will undoubtedly use various swaptions to hedge a callable exotic. It is a reasonable

expectation that the calibrated model we intend to use to price our exotic, will at least correctly price the market instruments that we intend to hedge with. Therefore, in an exotic derivative pricing situation, recovery of both cap and swaption markets might be desired.

The calibration of the LMM to caplet prices is quite straightforward. However, it is very difficult, if not impossible, to perfectly recover both cap and swaption markets. Fortunately for the LMM, there also exist extremely accurate approximate formulas for swaptions implied volatility, e.g., Rebonato's formula.

We introduced a parameter θ and set $\sigma_i = \theta \hat{\sigma}_i$ where $\hat{\sigma}_i$ denotes the market Black caplet volatility. One can choose different θ for different σ_i . For simplicity we describe one θ situation here. By choosing $\theta=1$, we have perfectly calibrated the LMM to the caplet prices in the market. However, our goal is to select a θ to minimize the sum of the squared differences of the volatilities derived from the market and the volatilities implied by our model for both caps and swaptions combined.

In the optimization, we use Rebonato's formula for an efficient expression of the model swaption volatilities, given by

$$\begin{aligned} (v_{\alpha,\beta}^{LMM})^2 &= \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{ij}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt \\ &= \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\hat{\sigma}_i\hat{\sigma}_j\theta^2}{S_{\alpha,\beta}(0)^2} = \theta^2 (v_{\sigma,\beta}^{Rebonato})^2 \end{aligned} \quad (31a)$$

where $\rho_{ij}=1$ under one-factor LMM. The swap rate $S_{\alpha,\beta}(0)$ is given by

$$S_{\alpha,\beta}(0) = \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(0) \quad (31b)$$

$$w_i(0) = \frac{\delta_i \prod_{j=\alpha+1}^{\beta} (1 + \delta_j F_j(0))^{-1}}{\sum_{k=\alpha+1}^{\beta} \delta_k \prod_{j=\alpha+1}^k (1 + \delta_j F_j(0))^{-1}} \quad (31c)$$

Assume the calibration containing M caplets and G swaptions. The error minimization is given by

$$\min \left| \sum_{i=1}^M (\theta \hat{\sigma}_i - \hat{\sigma}_i)^2 + \sum_{j=1}^G (\theta v_{\alpha+j,N}^{\text{Re bonato}} - \sigma_{\alpha+j,N}^{\text{sw}})^2 \right| \quad (32)$$

where $\sigma_{\alpha+j,N}^{\text{sw}}$ denotes the market Black swaption volatility. The optimization can be found at a stationary point where the first derivative is zero; that is,

$$\theta = \frac{\sum_{i=1}^M \hat{\sigma}_i + \sum_{j=1}^G \sigma_{\alpha+j,N}^{\text{sw}}}{\sum_{i=1}^M \hat{\sigma}_i + \sum_{j=1}^G v_{\alpha+j,N}^{\text{Re bonato}}} \quad (33)$$

In terms of forward volatilities, we use the time-homogeneity assumption of the volatility structure, where a forward volatility for an option is the same or close to the spot volatility of the option with the same time to expiry. The time-homogeneous volatility structure can avoid non-stationary behavior.

In the LMM, forward swap rates are generally not lognormal. Such deviation from the lognormal paradigm however turns out to be extremely small. Rebonato [1999] shows that the pricing errors of swaptions caused by the lognormal approximation are well within the market bid/ask spread. For most short maturity interest rate products, we can use the lattice model without calibration (33). However, for longer maturity or deeply in the money (ITM) or out of the money (OTM) exotics we may need to use the calibration and even some specific skew/smile adjustment techniques to achieve high accuracy.

IV. CONCLUSION

In this paper, we proposed a lattice model in the LMM to price interest rate products. Conclusions can be drawn, supported by the previous sections. First, the model is quite stable. The fast convergence behavior requires fewer discretization nodes. Second, this model has almost equivalent accuracy to the current pricing models in the market. Third, the implementation of the model is relatively easy. The

calibration is very simple and straightforward. Finally, the performance of the model is probably the best among all known approaches at the time of writing.

We use the following techniques in our model: shifted forward measure, drift approximation, probability distribution structure exploitation, long jump, numerical integration, incomplete information handling, and calibration. Combining these techniques, the model achieves sufficient accuracy in relatively few time steps and discrete nodes, which makes it a very efficient method.

For ease of illustration, we present the lattice model based on the Trapezoidal Rule integration. A better but slightly more complicated solution is to spline the payoff functions. The cubic spline of the option payoffs can achieve higher accuracy, especially for Greeks calculations, and higher speed. Although cubic spline takes some time, the lattice will require much fewer nodes (23 ~ 28 nodes are good enough) and can perform a much faster integration. In general, the spline method can provide a speedup factor around 3 ~ 5 times.

We have implemented the lattice model to price a variety of interest rate exotics. The algorithm can always achieve a fast convergence rate. The accuracy, however, is a bit trickier, depending on many factors: drift approximation approaches, numerical integration schemes, volatility selections, and calibration, etc. Some work, such as calibration, is more of an art than a science.

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APPENDIX A:

Proof of Proposition 1. We rewrite (9) as

$$Y(t) = \frac{1}{\sigma_j} \left(\ln \left(\frac{F_j(t)}{F_j(0)} \right) - \frac{\sigma_j^2 t}{2} \right) \quad (\text{A1})$$

In the general Brownian Bridge case when the Wiener process $Y(t)$ has $Y(t_1) = a$ and $Y(t_2) = b$, the distribution of $Y(t)$ at time $t \in (t_1, t_2)$ is normal given by

$$Y(t) \sim N \left(\mu_Y(t) = a + \frac{(t-t_1)(b-a)}{(t_2-t_1)}, \quad \nu_Y(t) = \frac{(t-t_1)(t_2-t)}{(t_2-t_1)} \right) \quad (\text{A2})$$

In our case: $t_1 = 0$, $t_2 = t$, $a=0$, $b=Y(t)$, $s \in (0, t)$, thus (A2) can be expressed as

$$Y(s) \sim N \left(\mu_Y(s) = \frac{s}{t} Y(t), \quad \nu_Y(s) = \frac{s(t-s)}{t} \right) \quad (\text{A3})$$

Let $A_j(s) = \sigma_j Y(s) + \sigma_j^2 s / 2$. According to the linear transformation rule, $A_j(s)$ is a normal given by

$$A_j(s) \sim N \left(\mu_{A_j}(s) = \sigma_j \mu_Y(s) + \frac{\sigma_j^2 s}{2} = \frac{s}{t} \ln \left(\frac{F_j(t)}{F_j(0)} \right), \quad \nu_{A_j}(s) = \sigma_j^2 \nu_Y(s) = \frac{\sigma_j^2 s(t-s)}{t} \right) \quad (\text{A4})$$

Let $B_j(s) = \exp(A_j(s))$. By definition, $B_j(s)$ is a lognormal given by

$B_j(s) \sim \text{LogN}(\mu_{A_j}(s), \nu_{A_j}(s))$. According to the characterizations of the lognormal

distribution, the mean and variance of $B_j(s)$ are

$$\mu_{B_j}(s) = E_0(B_j(s)) = \exp \left(\mu_{A_j} + \frac{\nu_{A_j}(s)}{2} \right) = \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{s}{t}} \exp \left(\frac{\sigma_j^2 s(t-s)}{2t} \right) \quad (\text{A5a})$$

$$\nu_{B_j}(s) = \left[\exp(\nu_{A_j}(s)) - 1 \right] \exp(2\mu_{A_j}(s) + \nu_{A_j}(s)) = \left(\exp \left(\frac{\sigma_j^2 s(t-s)}{t} \right) - 1 \right) \left(\frac{F_j(t)}{F_j(0)} \right)^{\frac{2s}{t}} \exp \left(\frac{\sigma_j^2 s(t-s)}{t} \right) \quad (\text{A5b})$$

We have the conditional expectation of the forward rate $F_j(s)$ as

$$E\left(F_j(s) \mid_{F_j(0), F_j(t)}\right) = F_j(0)E_0\left(B_j(s)\right) = F_j(0)\left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \quad (\text{A6})$$

Proof of Proposition 2. Let $C_j(s) = 1 + \delta_j F_j(s) = 1 + \delta_j F_j(0) B_j(s)$ where $B_j(s)$ is defined above. According to the linear transformation rule, $C_j(s)$ is a lognormal given by $C_j(s) \sim \text{LogN}(\hat{\mu}(s), \hat{\nu}(s))$. The mean and variance of $C_j(s)$ are

$$\mu_{C_j}(s) = 1 + \delta_j F_j(0) \mu_{B_j}(s) = 1 + \delta_j F_j(0) \left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \quad (\text{A7a})$$

$$\nu_{C_j}(s) = \delta_j^2 F_j(0)^2 \nu_{B_j}(s) = \delta_j^2 F_j(0)^2 \left(\exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) - 1 \right) \left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{2s}{t}} \exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) \quad (\text{A7b})$$

On the other hand, according to the characterizations of the lognormal distribution, the mean and variance of $C_j(s)$ are

$$\mu_{C_j}(s) = \exp\left(\hat{\mu}(s) + \frac{\hat{\nu}(s)}{2}\right) \quad (\text{A8a})$$

$$\nu_{C_j}(s) = [\exp(\hat{\nu}(s)) - 1] \exp(2\hat{\mu}(s) + \hat{\nu}(s)) \quad (\text{A8b})$$

Solving the equation (A8a) and (A8b), we get

$$\hat{\mu}(s) = \ln\left(\frac{\mu_{C_j}(s)}{\sqrt{1 + \nu_{C_j}(s) / \mu_{C_j}^2(s)}}\right) \quad (\text{A9a})$$

$$\hat{\nu}(s) = \ln\left(1 + \frac{\nu_{C_j}(s)}{\mu_{C_j}^2(s)}\right) \quad (\text{A9b})$$

We know the first negative moment of the lognormal is $E(C_j^{-1}(s)) = \exp(-\hat{\mu}(s) + \hat{\nu}(s)/2)$

and have the conditional expectation of the drift term as

$$\begin{aligned}
E_0 \left(\frac{\delta_j F_j(s)}{1 + \delta_j F_j(s)} \middle|_{F_j(0), F_j(t)} \right) &= E_0 \left(1 - \frac{1}{1 + \delta_j F_j(s)} \right) = 1 - E_0 \left(\frac{1}{C_j(s)} \right) \\
&= 1 - \exp \left(-\widehat{\mu}(s) + \frac{\widehat{\nu}(s)}{2} \right) = 1 - \frac{1 + \nu_{Cj}(s) / \mu_{Cj}^2(s)}{\mu_{Cj}(s)}
\end{aligned} \tag{A10}$$

where $\mu_{Cj}(s)$, $\nu_{Cj}(s)$ are given by (A7a) and (A7b).