Collateralization Study

Tim Xiao

ABSTRACT

This paper presents a framework that allows for collateralization adhering to bankruptcy laws. The framework is very useful for pricing outstanding defaultable derivatives. Our study shows that only a small portion of a swap premium is attributed to unsecured credit risk. We find empirical evidence that failure to properly account for collateralization may result in significant mispricing of financial contracts. The results indicate prima facie that the model performs quite well. The empirical tests corroborate the theoretical prediction on the impact of collateralization on swap rates.

Key words: collateralization, asset pricing, plumbing of the financial system, swap premium spread, OTC.

1. Introduction

Collateralization is an essential element in the so-called plumbing of the financial system that is the Achilles' heel of global financial markets. It allows financial institutions to reduce economic capital and credit risk, free up lines of credit, and expand the range of counterparties. All of these factors contribute to the growth of financial markets. The benefits are broadly acknowledged and affect dealers and end users, as well as the financial system generally.

Financial derivatives can be categorized into three types. The first category is over-the-counter (OTC) derivatives, which are customized bilateral agreements. The second group is cleared derivatives, which are negotiated bilaterally but booked with a clearinghouse. Finally, the third type is exchange-traded/listed derivatives, which are executed over an exchange. The differences between the three types are described in detail on the International Swap Dealers Association (ISDA) website (see ISDA (2013)).

The posting of collateral is regulated by the Credit Support Annex (CSA). The CSA was originally designed for OTC derivatives, but more recently has been updated for cleared/listed derivatives. For this reason people in the financial industry often refer to collateralized contracts as CSA contracts and non-collateralized contracts as non-CSA contracts.

From the perspective of collateral obligations, collateral arrangements can be unilateral or bilateral. In a unilateral arrangement, only one predefined counterparty has the right to call for collateral. Unilateral agreements are generally used when the other counterparty is much less creditworthy. In a bilateral arrangement, on the other hand, both counterparties have the right to call for collateral.

The use of collateral in financial markets has increased sharply over the past decade, yet analytical and empirical research on collateralization is relatively sparse. The effect of collateralization on financial contracts is an understudied area. Collateral management is often carried out in an ad-hoc manner, without reference to an analytical framework. Comparatively little research has been done to analytically and empirically assess the economic significance and implications of collateralization. Such a quantitative and empirical analysis is the primary contribution of this paper.

We obtain the CSA data from two investment banks. The data show that only 8.92% of CSA counterparties are subject to unilateral collateralization, while the remaining 91.08% are bilaterally collateralized. The data also reveal that all contracts in OTC markets are partially collateralized due to the mechanics of CSAs, which allow for the existence of limited unsecured exposures and set minimum transfer amounts (MTAs), whereas all contracts in cleared/listed markets are over-collateralized as all CCPs/Exchanges require initial margins. Therefore, full-collateralization does not exist in the real world¹. The reason for the popularity of full-collateralization is its mathematical simplicity and tractability.

In contrast to other collateralization models in current literature, we characterize a collateral process directly based on the fundamental principal and legal structure of the CSA agreement. A model is devised that allows for collateralization adhering to bankruptcy laws. As such, the model can back out differences in prices due to counterparty risk. This framework is very useful for valuing off-the-run or outstanding financial contracts subject to credit risk and collateralization, where the price quotes are not available. Given this model, we are able to explain credit-related spreads and provide an important tool for credit value adjustment (CVA).

Second, how can the model be empirically verified? To achieve the verification goal, this paper empirically measures the effect of collateralization on pricing and compares it with model-implied prices. This calls for data on financial contracts that have different collateral arrangements but are similar otherwise. We use a unique interest rate swap contract data set from an investment bank for the empirical study, as interest rate swaps collectively account for around two-thirds of both the notional and market value of all outstanding derivatives.

Prior research has primarily focused on the generic mid-market swap rates and results appear puzzling. Sorensen and Bollier (1994) believe that swap spreads (i.e., the difference between swap rates and par yields on similar maturity Treasuries) are partially determined by counterparty default risk. Whereas

¹ Singh (2010) and ECB (2009) come to a similar conclusion, although they do not provide any data to justify their statements.

Duffie and Huang (1996), Hentschel and Smith (1997), Minton (1997) and Grinblatt (2001) find weak or no evidence of the impact of counterparty credit risk on swap spreads. Collin-Dufresne and Solnik (2001) and He (2001) further argue that many credit enhancement devices, e.g., collateralization, have essentially rendered swap contracts risk-free. Meanwhile, Duffie and Singleton (1997), and Liu, Longstaff and Mandell (2006) conclude that both credit and liquidity risks have an impact on swap spreads. Moreover, Feldhütter and Lando (2008) find that the liquidity factor is the largest component of swap spreads. It seems that there is no clear-cut answer yet regarding the relative contribution of the liquidity and credit factors. Maybe, the recently revealed LIBOR scandal can partially explain these conflicting findings.

Unfortunately, we do not know the detailed allocation of a swap premium, i.e., what percentage of the adjustment is charged for each factor. Thus, a direct empirical verification is impossible.

Empirically, we find quite a number of CSA swap pairs in the data, where the two contracts in each pair have different counterparties but are otherwise the same. The test results demonstrate that the model-implied swap premium spreads are very close to the market swap premium spreads, indicating that the model is quite accurate.

Next, we assess the joint effect of both counterparty risk and collateralization on premium spreads. Since the model-generated swap premium spreads take into account both counterparty risk and collateralization, we present another regression model where the market swap premium spreads are regressed on the implied swap premium spreads. The estimation results show that the constant term is insignificantly different from zero; the slope coefficient is close to 1 and the adjusted R^2 is very high. This suggests that the implied premium spreads explains nearly all of the market premium spreads.

Although the practice recommended by CCPs is popular in the market, in which derivatives are continuously negotiated over-the-counter as usual but cleared and settled through clearinghouses, some market participants cast doubt on CCPs' economic equivalence claim. They find that cleared contracts have actually significant differences when compared with OTC trades. Some firms even file legal action against the clearinghouses, and accuse them of fraudulently inducing the firms to enter into cleared derivatives on the premise the contracts would be economically equivalent to OTC contracts (see Pengelly (2011)).

The remainder of this paper is organized as follows: Section 2 discusses unilateral collateralization. Section 3 elaborates bilateral collateralization. Section 4 presents empirical evidence. The conclusions and discussion are provided in Section 5. All proofs and some detailed derivations are contained in the appendices.

2. Unilateral Collateralization

A unilateral collateral arrangement is sometimes used when a higher-rated counterparty deals with a lower-rated counterparty, in which only one party, normally the lower-rated one, is required to deliver collateral to guarantee performance under the agreement. Typical examples of unilaterally collateralized contracts include: repos, cleared/listed derivatives, sovereign derivatives (see [AFME-ICMA-ISDA \(2011\)](#)), and some OTC derivatives².

Since the only reason for taking collateral is to reduce/eliminate credit risk, collateralization analysis is closely related to credit risk modeling. There are two primary types of models that attempt to describe default processes in the literature: structural models and reduced-form models. Many people in the market have tended to gravitate toward the reduced-form models given their mathematical tractability and market consistency. In the reduced-form framework, the stopping (or default) time τ of a firm is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined by,

$$\tau = \inf \left\{ t : \int_0^t h(s, \Gamma_s) ds \geq \Delta \right\} \quad (1)$$

where $h(t)$ or $h(t, \Gamma_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Γ_t , and Δ is a unit exponential random variable independent of Γ_t .

2.1 Valuation without collateralization

² Our CSA data from two investment banks show that 8.92% of CSA counterparties are subject to unilateral collateralization

Let valuation date be t . Consider a financial contract that promises to pay a $X_T > 0$ from party B to party A at maturity date T , and nothing before date T . We suppose that party A and party B do not have a CSA agreement. All calculations are from the perspective of party A . The risk free value of the financial contract is given by

$$V^F(t) = E[D(t, T)X_T | \mathcal{F}_t] \quad (3a)$$

where

$$D(t, T) = \exp\left[-\int_t^T r(u)du\right] \quad (3b)$$

where $E\{\bullet | \mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$, $D(t, T)$ denotes the risk-free discount factor at time t for the maturity T and $r(u)$ denotes the risk-free short rate at time u ($t \leq u \leq T$).

The survival and default probabilities of party B for the period $(t, t + \Delta t)$ are given by

$$\hat{p}(t) := p(t, t + \Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \quad (4a)$$

$$\hat{q}(t) := q(t, t + \Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \quad (4b)$$

The binomial default rule considers only two possible states: default or survival. For the one-period $(t, t + \Delta t)$ economy, at time $t + \Delta t$ the contract either defaults with the default probability $q(t, t + \Delta t)$ or survives with the survival probability $p(t, t + \Delta t)$. The survival payoff is equal to the market value $V^N(t + \Delta t)$ and the default payoff is a fraction of the market value³: $\varphi(t + \Delta t)V^N(t + \Delta t)$, where φ is the recovery rate. The non-CSA value of the contract at time t is the discounted expectation of all possible payoffs and is given by

$$V^N(t) = E\left\{\exp(-r(t)\Delta t) [\hat{p}(t) + \varphi(t)\hat{q}(t)] V^N(t + \Delta t) | \mathcal{F}_t\right\} \approx E\left\{\exp(-b(t)\Delta t) V^N(t + \Delta t) | \mathcal{F}_t\right\} \quad (5)$$

where $b(t) = r(t) + h(t)(1 - \varphi(t)) = r(t) + s(t)$ denotes the (short) risky rate and $s(t) = h(t)(1 - \varphi(t))$ denotes the (short) credit spread.

³ Here we use the recovery of market value (RMV) assumption.

Similarly, we have

$$V^N(t + \Delta t) = E\left\{\exp(-b(t + \Delta t)\Delta t)V^N(t + 2\Delta t)|\mathcal{F}_{t+\Delta t}\right\} \quad (6)$$

Note that $\exp(-b(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t + \Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V^N(t) &= E\left\{\exp(-b(t)\Delta t)V^N(t + \Delta t)|\mathcal{F}_t\right\} = E\left\{\exp(-b(t)\Delta t)E\left[\exp(-b(t + \Delta t)\Delta t)V^N(t + 2\Delta t)|\mathcal{F}_{t+\Delta t}\right]|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-\sum_{i=0}^1 b(t + i\Delta t)\Delta t\right)V^N(t + 2\Delta t)|\mathcal{F}_t\right\} \end{aligned} \quad (7)$$

By recursively deriving from t forward over T where $V^N(T) = X_T$ and taking the limit as Δt approaches zero, the non-CSA value of the contract can be obtained as

$$V^N(t) = E\left\{\exp\left[-\int_t^T b(u)du\right]X_T|\mathcal{F}_t\right\} \quad (8)$$

We may think of $b(u)$ as the risk-adjusted short rate. Equation (8) is the same as Equation (10) in Duffie and Singleton (1999), which is the market model for pricing risky bonds.

If we assume that a default may occur only on the payment date, the non-CSA value of the instrument in the discrete-time setting is given by

$$V^N(t) = E\left\{D(t, T)[p(t, T) + \varphi(T)q(t, T)]X_T|\mathcal{F}_t\right\} = E\left[I(t, T)X_T|\mathcal{F}_t\right] \quad (9)$$

where $I(t, T) = D(t, T)[p(t, T) + \varphi(T)q(t, T)]$ can be regarded as a risk-adjusted discount factor.

The difference between the risk-free value and the risky value is known as the *credit value adjustment* (CVA). CVA is required by regulators, such as, International Accounting Standard Board (IASB) and Basel committee. The CVA reflects the market value of counterparty risk or the cost of protection required to hedge counterparty risk and is given by

$$CVA(t) = V^F(t) - V^N(t) = E\left\{D(t, T)[q(t, T)(1 - \varphi(T))]X_T|\mathcal{F}_t\right\} \quad (10)$$

Since the recovery rate is always less than 1, we have $CVA(t) > 0$ or $V^N(t) < V^F$. In other words, the risky value is always less than the risk-free value. An intuitive explanation is that credit risk makes a financial contract less valuable.

2.2 Valuation with collateralization

The posting of collateral is regulated by the CSA that specifies a variety of terms including the threshold, the independent amount, and the minimum transfer amount (MTA), etc. The threshold is the unsecured credit exposure that a party is willing to bear. The MTA is used to avoid the workload associated with a frequent transfer of insignificant amounts of collateral. The independent amount plays the same role as the initial margin (or haircut).

Suppose that there is a unilateral CSA agreement between parties A and B in which only party B is required to deliver collateral when the mark-to-market (MTM) value arises over the collateral threshold H .

The default payment under a collateral agreement, therefore, can be mathematically expressed as

$$P^D(T) = \begin{cases} X_T & \text{if } C(T) \geq X_T \\ C(T) + \varphi(T)(X_T - C(T)) = \varphi(T)X_T + C(T)(1 - \varphi(T)) & \text{otherwise} \end{cases} \quad (11a)$$

or

$$P^D(T) = 1_{X_T \leq C(T)} X_T + 1_{X_T > C(T)} [C(T) + \varphi(T)(X_T - C(T))] \quad (11b)$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise, and $C(T)$ is the collateral amount at time T .

Let us consider repo/cleared/listed markets first. Contracts in these markets are always over collateralized, as the parties with collateral obligation are required to deposit initial margins and are also charged variation margins in response to changes in the market values. The total collateral (initial margin plus variation margin) posted at time t is given by

$$C(t) = V^C(t) - H(t) \quad (12)$$

where $V^C(t)$ is the CSA value of the contract at time t and $H(t)$ is the effective threshold. Note that for over-collateralization, the effective threshold is negative, i.e., $H(t) < 0$, which equals the negative initial margin. The collateral in equation (12) is a linear function of the asset value.

At time T , if the contract survives with probability $p(t, T)$, the survival value is the promised payoff X_T and the collateral taker returns the collateral to the collateral provider. If the contract defaults with probability $q(t, T)$, the collateral taker has recourse to the collateral and obtains a default payment up to the full value of the promised payoff X_T . The remaining collateral $C(T) - X_T$ returns to the collateral provider. The CSA value of the over collateralized contract is the discounted expectation of the payoffs and is given by

$$V^C(t) = E[D(t, T)(q(t, T)X_T + p(t, T)X_T) | \mathcal{F}_t] = V^F(t) \quad (13)$$

Equation (13) tells us that *the CSA value of an over-collateralized contract is equal to the risk-free value*. This result is consistent with the market practice in which market participants commonly assume that repos and cleared contracts are virtually free of default risk because of the implicit guarantee of the contracts provided by the clearinghouse and backup collateral.

Next, we turn to OTC markets. We obtain the CSA data from two investment banks. The data show that 61.32% of CSA counterparties have a zero threshold, and the remaining 38.68% use a positive threshold ranging from 25,000 to 750,000,000. Moreover, all CSA counterparties in the data maintain a positive MTA ranging from 500 to 60,000,000, which means that the effective thresholds are always greater than zero, i.e., $H(t) > 0$. If the value of the contract $V^C(t)$ is less than the effective threshold $H(t)$, no collateral is posted; otherwise, the required collateral is equal to the difference between the contract value and the effective threshold. The collateral amount posted at time t can be expressed mathematically as

$$C(t) = \begin{cases} V^C(t) - H(t) & \text{if } V^C(t) > H(t) \\ 0 & \text{otherwise} \end{cases} \quad (14a)$$

or

$$C(t) = 1_{V^C(t) > H(t)} (V^C(t) - H(t)) \quad (14b)$$

Since all CSA derivatives in OTC markets are partially collateralized, the default claim is almost certainly greater than the collateral amount. For a discrete one-period (t, T) economy, the collateral amount $C(t)$ posted at time t is defined in (14). At time T , if the contract survives, the survival value is the promised payoff X_T and the collateral taker returns the collateral to the collateral provider. If the contract defaults, the collateral taker possesses the collateral. The portion of the default claim that exceeds the collateral value is treated as an unsecured claim. Thus, the default payment is $C(T) + \varphi(T)(X_T - C(T))$, where $C(T) = C(t) / D(t, T)$ is the future value of the collateral. Since the most predominant form of collateral is cash according to ISDA (2012), it is reasonable to consider the time value of money only for collateral assets. The large use of cash means that collateral is both liquid and not subject to large fluctuations in value. It can be seen from this, that collateral does not have any bearing on survival payoffs; instead, it takes effect on default payments only. The CSA value of the partially collateralized contract is the discounted expectation of all the payoffs and is given by

$$V^C(t) = E\{D(t, T)[q(t, T)(C(T) + \varphi(T)(X_T - C(T))) + p(t, T)X_T] | \mathcal{F}_t\} \quad (15)$$

Proposition 1: The unilateral CSA value of the partially collateralized single-payment contract is given by

$$V^C(t) = E[F(t, T)X_T | \mathcal{F}_t] - G(t, T) \quad (16a)$$

where

$$F(t, T) = (1_{V^N(t) \leq H(t)} + 1_{V^N(t) > H(t)} / \bar{I}(t, T)) I(t, T) D(t, T) \quad (16b)$$

$$G(t, T) = 1_{V^N(t) > H(t)} H(t) \bar{q}(t, T) (1 - \varphi(T)) / \bar{I}(t, T) \quad (16c)$$

where $\bar{I}(t, T) = E(I(t, T) | \mathcal{F}_t)$. $I(t, T)$ and $V^N(t)$ are defined in (9).

Proof: See the Appendix.

We may think of $F(t, T)$ as the unilaterally CSA-adjusted discount factor and $G(t, T)$ as the cost of bearing unsecured credit risk. Proposition 1 tells us that the value of the unilaterally collateralized

contract is equal to the present value of the payoff discounted by the unilaterally CSA-adjusted discount factors minus the cost of taking unsecured counterparty risk.

We discuss the following two cases. Case 1: $H(t)=0$ corresponds to full-collateralization. We have $V^C(t)=V^F(t)$ according to (16) where $H(t)=0$ and $V^N(t)>0$. That is to say: the CSA value under full-collateralization is equal to the risk-free value, which is in line with the results of Johannes and Sundareshan (2007), Fujii and Takahashi (2012), and Piterbarg (2010). Due to the mathematical tractability and simplicity, previous modeling works focus on full-collateralization only. As we point out above, however, full-collateralization does not appear to exist in any markets.

Proposition 1 can be easily extended from one-period to multiple-periods. Suppose that a defaultable portfolio/contract has m netted cash flows. Let the m cash flows be represented as $X_i > 0$ with payment dates T_i , where $i = 1, \dots, m$. We derive the following proposition:

Proposition 2: *The unilateral CSA value of the partially collateralized multiple-payment contract is given by*

$$V^C(t) = \sum_{i=1}^m E \left[\prod_{j=0}^{i-1} (F(T_j, T_{j+1})) X_i \mid \mathcal{F}_t \right] - \sum_{i=0}^{m-1} E \left[\prod_{j=0}^{i-1} (F(T_j, T_{j+1})) G(T_i, T_{i+1}) \mid \mathcal{F}_t \right] \quad (17a)$$

where

$$F(T_j, T_{j+1}) = (1_{J(T_j, T_{j+1}) \leq H(T_j)} + 1_{J(T_j, T_{j+1}) > H(T_j)} / \bar{I}(T_j, T_{j+1})) I(T_j, T_{j+1}) D(T_j, T_{j+1}) \quad (17b)$$

$$G(T_j, T_{j+1}) = 1_{J(T_j, T_{j+1}) > H(T_j)} H(T_j) \bar{q}(T_j, T_{j+1}) (1 - \phi(T_{j+1})) / \bar{I}(T_j, T_{j+1}) \quad (17c)$$

$$J(T_j, T_{j+1}) = E \left[D(T_j, T_{j+1}) I(T_j, T_{j+1}) (V^C(T_{j+1}) + X_{j+1}) \mid \mathcal{F}_{T_j} \right] \quad (17d)$$

Proof: See the Appendix.

The valuation in Proposition 2 has a backward nature. The intermediate values are vital to determine the final price. For a payment period, the current price has a dependence on the future price. Only on the final payment date T_m , the value of the contract and the maximum amount of information needed to determine the $J(T_{m-1}, T_m)$, $F(T_{m-1}, T_m)$ and $G(T_{m-1}, T_m)$ are revealed. This type of problem can be best

solved by working backward in time, with the later value feeding into the earlier ones, so that the process builds on itself in a recursive fashion, which is referred to as *backward induction*. The most popular backward induction valuation algorithms are lattice/tree and regression-based Monte Carlo.

3. Conclusion

This article addresses a very important topic of the impact of collateralization on asset prices and risk management. This is the so called plumbing of financial system that affects many outcomes.

Empirically, we use a unique proprietary data set to measure the effect of collateralization on pricing and compare it with model-implied prices. We find a quite number of swap contracts that have different collateral arrangements but are similar otherwise. Presumably, differences in swap rates should be largely attributable to differences in counterparty risk. The empirical results show that model-implied prices are quite close to market-quoted prices, which suggests the model is fairly accurate in pricing collateralized contracts.

We also find evidence that financial institutions are sprinting to comply with the Dodd-Frank Act. In the new practice, contracts are continuously negotiated over-the-counter as usual, but cleared and settled via clearinghouses, as clearinghouses claim that cleared contracts would be economically equivalent to their OTC counterparts. As a result, swap premia in cleared market are determined in a similar way to those in OTC markets. We argue that this practice may not be appropriate because we notice that the clearing mechanics of a clearinghouse change the risk structure and thereby the asset prices. In fact, we find that cleared derivatives are not economically equivalent to their OTC counterparts.

Appendix

Proof of Proposition 1: If the contract survives, the survival value is the promised payoff X_T . If the contract defaults, the default payment is $C(T) + \varphi(T)(X_T - C(T))$. The CSA value of the contract is the discounted expectation of the payoffs and is given by

$$V^C(t) = E\{D(t, T)[q(t, T)(C(T) + \varphi(T)(X_T - C(T))) + p(t, T)X_T] | \mathcal{F}_t\} \quad (A1)$$

The collateral posted at time t is given by (14). The value of the collateral at time T becomes $C(T) = C(t) / D(t, T)$, where we consider the time value of money only for the collateral asset. If $V^C(t) > H(t)$ (i.e., $C(t) = V^C(t) - H(t)$), we have

$$V^C(t) = V^N(t) / \bar{I}(t, T) - H(t) \bar{q}(t, T) (1 - \varphi(T)) / \bar{I}(t, T) \quad (A2)$$

where $I(t, T) = D(t, T)[p(t, T) + \varphi(T)q(t, T)]$, $\bar{I}(t, T) = E[I(t, T) | \mathcal{F}_t]$, and $V^N(t) = E[I(t, T)X_T | \mathcal{F}_t]$.

In this case, $V^C(t) > H(t)$ is equivalent to $V^N(t) > H(t)$.

If $V^C(t) \leq H(t)$ (i.e., $C(t) = 0$), we have

$$V^C(t) = V^N(t) \leq H(t) \quad (A3)$$

Combining the two cases of $V^C(t) > H(t)$ and $V^C(t) \leq H(t)$, we have

$$V^C(t) = 1_{V^N(t) \leq H(t)} V^N(t) + 1_{V^N(t) > H(t)} [V^N(t) / \bar{I}(t, T) - H(t) \bar{q}(t, T) (1 - \varphi(T)) / \bar{I}(t, T)] \quad (A4)$$

or

$$V^C(t) = E[F(t, T)X_T | \mathcal{F}_t] - G(t, T) \quad (A5a)$$

where

$$F(t, T) = (1_{V^N(t) \leq H(t)} + 1_{V^N(t) > H(t)} / \bar{I}(t, T)) I(t, T) D(t, T) \quad (A5b)$$

$$G(t, T) = 1_{V^N(t) > H(t)} H(t) \bar{q}(t, T) (1 - \varphi(T)) / \bar{I}(t, T) \quad (A5c)$$

Proof of Proposition 2: Let $t = T_0$. On the first payment day, let $V^C(T_1)$ denote the CSA value of the contract excluding the current cash flow X_1 . According to Proposition 1, the CSA value of the contract at t is given by

$$V^C(t) = E[F(T_0, T_1)(X_1 + V^C(T_1)) | \mathcal{F}_t] - G(T_0, T_1) \quad (A6)$$

Similarly, we have

$$V^C(T_1) = E[F(T_1, T_2)(X_2 + V^C(T_2)) | \mathcal{F}_{T_1}] - G(T_1, T_2) \quad (A7)$$

Note that $G(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to the *taking out what is known* and tower properties of conditional expectation, we have

$$\begin{aligned} V^C(t) &= E[F(T_0, T_1)(X_1 + V^C(T_1)) | \mathcal{F}_t] - G(T_0, T_1) \\ &= \sum_{i=1}^2 E\left[\prod_{j=0}^{i-1} (F(T_j, T_{j+1})) X_i | \mathcal{F}_t\right] + E\left[\prod_{j=0}^1 (F(T_j, T_{j+1})) V^C(T_2) | \mathcal{F}_t\right] \\ &\quad - \sum_{i=0}^1 E\left[\prod_{j=0}^{i-1} (F(T_j, T_{j+1})) G(T_i, T_{i+1}) | \mathcal{F}_t\right] \end{aligned} \quad (\text{A8})$$

By recursively deriving from T_2 forward over T_m , where $V^C(T_m) = X_m$, we have

$$V^C(t) = \sum_{i=1}^m E\left[\prod_{j=0}^{i-1} (F(T_j, T_{j+1})) X_i | \mathcal{F}_t\right] - \sum_{i=0}^{m-1} E\left[\prod_{j=0}^{i-1} (F(T_j, T_{j+1})) G(T_i, T_{i+1}) | \mathcal{F}_t\right] \quad (\text{A9})$$

Proof of Proposition 3: At time T , there are a total of four possible states shown in Table 1. The joint distributions of A and B are given by (18) and the collateral amount posted at time t is given by (19). Depending on whether the value is in the money or out of the money at T , we have the following situations

Case 1: $V^C(t) > H_B(t) > 0$. We have $C(t) = V^C(t) - H_B(t)$ and

$$\begin{aligned} V^C(t) &= E\left\{D(t, T) \left[(p_{00}(t, T) + p_{10}(t, T)) X_T + (p_{01}(t, T) + p_{11}(t, T)) (C(T) + \varphi_B(T) (X_T - C(T))) \right] | \mathcal{F}_t \right\} \\ &= E\left\{D(t, T) \left[p_B(t, T) X_T + q_B(t, T) (\varphi_B(T) X_T + (V^C(t) - H(t)) (1 - \varphi_B(T)) / D(t, T)) \right] | \mathcal{F}_t \right\} \end{aligned} \quad (\text{A10a})$$

or

$$V^C(t) = V_B^N(t) / \bar{I}_B(t, T) - H_B(t) \bar{q}_B(t, T) (1 - \varphi_B(T)) / \bar{I}_B(t, T) \quad (\text{A10b})$$

where $V_B^N(t) = E[I_B(t, T) X_T | \mathcal{F}_t]$. In this case, $V^C(t) > H_B(t) > 0$ is equivalent to $V_B^N(t) > H_B(t)$.

Case 2: $0 \leq V^C(t) \leq H_B(t)$. We have $C(t) = 0$ and

$$V^C(t) = E\left\{D(t, T) (p_B(t, T) + q_B(t, T) \varphi_B(T)) X_T | \mathcal{F}_t\right\} = V_B^N(t) \quad (\text{A11})$$

In this case, $0 \leq V^C(t) \leq H_B(t)$ is equivalent to $0 \leq V_B^N(t) \leq H_B(t)$

Case 3: $V^C(t) < H_A(t) < 0$. We have $C(t) = V^C(t) - H_A(t)$ and

$$\begin{aligned} V^C(t) &= E\left\{D(t, T) \left[(p_{00}(t, T) + p_{01}(t, T)) X_T + (p_{10}(t, T) + p_{11}(t, T)) (C(T) + \varphi_A(T) (X_T - C(T))) \right] | \mathcal{F}_t \right\} \\ &= E\left\{D(t, T) \left[p_A(t, T) X_T + q_A(t, T) (\varphi_A(T) X_T + C(T) (1 - \varphi_A(T))) \right] | \mathcal{F}_t \right\} \end{aligned} \quad (\text{A12a})$$

or

$$V^C(t) = V_A^N(t) / \bar{I}_A(t, T) - H_A(t) \bar{q}_A(t, T) (1 - \varphi_A(T)) / \bar{I}_A(t, T) \quad (\text{A12b})$$

where $V_B^N(t) = E[I_B(t, T)X_T | \mathcal{F}_t]$. In this case, $V^C(t) < H_A(t) < 0$ is equivalent to $V_A^N(t) < H_A(t)$.

Case 4: $0 \geq V^C(t) \geq H_A(t)$. We have $C(t) = 0$ and

$$V^C(t) = E\{D(t, T)(p_A(t, T) + q_A(t, T)\varphi_A(T))X_T | \mathcal{F}_t\} = V_A^N(t) \quad (\text{A13})$$

In this case, $0 > V^C(t) \geq H_A(t)$ is equivalent to $0 \geq V_A^N(t) \geq H_A(t)$.

Combining the four cases together, we have

$$V^C(t) = E[L(t, T)X_T | \mathcal{F}_t] - M(t, T) \quad (\text{A14a})$$

Where

$$L(t, T) = D(t, T) \left[1_{0 \leq V_B^N(t) \leq H_B(t)} I_B(t, T) + 1_{V_B^N(t) > H_B(t)} I_B(t, T) / \bar{I}_B(t, T) + 1_{0 \geq V_A^N(t) \geq H_A(t)} I_A(t, T) + 1_{V_A^N(t) < H_A(t)} I_A(t, T) / \bar{I}_A(t, T) \right] \quad (\text{A14b})$$

$$M(t, T) = 1_{V_B^N(t) > H_B(t)} H_B(t) \bar{q}_B(t, T) (1 - \varphi_B(T)) / \bar{I}_B(t, T) + 1_{V_A^N(t) < H_A(t)} H_A(t) \bar{q}_A(t, T) (1 - \varphi_A(T)) / \bar{I}_A(t, T) \quad (\text{A14c})$$

where $\bar{I}_j(t, T) = E[I_j(t, T) | \mathcal{F}_t] = E(p_j(t, T) + \varphi_j(T)q_j(t, T) | \mathcal{F}_t)$, $V_j^N(t) = E[I_j(t, T)X_T | \mathcal{F}_t]$, and $j = A$ or B .

References

AFME, ICMA, and ISDA, 2011, “The impact of derivative collateral policies of European sovereigns and resulting Basel III capital issues.”

Cont, R. and T. Kokholm, 2011, “Central clearing of OTC derivatives bilateral vs. multilateral netting,” Working paper.

Duffie, D. and M. Huang, 1996, “Swap rates and credit quality,” *Journal of Finance*, 51, 921-949.

Duffie, D., and H. Zhu, 2011, “Does a central clearing counterparty reduce counterparty risk?” *Review of Asset Pricing Studies*, 74-95.

Feldhutter, P. and D. Lando, 2008, "Decomposing swap spreads," *Journal of Financial Economics*, 88 (2), 375-405.

FinPricing, 2018, Swap Data <https://finpricing.com/lib/EqCppi.html>

European Central Bank (ECB), 2009, "Credit default swaps and counterparty risk."

Garlson, D., 1991-1992, "Secured creditors and the early character of bankruptcy valuations," 41 *American University Law Review*, 63, 63-106.

He, H., 2001, "Modeling Term Structures of Swap Spreads," working paper, Yale School of Management, Yale University.

Hentschel, L. and C. W. Smith, Jr., 1997, "Risks in Derivatives Markets: Implications for the Insurance Industry." *Journal of Risk and Insurance*, 64, 323-346.

ISDA, 2013, Product Descriptions and Frequently Asked Questions (1. Major derivative categories; 2. How do cleared derivatives differ from OTC derivatives?),

Johannes, M. and S. Sundaresan, 2007, "The impact of collateralization on swap rates," *Journal of Finance*, 62, 383-410.

Liu, J., F. Longstaff, and R. Mandell, 2006, "The Market Price of Risk in Interest Rate Swaps: The Roles of Default and Liquidity Risks," *The Journal of Business*, vol. 79, pages 2337-2360.

Minton, B., 1997, "An Empirical Examination of Basic Valuation Models for Plain Vanilla U.S. Interest Rate Swaps." *Journal of Financial Economics*, 251-277.

O'Kane, D. and S. Turnbull, 2003, "Valuation of credit default swaps," *Fixed Income Quantitative Credit Research*, Lehman Brothers, QCR Quarterly, 2003 Q1/Q2, 1-19.

Pirrong, C., 2011, "The economics of central clearing: theory and practice," ISDA Discussion Papers Series.

Routh, R. and M. Douglas, 2005, "Default interest payable to oversecured creditor subject to reasonableness limitation," ABF Journal, Vol. 4, No. 3.

Sorensen, E. and T. Bollier, 1994, "Pricing swap default risk," Financial Analysts Journal, 50, 23-33.