A GENERALIZATION OF HASSE'S SERIES REPRESENTATION FOR THE RIEMANN ZETA FUNCTION

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ABSTRACT. We derive the following generalization of Hasse's series representation for the Riemann zeta function

$$\zeta(s) = \frac{(-1)^n}{s-1} \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{a=0}^k (-1)^a \cdot S(n, a) \cdot a! \cdot \sum_{m=0}^{k-a} \binom{k}{m} \cdot \frac{(-1)^m}{(m+1)^{s+n-1}}$$

where $s \neq 1$, n being any non-negative integer, and $S(n, k)$ denotes the Stirling number of the second kind. The special cases of the above representation are Ser's and Hasse's global series representation for the Riemann zeta function.

Let $\zeta(s)$ represent the Riemann zeta function. H. Hasse [1] proved that

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^{s-1}} \quad (\text{where } s \neq 1). \quad (1)$$

Joseph Ser [2] proved that

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^s} \quad (\text{where } s \neq 1). \quad (2)$$

We prove the following generalization of the above two results.

Theorem 1. *Let $\zeta(s)$ represent the Riemann zeta function. Then we have*

$$\zeta(s) = \frac{(-1)^n}{s-1} \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{a=0}^k (-1)^a \cdot S(n, a) \cdot a! \cdot \sum_{m=0}^{k-a} \binom{k}{m} \cdot \frac{(-1)^m}{(m+1)^{s+n-1}}$$

where $s \neq 1$, n being any non-negative integer, and $S(n, k)$ denotes the Stirling number of the second kind.

Letting $n = 0$ and $n = 1$ in the above representation gives us Hasse's representation (1) and Ser's representation (2), respectively.

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Proof. We start with the following integral [3]

$$(\alpha + \beta)\zeta(\alpha + \beta + 1) = \int_0^\infty \frac{\mathbf{Li}_\alpha(-1/t) \mathbf{Li}_\beta(-t)}{t} dt \quad (\alpha + \beta \neq 0), \quad (3)$$

where $\zeta(\cdot)$ is the Riemann zeta function, and $\mathbf{Li}_\beta(-t)$ is the polylogarithm function.

Letting $\alpha = -n$, a non-negative integer, and $\beta = s$, a complex number, in equation (3) we have

$$(s - n) \cdot \zeta(s - n + 1) = \int_0^\infty \frac{\mathbf{Li}_{-n}(-1/t) \mathbf{Li}_s(-t)}{t} dt. \quad (4)$$

We will use the following representation [4]

$$\mathbf{Li}_{-n}(-1/t) = (-1)^{n-1} \frac{1}{t} \sum_{k=0}^n k! S(n, k) \left(\frac{t}{1+t} \right)^{k+1} (-1)^k. \quad (5)$$

The *polylogarithm function*, $\mathbf{Li}_s(z)$, has the following series representation [5, Theorem 2.1]

$$\mathbf{Li}_s(z) = \sum_{k=0}^\infty \left(\frac{-z}{1-z} \right)^{k+1} \sum_{m=0}^k \binom{k}{m} \frac{(-1)^{m+1}}{(m+1)^s}, \quad (6)$$

for all $s, z \in \mathbb{C}$ with $\Re(z) < 1/2$. These representations allow us to evaluate right side of equation (4) as

$$\begin{aligned} & \int_0^\infty \frac{\mathbf{Li}_{-n}(-1/t) \mathbf{Li}_s(-t)}{t} dt \\ &= (-1)^{n-1} \sum_{k=0}^\infty \int_0^\infty \frac{t^k}{(1+t)^{k+2}} dt \sum_{a+b=k} (-1)^a \cdot S(n, a) \cdot a! \cdot \sum_{m=0}^b \binom{k}{m} \cdot \frac{(-1)^{m+1}}{(m+1)^s} \\ &= (-1)^{n-1} \sum_{k=0}^\infty \frac{1}{k+1} \sum_{a+b=k} (-1)^a \cdot S(n, a) \cdot a! \cdot \sum_{m=0}^b \binom{k}{m} \cdot \frac{(-1)^{m+1}}{(m+1)^s} \\ &= (-1)^{n-1} \sum_{k=0}^\infty \frac{1}{k+1} \sum_{a=0}^k (-1)^a \cdot S(n, a) \cdot a! \cdot \sum_{m=0}^{k-a} \binom{k}{m} \cdot \frac{(-1)^{m+1}}{(m+1)^s}. \end{aligned}$$

□

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