A Special One-Point Connectification that Characterizes T_0 Spaces

Yu-Lin Chou*

Abstract

We show that every T_0 space X has some T_0 “special” one-point connectification X_∞ , unique up to a homeomorphism, such that X is a closed subspace of X_∞ ; moreover, having such a one-point connectification characterizes T_0 spaces. As an application, it is also shown that our one-point connectification of every given topological n -manifold is a space more general than but “close to” a topological n -manifold with boundary.

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Once the elegance, conceptual importance, and applicability of the Alexandroff one-point compactification for (noncompact) locally compact Hausdorff spaces are recognized, it is then natural to intend to extend the idea for other topological invariants; this has been rightfully so.

By a one-point connectification of a given topological space we mean precisely a connected space including (set-theoretically) the given space with exactly one additional element; thus no additional requirement such as denseness is built in.

Every uniqueness statement is taken as a uniqueness statement modulo a homeomorphism.

We wish to give with a surprisingly simple construction a one-point connectification result for T_0 spaces that, to a great extent, resembles the classical Alexandroff one-point compactification result for (noncompact) locally compact Hausdorff spaces:

Theorem 1. *Let X be a topological space. Then X is T_0 if and only if there is some T_0 one-point connectification X_∞ of X such that X is a closed subspace of X_∞ .*

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*Institute of Statistics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C.; Email: y.l.chou@gapp.nthu.edu.tw.

Moreover, if $X_{\infty'}$ is a topological space with the property i) a closed subset of X is precisely a closed subset of $X_{\infty'}$ containing not ∞' and with the property ii) $X_{\infty'} \setminus X$ is a singleton, in particular if $X_{\infty'}$ shares all the properties of X_{∞} , then $X_{\infty'}$ is homeomorphic to X_{∞} .

Proof. We first deal with the T_0 characterization statement.

For the “if” part, suppose X is not T_0 , and choose a pair of distinct points $x, y \in X$ such that every neighborhood of each one of x and y contains the other point. Since X_{∞} includes X as a closed subspace by assumption, it follows that a closed subset of X is precisely a closed subset of X_{∞} that does not contain ∞ .

If V_x, V_y are neighborhoods of x and y respectively, then the sets $V_x \cup \{\infty\}, V_y \cup \{\infty\}$ are neighborhoods of x and y respectively in X_{∞} that contain both x and y . Since X_{∞} is T_0 by assumption, choose without loss of generality some open subset V_{∞} of X_{∞} that contains x but not y . Then the set V_{∞} is either an open subset of X or the union of $\{\infty\}$ and some open subset of X . But in either case V_{∞} contains y ; the “if” part follows from this contradiction.

To prove the converse, let $\mathcal{T}$ be the given topology of X . Topologize the set $X_{\infty} := X \cup \{\infty\}$ by the “naive” construction

$$\mathcal{T}_{\infty} := \{\emptyset\} \cup \{G \cup \{\infty\} \mid G \in \mathcal{T}\}.$$

Then, since $\{\infty\} \in \mathcal{T}_{\infty}$, the collection $\mathcal{T}_{\infty}$ is in particular a connected T_0 topology of X_{∞} , and X is a closed subspace of X_{∞} .

There remains to prove the uniqueness of X_{∞} . Let $\infty' \notin X$; let $f : X_{\infty} \rightarrow X_{\infty'}$ be defined by $f|_X := \text{id}_X$ with $f(\infty) := \infty'$. Then f is a bijection. If $G \in \mathcal{T}$, then, since

$$X_{\infty'} \setminus f^{-1}(G \cup \{\infty'\}) = f^{-1}(X \setminus G) = X \setminus G$$

is by assumption closed in $X_{\infty'}$, the map f is by symmetry a homeomorphism. This completes the proof. $\square$

Remark 1. In Theorem 1, the topology $\mathcal{T}_{\infty}$ of X_{∞} is by construction never T_1 even if X is Hausdorff. And X is not $\mathcal{T}_{\infty}$ -dense in X_{∞} as X is by construction closed- $\mathcal{T}_{\infty}$.

Thus, apart from the apparent nominal differences, Theorem 1 differs from the classical Alexandroff’s one-point compactification result precisely in replacing the openness (and denseness, depending on how one defines one-point compactification) of X in X_{∞} with closedness. $\square$

Due to Theorem 1, we may fix some terminology:

Definition 1. Let X be a T_0 space; let X_∞ be the specific topological space obtained from X in Theorem 1. Then X_∞ is called the *special one-point connectification* of X .

In general, a special one-point connectification is by definition precisely a topological space X_∞ such that there is some T_0 space X for which X_∞ is the special one-point connectification of X . $\square$

Nonexample 1. Let $X := \{0, 1\}$ be the Sierpiński space with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$. Since X is T_0 , we may by Theorem 1 consider the special one-point connectification X_∞ of X . And we evidently have

$$X_\infty = \{\emptyset, \{\infty\}, \{0, \infty\}, \{0, 1, \infty\}\}.$$

Thus a connected compact T_0 space can still admit its special one-point connectification, which is not possible in the regular context such as [1] or [2] where a one-point connectification is understood in a more stringent sense. $\square$

Remark 2. If X is a (noncompact) locally compact separable metrizable space, and if X is connectible in the sense of [1], then the Alexandroff one-point compactification of X is not the special one-point connectification of X even though it is a one-point connectification (in the sense of [1]) of X by Proposition 13 in [1]; the special one-point connectification of X is not Hausdorff. $\square$

There are some additional minor nice properties of a special one-point connectification:

Proposition 1. *If X is a locally compact T_0 space, then the special one-point connectification X_∞ of X is a locally compact T_0 space; if X is a compact T_0 space, then X_∞ is a compact T_0 space.*

Proof. Let X be a T_0 space, so that the special one-point connectification X_∞ of X exists by Theorem 1.

Fix any point $x \in X_\infty$. If $x = \infty$, then, since ∞ lies in every nonempty open subset of X_∞ , the set $\{\infty\}$ is also compact in X_∞ . It then suffices to consider elements of X .

Let $x \in X$. Suppose X is locally compact, and choose (in X) some neighborhood G of x and some compact superset K of G . Since $G \cup \{\infty\}$ is open in X_∞ , and since the set $K \cup \{\infty\}$ is compact in X_∞ by the construction of the topology of X_∞ , the space X_∞ is locally compact.

Every open cover of X_∞ is of the form $\{G_\theta \cup \{\infty\}\}_\theta$ where $\{G_\theta\}$ is an open cover of X ; fix any open cover $\{G_\theta \cup \{\infty\}\}$ of X_∞ . If X is compact, choose a finite subcover $\{G_{\theta_1}, \dots, G_{\theta_n}\}$ of $\{G_\theta\}$; but then $\{G_{\theta_i} \cup \{\infty\} \mid 1 \leq i \leq n\}$ is a desired finite subcover. $\square$

By an n -manifold we always mean a topological n -manifold, i.e. a second-countable Hausdorff space where every point has some neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$. For our purposes, we introduce the following

Definition 2. Let $n \in \mathbb{N}$. A topological space X is called a *pseudo locally n -Euclidean space with boundary* if and only if X is a second-countable T_0 space where every point has some neighborhood equinumerous to some subset of the upper-half plane $\mathbb{H}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ such that this subset less the topological boundary $\partial\mathbb{H}_+^n$ of $\mathbb{H}_+^n$ is open in $\mathbb{R}^n$.

We take the phrase “pseudo locally n -Euclidean with boundary” as a modifier. $\square$

Thus every n -manifold (with boundary or not) is a pseudo locally n -Euclidean space with boundary.

An application of Theorem 1 is the following

Theorem 2. *If X is an (resp. compact) n -manifold, then the special one-point connectification X_∞ of X is a (resp. compact) locally compact connected (and nonHausdorff) space that is pseudo locally n -Euclidean with boundary.*

Proof. Since X is by definition Hausdorff and hence T_0 , the special one-point connectification X_∞ of X exists as a T_0 space by Theorem 1. If X is compact, then X_∞ is compact by Proposition 1. Since X is in addition locally compact from assumption, Proposition 1 implies also that X_∞ is locally compact. The connectedness, nonHausdorffness, and second-countableness of X_∞ are evident.

For the remaining desired property of X_∞ , fix any $x \in X$ and choose by assumption some chart (G, φ) of X about x such that G is φ -homeomorphic to the Euclidean subspace $\mathbb{R}_{++}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i > 0 \text{ for all } 1 \leq i \leq n\}$. We are considering only those points of X as ∞ by construction lies in every nonempty open subset of X_∞ . Let $G_\infty := G \cup \{\infty\}$, and define the map $\varphi_\infty : G_\infty \rightarrow \mathbb{R}_{++}^n \cup \{(0)_{i=1}^n\}$ by $\varphi_\infty|_G := \varphi$ with $\varphi_\infty(\infty) := (0)_{i=1}^n$. Then φ_∞ is a bijection. Since $(\mathbb{R}_{++}^n \cup \{(0)_{i=1}^n\}) \setminus \partial\mathbb{H}_+^n = \mathbb{R}_{++}^n$ is open in $\mathbb{R}^n$, our proof is complete. $\square$

References

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