The Electrostrong Relation

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Abstract

In an attempt to eliminate the Landau Pole from QED by “borrowing” asymptotic freedom from QCD, I was successful in uniting the coupling constants of the two, respectively, in the “Electrostrong relation”. The Landau pole, estimated at around 10^{286} eV, leads to a value of $\alpha^{-1}(Q) = 43.06(33)$ in the Electrostrong Relation, instead of infinity as was the case before. In addition, the Strong CP problem is also solved proving that there is no need for fine tuning in QCD. The first step however, was improving the measurement for the running of the strong coupling constant $\alpha_s(Q)$ that can be tested experimentally for value of $\alpha_s(\Lambda_{\text{QCD}})$ for the energy scale, QCD integration parameter Λ_{QCD} offering such a prediction for the first time.

I Introduction

The two dimensionless running constants $\alpha(Q)$ and $\alpha_s(Q)$ will be united in a single equation, the Electrostrong Relation, allowing us to both provide a solution for the Strong CP problem and remove the problem arising due to infinities in $\alpha(Q)$ known as the Landau pole [1] when on a very large value of Q the fine-structure constant $\alpha \rightarrow \infty$ therefore we will effectively make QED a mathematically complete theory. The Landau pole will be eliminated since QCD enjoys asymptotic freedom [2], which will be “borrowed” offering us exactly what we need since β -expansion (see Appendix 1 for a short introduction to β -expansion) describes the asymptotic behavior of a denominator of convergents of continued fractions. The paper will also provide the first prediction for the value $\alpha_s(\Lambda_{\text{QCD}})$ and improve the accuracy for the measurement of the running values of the strong coupling constant.

In QED vacuum polarization corrections to processes involving the exchange of virtual photons result in a Q dependence, of the effective fine structure constant $\alpha(Q)$, which is known as the running of a coupling constant which is, for the running of the fine structure constant, parameterized as:

$$\alpha^{-1}(Q) = \alpha^{-1}(0) - \Delta\alpha(Q) \quad (1.1)$$

where at the scale of the Landau pole, approximately around 10^{286} eV , the problem of infinities arises. This problem does not exist within QCD since in the running of the strong coupling constant $\alpha_s(Q)$ there is asymptotic freedom.

The strong interactions do not violate CP symmetry in the manner of the weak interactions since the gauge interactions in QCD involve vector currents instead of chiral currents. Experiments too do not indicate CP violation in the QCD sector. However, if one writes the additional term in the Lagrangian density as:

$$\mathcal{L} = \frac{\theta}{16\pi^2} F_{\mu\nu}^A \tilde{F}^{A\mu\nu} \quad (1.2)$$

it contains the topological term with the QCD vacuum angle θ , which violates CP symmetry and it can, if non-zero, lead to experimentally detectable CP violating effects. In short this is referred to as the Strong CP problem.

II Running of $\alpha_s(Q)$ and $\alpha(Q)$

The QCD Lagrangian is $\mathcal{L}_{\text{QCD}} = \sum_f \bar{\Psi}_{f,a} (i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t_{ab}^C A_\mu^C - m_f \delta_{ab}) \Psi_{f,b} - \frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu}$

where γ^μ are the Dirac γ -matrices, the $\Psi_{q,b}$ are quark-field spinors for a quark of flavor f and mass m_f , with a color-index a that runs from $a = 1$ to $N_C = 3$ and $g_s = \sqrt{4\pi\alpha_s}$ is the strong gauge coupling. Let's remember that Lebesgue measure was used in the introduction to β -expansion. A generalization of the Lebesgue measure for any locally compact group is known as the Haar measure [4]. We know that the simple compact Lie group we need is $SU(N)$ (see the paper by G. Nagy [5]). From the invariant metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ with a unitarity, provided that all eigenvalues λ_i different. For matrixes W and $W^\dagger$ we get $WdW^\dagger + dWW^\dagger = 0$ and in a given set of coordinates we attain the invariant measure:

$$d\mu(x) = \sqrt{\det g(x)} \prod_{\alpha} dx^\alpha \quad (2.1)$$

we parameterize the matrix W as $W = \exp(it_\alpha T_\alpha)$ therefore we conclude that:

$$\det g \sim (\det Q)^2 \prod_{i>j} |\lambda_i - \lambda_j|^4 \quad (2.2)$$

the distribution of eigenvalues on the unitary group is given by the invariant measure:

$$d\mu(\theta) = \prod_{i>j} |\exp(i\theta_i) - \exp(i\theta_j)|^2 \prod_i d\theta_i \quad (2.3)$$

which is valid for $U(N)$ groups. If we impose the constraint $\sum_i \theta_i = 0 \mod 2\pi$ and implement it by a δ -function, the density distribution of the eigenvalues in $SU(N)$ is given by the formula above as well. For the $SU(3)$ group this would be of the form:

$$H(\theta) = \prod_{i>j} |\exp(i\theta_i) - \exp(i\theta_j)|^2 = \left| \begin{array}{ccc} 3 & \sum e_q & \sum e_q^2 \\ \sum \bar{e}_q & 3 & \sum e_q \\ \sum \bar{e}_q^2 & \sum \bar{e}_q & 3 \end{array} \right| \quad (2.4)$$

We must first describe the running of $\alpha_s(Q)$ and then proceed to connect it to the running of $\alpha(Q)$. Using β -functions for QCD, a $SU(3)$ group theory we have for $i = 0, 1, 2, \dots, n$

$$2\beta(\alpha_s) = \frac{b_0}{2\pi} \alpha_s^2 - \frac{b_1}{4\pi^2} \alpha_s^3 - \frac{b_2}{64\pi^3} \alpha_s^4 - \dots \quad (2.5)$$

where the QCD β -function has a negative sign due to gluons carrying color charge which leads to self-interactions. At one-loop order we determine:

$$\alpha_S(\mu) = \frac{\alpha_S(\mu_0)}{1 - b_0 \alpha_S(\mu_0) \ln\left(\frac{\mu}{\mu_0}\right)} \quad (2.6)$$

Now we introduce the β -expansion for a real number q :

$$x = \sum_{n=0}^{\infty} c_n q^{-n} \quad (2.7)$$

where for all $n \geq 0$ we have $0 \leq c_n \leq [q]$ where c_n doesn't have to be an integer and $[q]$ is a floor function [6]. We connect equations 2.6 and 2.7 which yield:

$$q^{-n} = \left(\frac{\alpha_S(Q^2)}{\pi}\right)^n + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^4}{Q^4}\right) \quad (2.8)$$

meaning that x from equation 2.7 equals $x = \delta_{\text{QCD}}$ which is the correction due to QCD effects and $\Lambda_{\text{QCD}} = 217(30)$ is the integration constant which corresponds to the scale where the perturbatively defined coupling would diverge and we use a value of 240 MeV but other values can be used as well. Knowing the values $c_1 = 1$, $c_2 = 1.9857 - 0.1152n_f$, $c_3 = -6.63694 - 1.20013n_f - 0.00518n_f^2 - 1.24\eta$, $c_4 = -156.61 + 18.77n_f - 0.797n_f^2 + 0.0215n_f^3 + C\eta$ where the coefficient C of the η -dependent piece in the α_S^4 term is yet to be determined and n_f is the number of flavors, allows us to establish that $\eta = (\sum e_q)^2 / 3 \sum e_q^2$ is consistent with the Vandermonde determinant of the Haar measure $H(\theta)$ for $SU(3)$. We introduce $v_0 = 246$ GeV the vacuum expectation value of the Higgs field [7]. When a local rather than global symmetry is spontaneously broken the gauge field of the broken symmetry becomes massive, leading to what should be a Goldstone scalar to become a longitudinal mode of the massive vector, which works for both abelian and non-abelian local symmetries. In the non-abelian case, for each spontaneously broken generator T^a of the local symmetry the corresponding gauge field $A_\mu^a(x)$ becomes massive. The Higgs mechanism gives quarks mass and leaves photons and gluons massless therefore we measure $\alpha_S(\Lambda_{\text{QCD}})$ for the first time, being that no other theory predicted the running value of α_S for the Λ_{QCD} parameter:

$$\alpha_S(\Lambda_{\text{QCD}}) = \frac{\ln\left(\frac{v_0}{\Lambda_{\text{QCD}}}\right)}{\pi^2} \quad (2.9)$$

This allows experimental tests to determine if the predicted value of $\alpha_s(\Lambda_{\text{QCD}})$ is accurate and therefore test the β -expansion method. The equation for the running value of $\alpha_s(Q)$ is then:

$$\alpha_s(Q) = \frac{\alpha_s(\Lambda_{\text{QCD}})}{\ln\left(\frac{Q}{\Lambda_{\text{QCD}}}\right)} \quad (2.10)$$

where it is evident that QCD enjoys asymptotic freedom. For $\alpha_s(Q^2)$ we add squared values for v, Λ_{QCD} and Q . The numerical results are provided in the table 1 below. All the values are in great agreement with experimental values such as provided by [8], [9], [10], [11]. Particle Data Group [12] provides the world average $\alpha_s(M_z) = 0.1184(07)$. The method offered by β -expansion offers much desired [13] accuracy and simplicity.

Using the Lyapunov exponent for the rate of exponential divergence from the initial perturbed conditions, we have $\mathcal{L}(x) := \lim_{n \rightarrow \infty} n^{-1} \log |(T^n)'(x)|$ where the constant θ has the form $\theta = (\tau(\log \beta_i) - 1) \log \beta_i$ and $\tau(\gamma) := \dim_{\text{H}}\{x \in [0,1): \mathcal{L}(x) = \gamma\}$ where we used the Housdorff dimension [14]. See C. Gong, Lyapunov Exponent of Classical SU(3) Gauge Theory [15] for a more detailed explanation on the relationship of the gauge coupling and the SU(3) Lyapunov exponent. From this we can deduce that in eq. (1.1) for the running of the fine-structure constant:

$$\Delta\alpha(Q) = \left(\frac{\pi^2}{\alpha_s(Q)}\right)^{\frac{1}{2}} \quad (2.11)$$

Allowing us to eliminate the Landau pole in the running of $\alpha(Q)$ by establishing a link between $\alpha(Q)$ and $\alpha_s(Q)$ which eliminates the infinities since QCD enjoys asymptotic freedom by establishing:

$$\alpha^{-1}(Q) = \alpha^{-1}(0) - \left(\frac{\pi^2}{\alpha_s(Q)}\right)^{\frac{1}{2}} \quad (2.12)$$

The results are in good agreement with experimental values [16], [17].

Energy [GeV]	$\alpha_s(Q)$	$\alpha^{-1}(Q)$
$\Lambda_{\text{QCD}} = 0.24$	0.702404(06)	133.287509(06)
1	0.492184(06)	132.557978(12)
10	0.188327(08)	129.796749(15)
$M_z = 91.1876(23)$	0.118249(13)	127.90012(24)
$M_{\text{GUT}} = 10^{13}$	0.0224(21)	116.0454(33)
$M_{\text{GUT}} = 10^{17}$	0.0173(24)	113.151(26)

Table 1: Measurements for the electromagnetic and strong nuclear couplings for the values of $\Lambda_{QCD} = 240 \text{ MeV}$.

In order to prove the lack of Landau pole in my equations I provide the measurements albeit the Landau pole is not relevant for particle physics but is purely of academic interest. The value of the Landau pole is estimated roughly to be around 10^{286} eV , for this value $\alpha_s(Q) = 0.00112(24)$ and $\alpha^{-1}(Q) = 43.06(33)$ where the uncertainties are high since we are dealing with such a high energy level.

When it comes to the Strong CP problem we will look at the θ -dependence on the vacuum energy. If we take θ to be the phase of the overall Lagrangian $\mathcal{L}_m = (m_u \bar{u}u + m_d \bar{d}d) \cos(\theta/2) + \text{h.c.}$ where m_u is the up quark mass and is m_d the down quark mass and we can treat the Lagrangian term as a perturbation. The only degrees of freedom which can readily be excited in QCD are pions and as M. Dine shows in his paper TASI Lectures on The Strong CP Problem [18], the relation of this dependence is $E(\theta) = E_\pi^2 f_\pi^2 \cos \theta$ where E_π is the rest energy of a pion, however we will only be working with charged pions π^+ and π^- and as such we obtain:

$$E_\pi = \left(\frac{E(\theta)}{f_\pi^2 \cos \theta} \right)^{\frac{1}{2}} = \frac{n_{q\bar{q}} \text{ Ry}}{\alpha^\pi(Q = M_\pi)} \quad (2.13)$$

where $n_{q\bar{q}}$ is the number of constituent quarks/anti-quarks and for charged pions $n_{q\bar{q}} \equiv 2$ and the Ry is the Rydberg unit of energy that is measured at $13.605693009(84) \text{ eV}$. Since the rest energy of charged pions is measured to be $E_\pi = 139.57018(35) \text{ MeV}$ we can use it to measure the running of the fine structure constant $\alpha^{-1}(Q = M_\pi) = 136.7328(26)$ effectively solving the Strong CP problem. For other Electrostrong relations, such as for electrons, muons and their relationship with the Koide formula [19] see Appendix 2.

III Conclusion and Debate

In the $SU(3) \times SU(2) \times U(1)$ Standard Model, constants in the one-loop β -functions are given as $b_i = \frac{2}{3} T(R_i) d(R_j) d(R_k) + \frac{1}{3} T(S_i) d(S_j) d(S_k) - \frac{11}{3} C_2(G_i)$ we can summarize these results and insert the number of fermion generations $N_G = 3$ and Higgs doublets $N_H = 1$ obtaining us $b_1 = 41/10$, $b_2 = -19/6$ and $b_3 = -7$.

These three intersections point towards a range from $M_{GUT} = 10^{13} \text{ GeV}$ to $M_{GUT} = 10^{17} \text{ GeV}$ and correspond to a coupling ranging from $\alpha_{GUT}^{-1} = \alpha_S^{-1} \approx 40$ to 47 where we see an agreement for $M_{GUT} \approx 10^{13} \text{ GeV}$ in the table where $\alpha_{GUT}^{-1} = \alpha_S^{-1} = 44.64(18)$. This means that, regardless of unification, the equations measure the running of the couplings in excellent agreement with both the experimental values and the theoretical predictions of the SM.

Appendix 1: β -expansion

Let $\beta_i > 1$ be a real number and $T_{\beta_i}: [0, 1) \rightarrow [0, 1)$ be the β -transformation such that $T_{\beta_i}(x) = \beta_i x - [\beta_i x]$ where $[x]$ is the largest integer that does not exceed x . Every $x \in [0, 1)$ can be uniquely expanded into a finite or infinite series:

$$x = \frac{\varepsilon_1(x)}{\beta_i} + \frac{\varepsilon_2(x)}{\beta_i^2} + \frac{\varepsilon_3(x)}{\beta_i^3} + \dots + \frac{\varepsilon_n(x)}{\beta_i^n} + \dots \quad (\text{A.1})$$

where $\varepsilon_i(x) = [\beta_i x]$ and $\varepsilon_{n+1}(x) = \varepsilon_1(T_{\beta_i}^n x)$ for all $n \geq 1$. For any Borel set $A \subseteq [0, 1)$ where C is a constant that depends on β_i such that $C^{-1}\lambda(A) \leq \mu(A) \leq \lambda(A)C$ where the measure of the length/energy scale μ is T_{β_i} -invariant meaning that β_i does not implicitly depend on μ and if A is Lebesgue measurable [3], then $\mu(A) = \sup\{\mu(k): k \subset A\}$ where k is compact.

If $x \in [0, 1)$ then any irrational number can be written as $p_n(x)/q_n(x) = [c_1(x), c_2(x), \dots, c_n(x), \dots]$ where $c_1(x) = [x^{-1}]$ and $c_{n+1}(x) = a_1(T^n x)$ for any $n \geq 1$ is a convergent of the continued fraction expansion of x where $p_n(x)$ and $q_n(x)$ are relatively prime which allows us to study the asymptotic behavior of the denominator k_n

$$\lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta_i}{\pi^2} \quad (\text{A.2})$$

For any $\varepsilon > 0$ there exists two positive constants A and α which depend on β_i thus for all $n \geq 1$:

$$\lambda \left\{ x \in [0, 1): \left| \frac{k_n}{n} - \frac{6 \log 2 \log \beta_i}{\pi^2} \right| \geq \varepsilon \right\} \leq A \cdot e^{-\alpha n} \quad (\text{A.3})$$

If $\log \beta_i > \pi^2/6 \log 2$ then for λ almost all $x \in [0, 1)$ exists a positive integer N_1 that depends on x , such that for all $n \geq N_1$:

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| > x - x_n \quad (\text{A.4})$$

In the opposite case when $\log \beta_i < \pi^2/6 \log 2$ we have an integer N_2 and for all $n \geq N_2$:

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| < x - x_n \quad (\text{A.5})$$

where the convergents of the β -expansion of x are $x_n = \varepsilon_1(x)/\beta_i + \varepsilon_2(x)/\beta_i^2 + \varepsilon_3(x)/\beta_i^3 + \dots + \varepsilon_n(x)/\beta_i^n$.

Appendix 2: Electrostrong Relations for Electrons, Muons and the Higgs Boson

The relationship between the fine-structure constant and the electron rest energy/mass is well known and established but its relationship with the rest energy of muons, taus and the Higgs boson is not detailed enough. For the time being it has been well established that at the energy level of Z bosons, the fine-structure constant is approximately 1/128 and the strong coupling is approximately 0.1184 and its relationship with electron, muon and Higgs boson rest energy/mass is not explained at all. For the rest energy of electrons, we have an equation:

$$E_e = \frac{Q_1 \text{ Ry}}{\alpha_s(0) \alpha^{Q_1}(0)} = \frac{2 \text{ Ry}}{\alpha_s(0) \alpha^2(0)} \quad (2)$$

where Ry is the Rydberg unit of energy, $\alpha(M_e)$ is the fine structure constant that equals 0.0072973525664(17), value provided by NIST, $\alpha_s(0) \equiv 1$ is the strong coupling constant at zero Q momentum transfer and evidently $Q_1 \equiv 2$ is part of the Koide formula and shouldn't be confused with the momentum scale Q. For muons, we have a similar equation where $Q = M_\mu$ the rest energy equals:

$$E_\mu = \frac{Q_2 \text{ Ry}}{\alpha_s(M_\mu) \alpha^{Q_2}(M_\mu)} = \frac{3 \text{ Ry}}{\alpha_s(M_\mu) \alpha^3(M_\mu)} \quad (3)$$

where we have to account for the running of the coupling constants. For the energy level of muons the fine-structure constant is approximated to be around 1/136 in which case the strong coupling constant $\alpha_s(M_\mu) = 0.994(02)$ and evidently the second part of the Koide formula $Q_2 = 3$. The Koide formula relates the rest energies/masses of three leptons, namely electrons, muons and taus'. It's been discovered by dr. Yoshio Koide in 1981. it states that $Q = 2/3$ in the formula:

$$E_e + E_\mu + E_\tau = \frac{Q_1}{Q_2} \left(\sqrt{E_e} + \sqrt{E_\mu} + \sqrt{E_\tau} \right) = \frac{2}{3} \left(\sqrt{E_e} + \sqrt{E_\mu} + \sqrt{E_\tau} \right) \quad (4)$$

where the parameters Q_1 and Q_2 are the addition of this paper. Additionally we have a formula for rest energy of the Higgs boson that has been measured to be approximately 125.36(41) GeV [20]. As such, we have:

$$E_{Higgs} = \frac{4 \text{ Ry}}{\alpha_s(M_{Higgs}) \alpha^4(M_{Higgs})} \quad (5)$$

where the running of the two coupling has been measured with experimental values available at [21]. The results are in great agreement with the experimental result provided by ATLAS in ref. [20].

IV References

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