

# MULTI-DISTRICT PREFERENCE MODELLING

ABSTRACT. Generating realistic artificial preference distributions is an important part of any simulation analysis of electoral systems. While this has been discussed in some detail in the context of a single electoral district, many electoral systems of interest are based on multiple districts. Neither treating preferences between districts as independent nor ignoring the district structure yields satisfactory results. We present a model based on a multi-urn extension of the classic Eggenberger-Pólya urn, in which each district is represented by an urn and there is correlation between urns. We show in detail that this procedure has a small number of tunable parameters, is computationally efficient, and produces “realistic-looking” distributions. We present applications to retrospective analysis and forecasting of real elections, and intend to use the methodology to help set optimal parameters for electoral systems.

## 1. INTRODUCTION

In order to test the average-case performance of electoral systems, or to test models of electoral phenomena such as inter-election swing, it is useful to be able to generate artificial preference distributions for a society.

We are particularly interested in multi-winner elections of parliamentary type, in which the electorate is partitioned into districts, one or more representatives are chosen from each district, and these representatives together make up the parliament. It is crucial to consider the variability between districts, as otherwise we obtain very unrealistic results. For example, if simulating plurality voting in single-winner districts (“First Past the Post”) where Party A has nationally 55% of the vote, if districts behave identically (so each is a microcosm of the overall national electorate) then Party A will get 55% of the vote in each

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*Key words and phrases.* simulation, Pólya urn model, electoral systems.

FIGURE 1. UK election 2015, Labour party 2-party percentage in England &amp; Wales

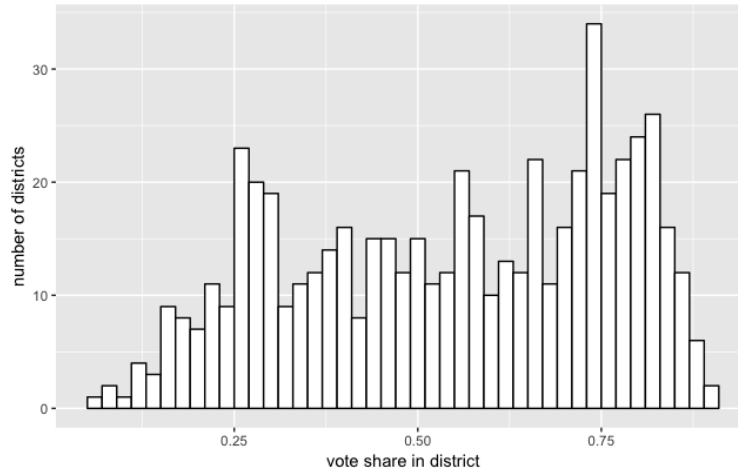
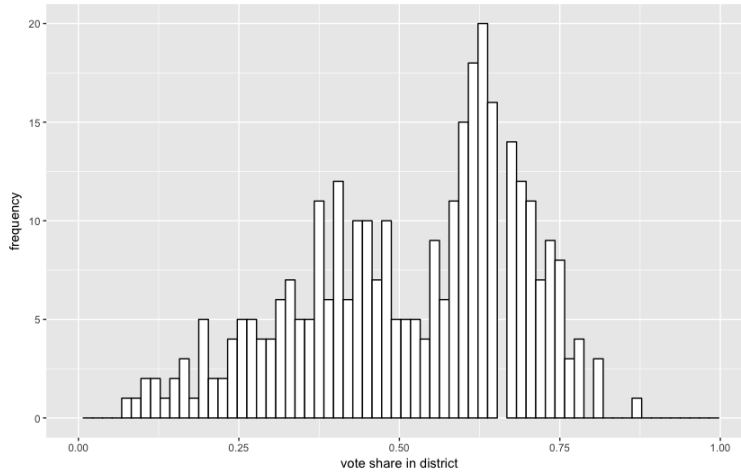


FIGURE 2. US Congressional elections 2014, Republican party 2-party percentage



district, and hence win 100% of seats. At another extreme, if all voters within a district behave identically (so there is extreme geographic polarization) then Party A will have vote share 100% in some districts and 0% in the others, and win 55% of the seats (ignoring integer rounding issues). However, real-world FPTP elections are obviously not consistent with these assumptions. The way districts are drawn typically assures each of the major parties dominance in some number of seats, with a smaller number of marginal ones, and so party A will typically have a vote share distribution across districts that looks very different from either of those described above.

For real-world examples, see Figures 1 and 2. The first hypothetical example above would have a vote share graph with a single peak at 55% and zero height everywhere else, while the second hypothetical example would have a graph that has two peaks, a larger one at 100% and a smaller one at 0%, with zero height everywhere else.

In social choice theory (basically corresponding to the single-district case), the theoretical performance of voting systems, for example with respect to manipulability, has been studied in detail using various preference distributions [BL94; Reg+06]. This kind of work may be useful for studying, for example, party list proportional systems. However, we are not aware of similar work for district-based electoral systems. In this article we present a general method for generating artificial voting data in districts which has a small number of tunable parameters, is computationally efficient, and produces “realistic-looking” distributions. We cover only the case of plurality ballots.

The rest of the paper is as follows. In Section 2 we introduce the basic probabilistic models that we will use for generating artificial data, and discuss their theoretical properties. In Section 3 we report on detailed experiments and explain the effect of varying parameters in the model. Most of this involves the case of two parties, but we also go beyond this in Section 3.3. Section 4 shows how the model can cast light on phenomena such as the cube law for the vote-seat mapping and theories of inter-election swing, as well as election forecasting and campaign management. In Section 5 we discuss possible future work.

## 2. URN MODELS

**2.1. Our basic model.** Our process of generating a preference distribution among voters depends on parameters  $N$  (a positive integer),  $K$  (a nonnegative integer) and  $p$  (a real number in the closed interval  $[0, 1]$ ). The procedure starts with  $N$  urns (each representing a different electoral district). Each urn contains colored balls, where each color corresponds

to a different political party and the number of balls of a given color corresponds to the number of voters supporting the corresponding party.

We draw an urn  $u$  uniformly at random. With probability  $1 - p$ , we draw a ball from  $u$ , and with probability  $p$  we draw from a different urn  $v$  chosen uniformly at random. Then  $K$  balls of that same color are added to  $u$  (and the ball that we drew is replaced in its urn).

Thereby, the preference distribution in one district (urn) is influenced by those in other districts (the other urns). When  $K = 0$ , this is simply sampling with replacement, but for larger values of  $K$ , the number of balls increases at each iteration with a bias toward colors that already occur frequently. The initial distribution of balls of course influences the distribution at later times. Most obviously, no new parties are created during this process, and a party cannot vanish completely from a district once it is present there.

The probability  $p$  is a parameter measuring the likelihood of inter-district imitation by voters, so when  $p = 0$  the urns evolve independently. The parameter  $K$  controls the strength of imitation, whether within a district or between two districts. The urn models are dynamic, and in principle could be used to model the evolution of voting behavior over time. We adopt a static interpretation here — we think of an urn model with given parameters as a stochastic process that we run for a long time until the situation is almost stable and we have a total number of voters suitable for the situation under study, and then use the resulting vote distribution as a simplified description of an artificial society. Adding voters can be interpreted as meaning that eligible but previously uncommitted voters make a choice, or that new voters become eligible, for example by attaining the minimum voting age.

In the case of a single urn and  $p = 0$ , we are dealing with a standard Eggenberger-Pólya urn model [EP23] (they in turn drew on the work of still earlier authors). This case has been widely used in the preference modeling context (see, e.g. [BL94]). In particular when

$K = 1$ , we can interpret it as adding new voters one at a time, and each voter imitating the vote of a randomly selected previous voter. The case  $K = 1$  can be used to generate data from the so-called Impartial Anonymous Culture distribution of preferences [BL94], while  $K = 0$  corresponds to the Impartial Culture distribution.

Gudgin & Taylor [GT79, Sec. 3.2] develop a probabilistic model for voting in districts that resembles our model in that it considers voters one at a time and has them imitate previously added voters. However in their model, imitation is only of recently added voters and the model does not explicitly use inter-district imitation. In contrast to our model, their parameters depend on the desired population size. We are not aware of further use of their model.

We note that a variant of our model in which the total voting population of each district remains fixed, and voters periodically change their state (that is, their opinion) by imitating others as in the present paper, is essentially a version of the well-known *voter model* ([Lig85]). The standard voter model is defined on the integer lattice and emphasizes very local interactions between voters (each voter is influenced only by a handful of immediate neighbours). The possibility of a more general version defined on a graph, with a voter's range of immediate influence extending throughout a large "district" and even beyond, has not, to our knowledge, been explored.

**2.2. Theoretical properties of the model.** The known theoretical properties of the single Eggenberger-Pólya urn can provide some insight into the behaviour of the more complex multiple-urn model. Suppose a single urn initially contains  $a_1, \dots, a_m$  balls of colors  $1, \dots, m$  respectively, and each ball drawn is replaced along with one more ball of the same color. Let  $X_1, X_2, \dots$  be random variables giving the colors of the balls drawn, and  $Y_1, Y_2, \dots$  be random  $m$ -vectors giving the proportions of the colors after each draw (*i.e.*  $Y_n(i)$  is the proportion of the balls that have color  $i$  after the first  $n$  draws). Then

- (1) With probability 1, the sequence  $(Y_n)$  converges to a limit  $Y$ ;
- (2)  $Y$  is a random variable with the Dirichlet( $a_1, \dots, a_m$ ) distribution;
- (3) Conditional on  $Y$ , the random variables  $X_1, X_2, \dots$  are independent with probability distribution  $Y$ .

(See [BM73].) If there are only two colors, we may consider  $Y_n$  to be scalar-valued (since then  $Y_n(2)$  is simply  $1 - Y_n(1)$ , adding no further information); in this case the limit  $Y$  has a Beta( $a_1, a_2$ ) distribution.

A particularly simple special case occurs when  $a_1 = \dots = a_m = 1$  *i.e.* the urn begins with one ball of each color. The limiting Dirichlet( $1, \dots, 1$ ) distribution is then the uniform distribution on the  $m$ -simplex; for the two-color case this is the uniform distribution on the interval  $[0, 1]$ .

Turning to multiple urns but remaining in the case where the initial numbers of balls in each urn are equal, we can make some observations about what to expect. Firstly, since each step adds a ball to one randomly selected urn, the urns will likely end up containing different numbers of balls. However, once the number  $n$  of steps taken has become large, there will be similar numbers of balls in each urn (the differences between the numbers of balls in different urns will be on the order of  $\sqrt{n}$ , but the numbers themselves are of order  $n$ ).

Second, when  $p = 0$ , the urns do not interact. Each urn behaves as a single Eggenberger-Pólya urn, independently of the others.

Thirdly, when  $p = 1$  and  $N$  is not small, the behaviour will be close to that exhibited in the case where there is a single large urn, leading to substantial correlation between the ball counts in urns, for each color.

### 3. EXPLORING THE MODEL

The theoretical results above show that we expect the districts to have roughly equal numbers of voters at all times during the process, assuming that we start with equal numbers of voters in each district. We now want to explore the effect on the final state of the process of varying  $p, K$  and the initial distribution of voter preferences across parties in each district.

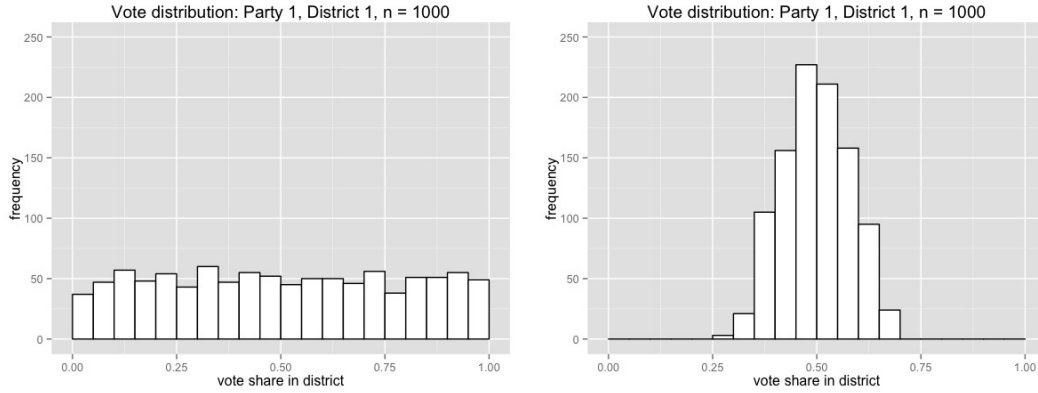
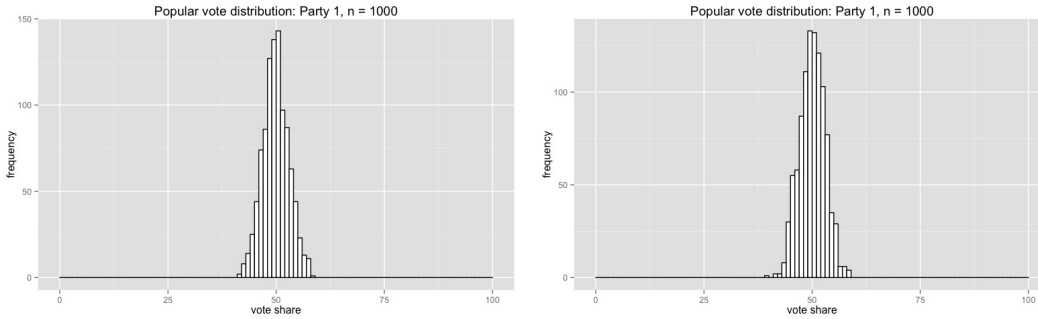
In order to test the realism of the model, we carried out extensive simulation (C++ program available at [give URL according to publisher instructions](#)). The baseline computations involved  $N = 100$ , and we also performed some for  $N = 1000$ . We concentrated on values of  $m$  from 2 to 4, and varied  $p$  from 0 to 1 in steps of 0.1. We experimented with various initial conditions. For each case (value of  $N, m, p$  and initial conditions) we ran 1000 simulations. We mostly used  $K = 1$  but also considered some higher values of  $K$ .

We display the results using several types of graphs. For party  $A$  we fix a district and show a histogram of the vote share in that district. Note that this is similar to what is shown in Figure 1, but different in that it shows vote shares in a single district across many replications of the experiment, rather than vote shares across all districts for a single replication.

We then show histograms of the party's total vote share  $V$  and the number of seats  $S$  that it wins under FPP. We also compute the correlation between  $A$ 's share of the total vote in the first 50 (arbitrarily labelled "north") districts and in the second 50 (labelled "south") districts. We then give a scatterplot of the mapping from  $V$  to  $S$ .

#### 3.1. **Two parties.** We fix $m = 2$ .

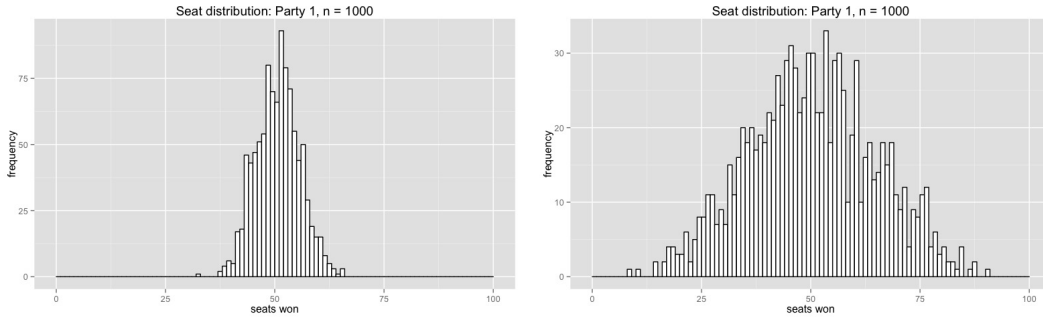
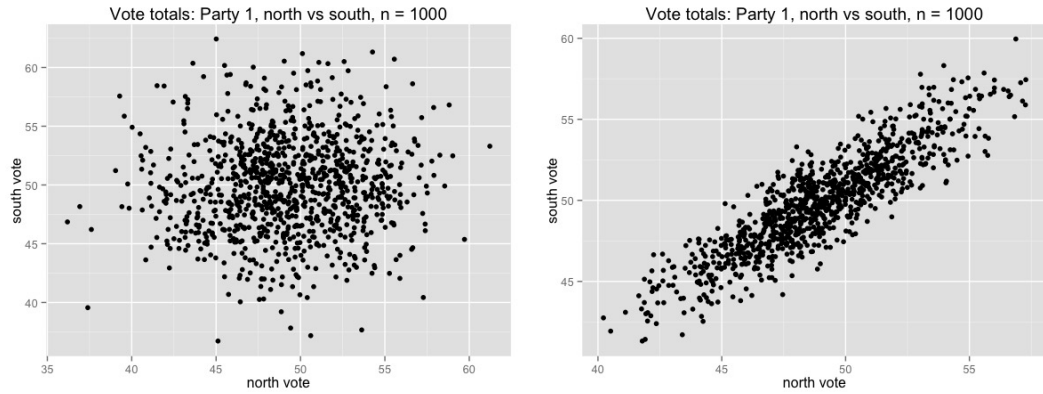
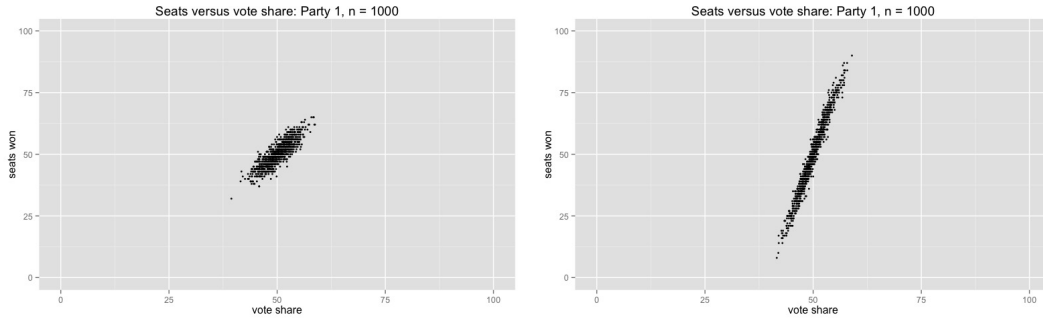
3.1.1. *Symmetric initial conditions.* Here we started with one voter of each party in each district. When  $p = 0$ , the distribution of the vote share of Party 1 in District 1 is shown in

FIGURE 3. District vote share distribution when  $p = 0$  (L) and  $p = 0.2$  (R)FIGURE 4. Total vote share distribution when  $p = 0$  (L) and  $p = 0.2$  (R)

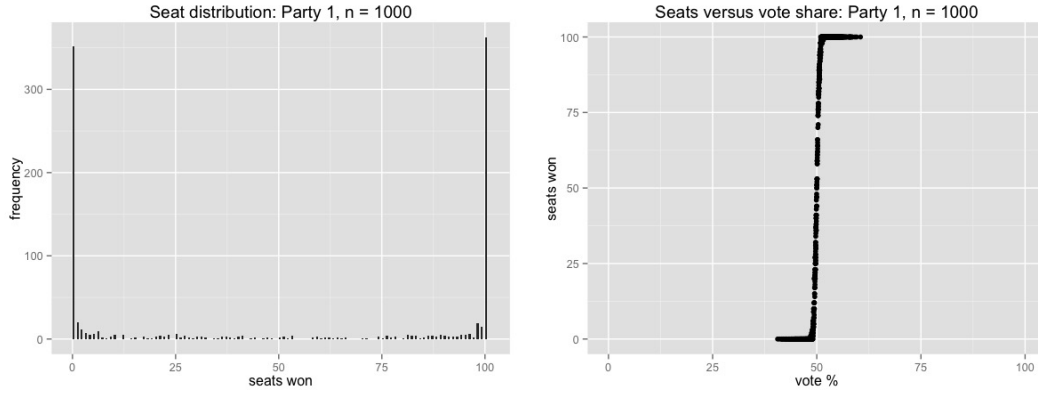
the left picture in Figure 3 (as predicted by the theory, the urn process produces something drawn from a uniform distribution) and the distribution of this party's seat share in the left picture of Figure 5 (as predicted by the theory, the urn process produces something drawn from a binomial distribution). The total vote share of Party 1 across all districts is shown in the left panel of Figure 4. As expected, we see no correlation between the party's vote in the northern and southern districts — see the left picture in Figure 6. The left picture in Figure 7 shows a positive association between votes and seats, again as expected.

Increasing the value of  $p$  to 0.2 leads to quite different results (see the right-hand pictures in Figures 3-7). The distribution of the party's vote in a given district becomes more concentrated; the influence of other districts now makes it much less likely that this district will deliver a result wildly different from the national average. At the same time, the seat distribution becomes *less* concentrated. The stabilization of the within-district vote



FIGURE 5. Seat share distribution when  $p = 0$  (L) and  $p = 0.2$  (R)FIGURE 6. North-south correlation when  $p = 0$  (L) and  $p = 0.2$  (R)FIGURE 7. Vote - seat mapping when  $p = 0$  (L) and  $p = 0.2$  (R)

puts many more districts in play, while inter-district imitation has a self-reinforcing effect: a high vote in one district not only wins that seat, but improves the party's prospects in every other district as well. Another view of this effect is provided by the north-south correlation graph (Figure 6). Also, the mapping of votes to seats has a larger slope, a point we shall return to in Section 4.1. Note that the distribution of total vote share changes very little, because changing the value of  $p$  has large district level effects (whether a new voter

FIGURE 8. The case  $p = 1$ : seats (L) and seats vs votes (R)

imitates a voter in the same district) but total vote share depends only on whether they imitate *some* voter.

As  $p$  increases, the trend observed above continues. Figure 8 shows the behaviour in the extreme case  $p = 1$ : the entire electorate is now behaving (approximately) like a single well-mixed district, and it is quite likely that all of the parliamentary seats will be won by the same party. (Due to the symmetry of the initial conditions, each of our two parties has the same probability of achieving such a total victory.) Note that the vote shares (both nationally and in each district) are still quite close to 50-50; it is the lack of diversity among districts that makes such lopsided seat-shares possible. This case could be thought of as resembling a single district with an initial content of 50 voters of each type.

We repeated the above experiments for the case where each party initially has 2 voters in each district (Figures 9 and 10). When  $p = 0$ , the district vote distribution is Beta(2,2) (density proportional to  $x(1-x)$ ); consequently, the total vote is a little less variable than in Figure 4. The seat-share distribution is still binomial, unchanged from Figure 5, and the north-south correlation is still zero (graph omitted). The seat-vote relationship is shown in Figure 11. Introducing inter-district mimicry again has the effect of homogenizing districts, thereby making the seat-share distribution more variable. When we increase the

FIGURE 9. District vote (L), total vote (M) and seat (R) distribution when

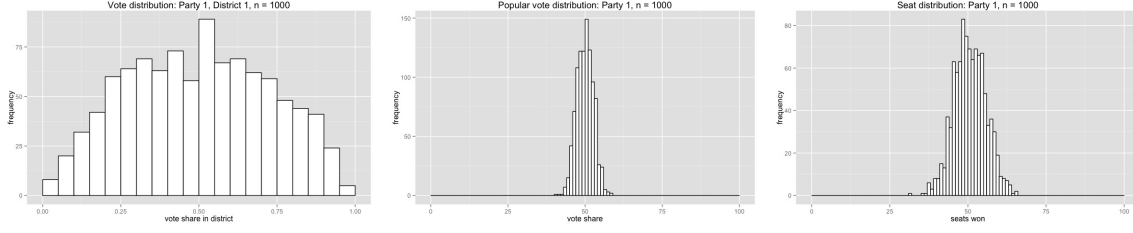
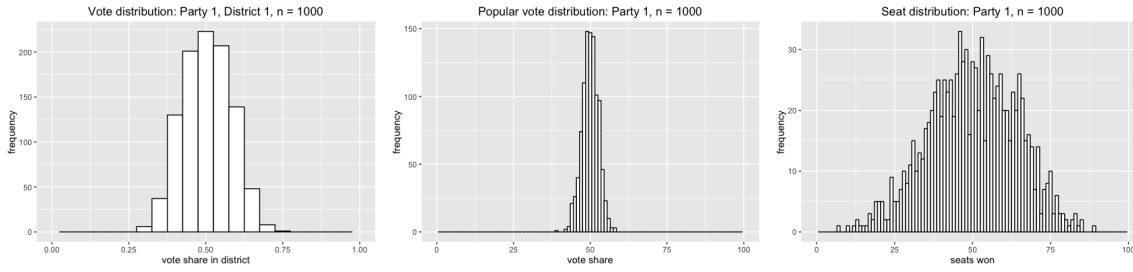
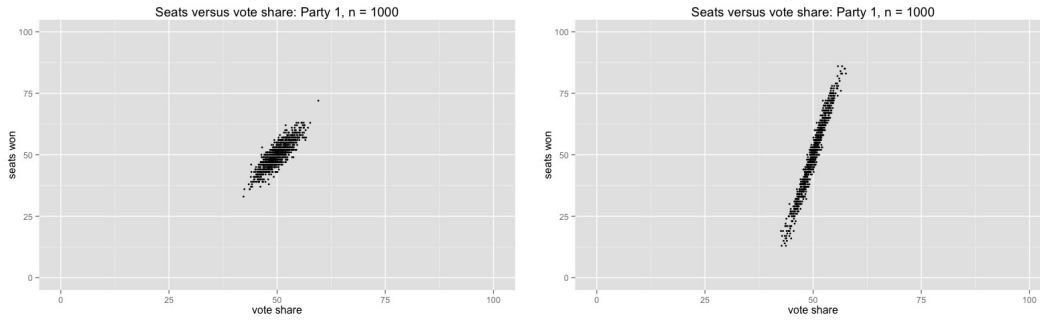
 $p = 0$ : initial conditions 2 : 2.

FIGURE 10. District vote (L), total vote (M) and seat (R) distribution when

 $p = 0.2$ : initial conditions 2 : 2.FIGURE 11. Vote - seat mapping when  $p = 0$  (L) and  $p = 0.2$  (R), initial conditions 2 : 2.

initial number of voters even more, the trend continues, with the district vote share and total vote share both approaching a point mass at  $1/2$ .

Going in the other direction, we considered the case where the initial conditions can be rapidly forgotten as districts become swamped by imitated voters (alternatively, the initial number of voters is a small fraction of 1). For example, we recomputed the above results when each party has 1 voter in each district initially, but  $K = 5$  new voters are added at each step. These are shown in Figures 12–14. When  $p = 0$ , note the very different vote

FIGURE 12. District vote (L), total vote (M) and seat (R) distribution when

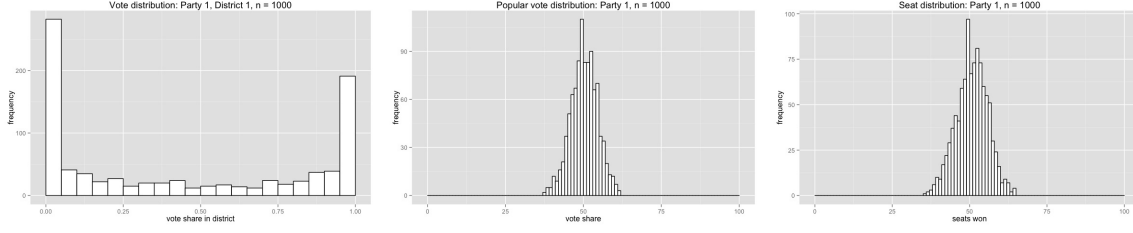
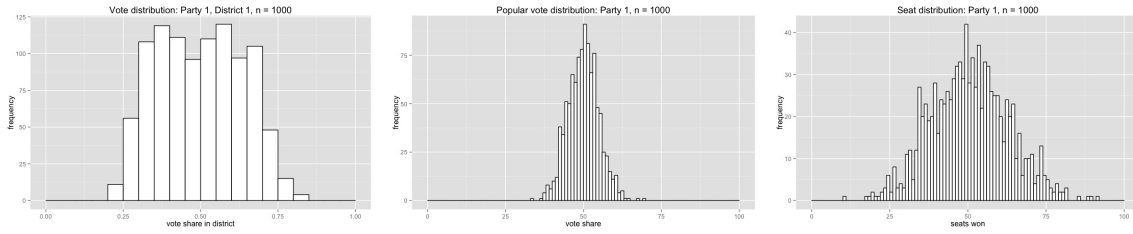
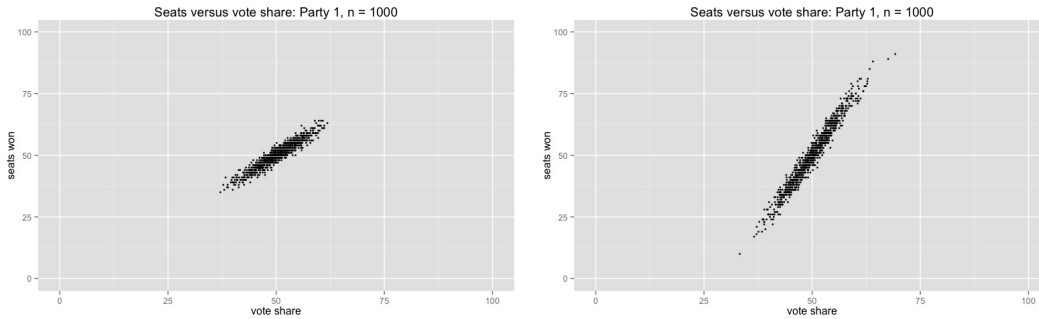
 $p = 0$ : initial conditions 1 : 1,  $K = 5$ 

FIGURE 13. District vote (L), total vote (M) and seat (R) distribution when

 $p = 0.2$ : initial conditions 1 : 1,  $K = 5$ FIGURE 14. Vote - seat mapping when  $p = 0$  (L) and  $p = 0.2$  (R), initial con-ditions 1 : 1,  $K = 5$ 

share distribution but the similarity of the other quantities presented. Also note that when  $p = 0.2$  we get results quite similar to the other cases presented above — imitation washes out the effect of initial conditions.

**3.2. Other initial conditions.** We next considered initial conditions in which for the northern districts, Party 1 starts with the advantage of two initial voters to Party 2's single voter, while this advantage is reversed in the southern districts. In the absence of inter-district imitation (*i.e.*  $p = 0$ ), this means that the district-level vote distribution is now Beta(2, 1),

FIGURE 15. District vote (L), total vote (M) and seat (R) distribution when

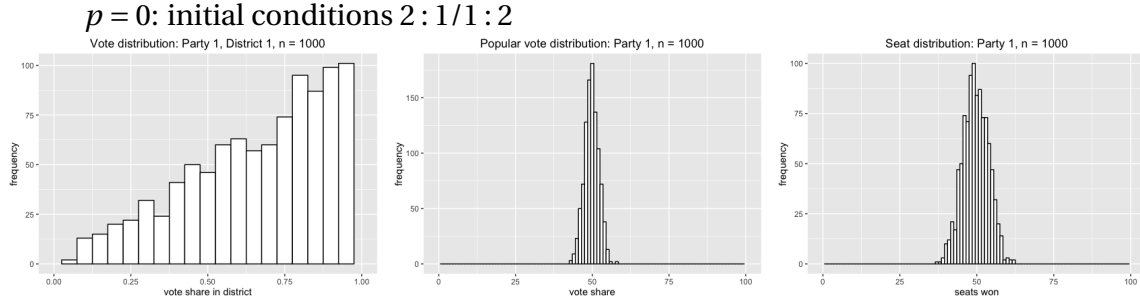
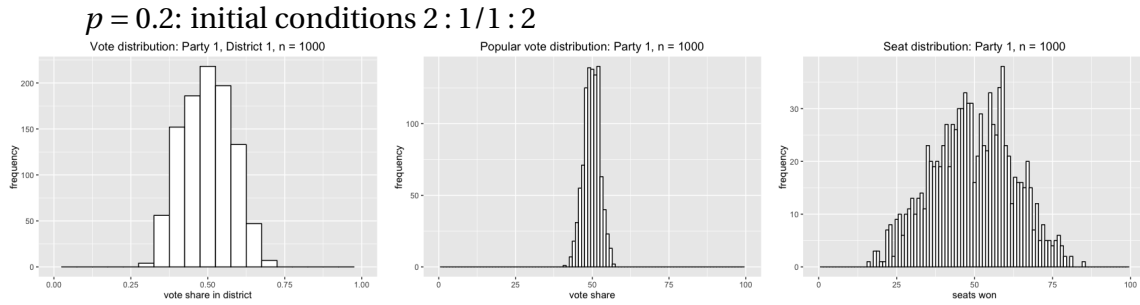


FIGURE 16. District vote (L), total vote (M) and seat (R) distribution when



or triangular (Figure 15, left panel), with each district having probability  $\frac{3}{4}$  of being won by its dominant party. The resulting seat-share distribution resembles Figure 5 (left panel), but with less variance.

The polarisation of this electorate along geographical lines can be reduced by increasing the value of  $p$ . Imitation across the north-south boundary has the effect of making at least some districts more competitive, and increasing the variance of the seat-share distribution. This is perhaps a step towards a more realistic version of our model: real-world elections typically have many districts in which the mild initial dominance of one party can be overcome by national-level effects.

We repeated the analysis with initial bias of 3 : 1 instead of 2 : 1, and the results were as expected. For  $p = 0$ , there is a continuation of the trends in district vote and seat distribution, and a shift in the mean of the total vote distribution but little change in spread. The

slope of the seat-vote curve decreases. With  $p = 0.2$ , the effect of the initial conditions has been largely washed out.

In summary, we have found that increasing  $p$  has the effect of concentrating the vote distribution in a single district, leaving the total vote distribution essentially unchanged, and spreading out the seat distribution. Furthermore it increases north-south correlation and steepens the seat-vote curve. Changing the initial conditions of the process (but keeping the symmetry between parties overall) has a strong effect on the shape of the vote distribution in a given district, but does not change anything else substantially.

**3.3. Modelling third parties.** In order to investigate scenarios involving a smaller third party competing with two roughly equal larger ones, we ran our model with the following sets of initial conditions.

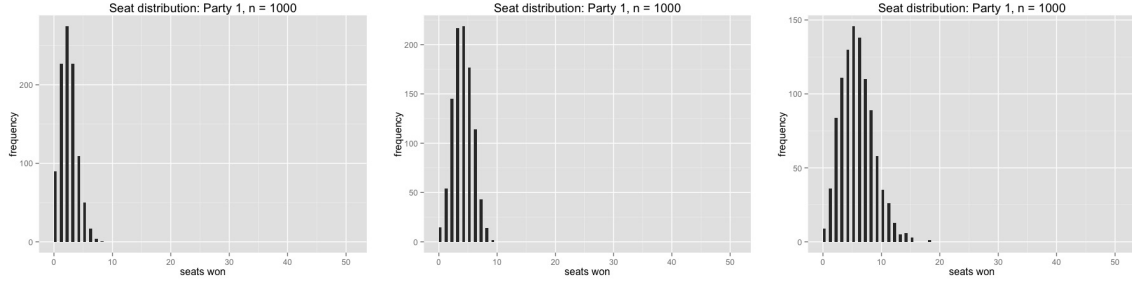
- (i) in 80 districts Parties 1,2,3 have respectively 0,2,2 voters; in 10 districts they have 1,2,2 and in 10 districts they have 2,1,1.
- (ii) as above, but in the last 10 districts (the “regional base”) the initial conditions are 3,1,1.
- (iii) in all districts parties 1,2,3 have respectively 1,2,2 voters.

We expect that the small party will fail to become large. When  $p = 0$ , it cannot extend to any of the first 80 seats in the first two cases. As  $p$  increases, the imitation means that although it attracts voters in districts away from its base, it loses more in its base to the other parties. We present graphs on seat distribution for Party 1 under the three scenarios in Figure 17.

## 4. APPLICATIONS

**4.1. The cube law.** It has long been observed (see *e.g.* [KS50]) that in two-party electoral systems where each district elects a single representative by simple plurality voting (“First

FIGURE 17. Seat shares for small party under Scenarios (i) – (iii) in Section 3.3 ( $p = 0.1$ )



Past the Post", or FPP), there is a relationship between the fraction  $x$  of all the votes and the fraction  $y$  of all the districts won by a party along the lines of

$$(1) \quad \frac{y}{1-y} \approx \left( \frac{x}{1-x} \right)^k.$$

When  $k = 3$  this is sometimes known as the “cube law”, or “law of the cubic proportion”.

For a more comprehensive discussion of this “law” and some alternatives to it, see [Tuf73; Taa73; Taa86; GT79].

Taking (1) as an equality gives

$$(2) \quad y = \frac{x^k}{x^k + (1-x)^k} \quad \text{and in particular,} \quad \left. \frac{dy}{dx} \right|_{x=\frac{1}{2}} = k.$$

It is principally this last result that has an empirical basis: for a party winning around 50% of all the votes, each additional 1 percent of vote-share translates into an additional  $k$  percent of the parliamentary seats.

The two-party version of our model also exhibits behaviour of this kind, with the value of  $k$  depending on the inter-district imitation parameter  $p$ . Figure 18 shows our results when initially there is 1 voter of each of 2 parties in each district, and  $p = 0.5$ . The fitted (red) line corresponds to the formula (2) with  $k = 30$ . The full curve is visible only in rather unrealistic situations like this one where strong self-reinforcement (*i.e.* large  $p$ ) makes extreme (“landslide”) election victories for either party possible. The seat-vote mappings

TABLE 1. Best fit slope at (50,50)

$p$	initial conditions 1:1	initial conditions 2:2	initial conditions 2:1/1:2	initial conditions 3:1/1:3
0.0	1.48	1.67	1.52	1.25
0.1	2.59	2.91	2.58	2.10
0.2	4.79	5.25	4.82	4.29

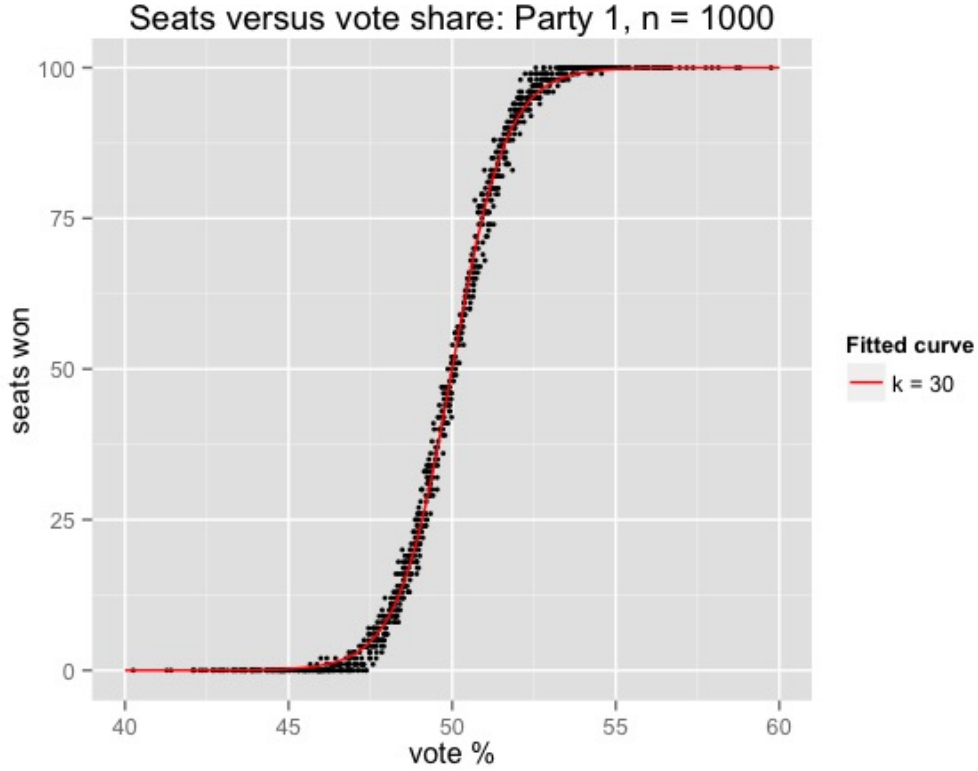
we have seen in Figure 7 and Figure 11 show only a small part of the full curve, because the values of  $p$  are small (and hence the inter-district correlation is too small). This means that extreme landslide elections have too low a probability to be observable in our sample. Similarly, data derived from real-world elections [GT79, Ch. 1] does not show the extremes of the seat-vote mapping (there have been simply too few real elections in history, for one thing).

In general,  $k$  increases with  $p$ : imitation between districts increases the advantage in seats to a party of gaining extra vote share overall, as can be seen in Figure 7. This seems intuitively reasonable: even “useless” extra voters in districts that their chosen party has no real chance of winning (or no real chance of losing) may still do the party some good if they can influence other voters elsewhere on the electoral map.

Table 1 shows the best fit for  $k$  obtained by ordinary linear regression, for various initial conditions and values of  $p$ . The value  $k = 3$  corresponds then to a value of  $p$  slightly more than 0.1. Note that each additional increase of one unit in total vote share gives an increase in seat share that is more than one unit, even when the districts are independent ( $p = 0$ ).

For the simplest independent-district cases, it is possible to do some direct calculations of the model’s seat-vote relationship. Let  $X_1, \dots, X_N$  be independent, identically distributed random variables representing a party’s vote shares in each of  $N$  identically-sized districts. Assume that the common distribution of the  $X_i$  is the uniform distribution; this corresponds to the large- $n$  limit of the initial model depicted in Figure 3. Let



FIGURE 18. Seats versus votes when  $p = 0.5$ , initial conditions 1:1

$\bar{X}_N = \frac{1}{N} \sum_{i=1}^N X_i$ , the total vote share. Then the theoretical expected seat-share  $y$  corresponding to a total-vote share  $x$  is

$$(3) \quad y = P\left(X_1 > \frac{1}{2} \mid \bar{X}_N = x\right)$$

(since the expected seat share is the same as the probability of winning the first (or any given) district). For  $N = 2$ , (3) yields the exact result

$$(4) \quad y = \begin{cases} 0 & , \text{if } x \leq \frac{1}{4} \\ 1 - \frac{1}{4x} & , \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ \frac{1}{4(1-x)} & , \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 1 & , \text{if } x \geq \frac{3}{4}. \end{cases}$$

From this, and a similar but lengthier calculation for  $N = 3$ , we obtain

$$(5) \quad \left. \frac{dy}{dx} \right|_{x=\frac{1}{2}} = \begin{cases} 1 & , \text{if } N = 2 \\ 2 & , \text{if } N = 3. \end{cases}$$

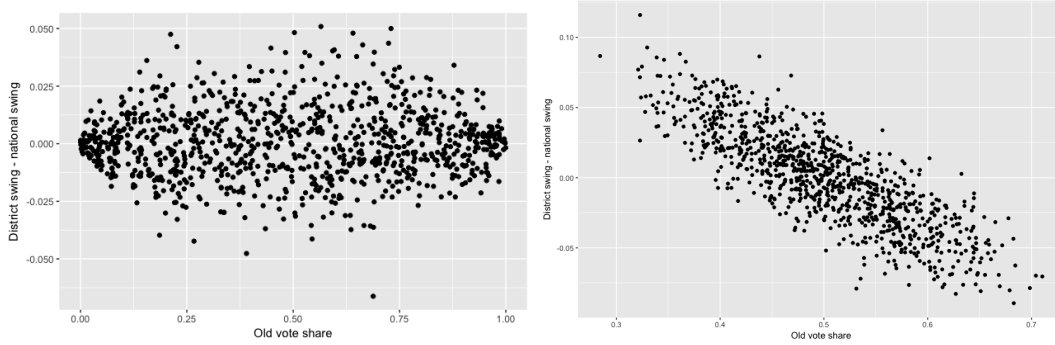
Our simulations suggest that these are extreme cases, with the central slope being close to 1.5 for  $N \geq 4$ .

**4.2. Inter-election swing.** In many situations, repeated opinion polling at the level of districts is infeasible. Instead, preference changes at district level are often imputed from the overall national *swing* using a model. The most common is the *uniform swing* model. These models are used, for example, for election prediction — knowing the exact district-level information from a base election and an estimate of the national swing, we infer the district level information for the next election and hence the election result.

We can test the uniform swing model against our simulated data as follows. We first run the urn process with 2 parties and 100 districts, starting with 1 voter of each party in each district, until we have 1000000 voters. We then rescale down to 600 voters, keeping the party percentages the same in each district (up to rounding error) and re-run the process. This typically results in a small difference in each district and overall, which gives the local and overall swing. We did this 1000 times and recorded the results from District 1 each time, yielding 1000 data points.

To analyse the data, we plotted the difference between local and national swing against the original vote share in the district. Results are shown in Figure 19. The first scatter-plot show clearly that uniform swing is not obviously inconsistent with the data for  $p = 0$  (although the concept of swing has less relevance in a situation where all districts are completely independent of the others). However for  $p = 0.1$  (a more realistic value, with some inter-district correlation) we clearly obtain a decreasing relationship between original vote

FIGURE 19. Local minus national swing versus original vote share in District 1:  $p = 0$  (L) and  $p = 0.1$  (R)



share and swing in a district, instead of the horizontal line through  $(0,0)$  expected from uniform swing.

**4.3. Retrospective simulations of real elections.** We investigated the 2017 UK general election from the point of view of what might have happened according to our model. We focused on the Conservative party, who unexpectedly failed to achieve an outright majority in the 2017 election. The approach is as follows. We deal with England, Scotland and Wales separately, because of their very different voting patterns (we do not consider Northern Ireland in detail because the Conservatives did not stand candidates there, but the same methodology could be used). We use the real vote data from these three nations, for each party in each district [Com].

We then “downscale” these vote numbers so that each district has 50 voters and the vote shares of parties change as little as possible (we achieve this by using the Webster–St-Lagüe allocation method, which has no bias toward small or large parties). We then run our urn model with  $K = 1$  and  $p = 0.08$  until the exact number of voters in each district is achieved (in practice, the process converges quickly enough that it may be terminated once the winner of a district is clear). We performed 100 runs. Each run gives a counterfactual election where the vector of vote shares is sampled from a distribution whose mean is the actual election result. Note that the variance is a decreasing function of the size of the

TABLE 2. Seats won by Conservative party under our model (100 simulated elections)

Nation	Real	Min	Median	Maximum
England	296	278	299	313
Scotland	13	6	12	18
Wales	8	3	7	13

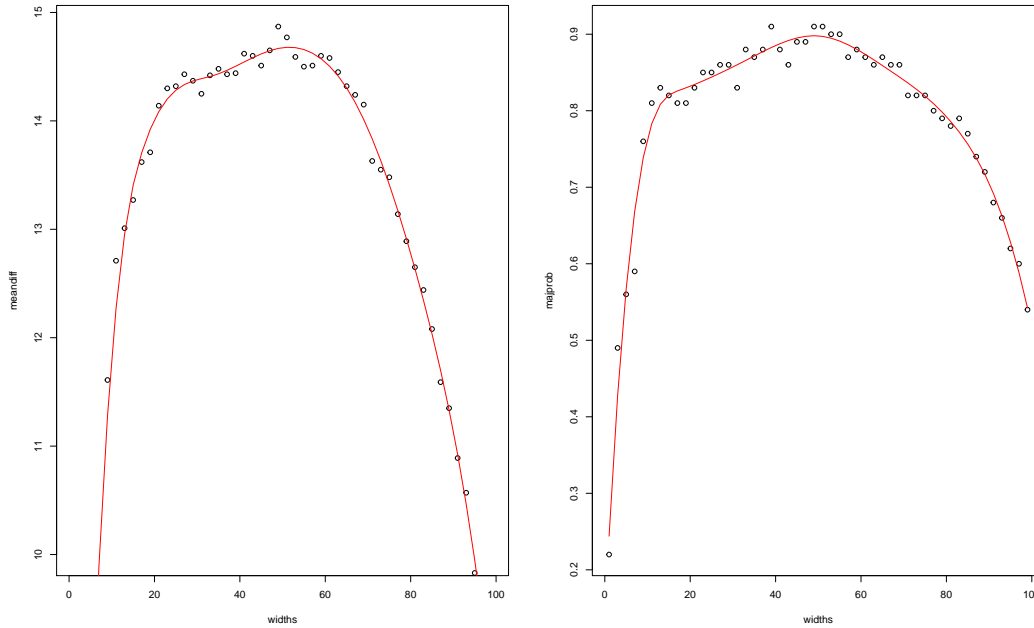
downscaled district and there is some choice available. We chose the downscaled district size to yield a variance similar to that of the margin of error obtained from traditional opinion polling. Results are shown in Table 2.

More detailed analysis, adding the results for the individual nations, shows that among the 100 simulations, exactly 13 have the Conservatives achieving a majority of seats in the UK as a whole.

*Campaign management.* How might the overall result have changed with more resources? We investigated the following “campaign management” question. Suppose that the Conservatives had resources equivalent to adding 240 extra voters to the downscaled England electorates (recall that there are 533 of these, each with 50 voters). Is it better to concentrate all the voters in the most marginal seats, or spread them more thinly? To investigate this, we simulated what would happen for various values of “width”  $w$ . This means that for each odd number  $w$  from 1 to 99, we added voters only to districts in which the number of wins out of the 100 simulations above was between  $50 - w/2$  and  $50 + w/2$ . As  $w$  increases, we distribute resources over more and more districts which include those that are more and more lopsided.

We computed two measures: the expected extra number of seats won in this way, and the probability of achieving a number of seats in England that would achieve an overall majority in UK given no change in the results in the other nations (in this case, 305).

FIGURE 20. Campaign management simulations. Extra seats (L) and probability of majority (R), by width.



Results are shown in Figure 20, which clearly indicates that neither extreme yields best results. Instead, resources should be spread across seats where the probability of winning before added resources is in the approximate range 30-70%. The probability of winning exceeds 0.9 and the expected number of extra seats is close to 15 with optimal distribution of resources, and both are much lower at the extreme widths.

*Forecasting.* Using historical election results and estimating the real national vote distribution by combining opinion polling with a swing model, we can make simple simulation predictions of elections. For example, for the 2017 UK election in England, we can use the exact 2015 results, a national poll estimate, and the proportional swing assumption as in Section 4.2. Downscaling the resulting data to 50 voters per district and running simulations gives a range of possible outcomes. In addition to the probabilistic approach we compute what we call the point prediction, which is the result of a single election predicted using the proportional swing assumption but without any downscaling or running

TABLE 3. Forecast seats won by Conservative party using perfect national exit poll (100 simulated elections)

Nation	Real	Point	Minimum	Median	Maximum
England	296	299	280	295.5	309
Scotland	13	16	6	12	19
Wales	8	9	4	9	14

TABLE 4. Forecast seats won by Conservative party using pre-election opinion poll (100 simulated elections)

Nation	Real	Point	Minimum	Median	Maximum
England	296	308	295	312	326
Scotland	13	16	8	12	18
Wales	8	9	4	9	13

the stochastic process. Note that opinion polling in districts is expensive and rarely performed, while national polls are much more readily available.

Table 3 shows the results obtained when the 2017 national level poll is the actual election result, while Table 3 shows the results obtained when the 2017 national level poll is a pre-election poll from election eve (we used an overall UK poll aggregation [Fis] for England because we did not find any England-specific polls, while for Scotland and Wales we used information from Wikipedia [con].)

Results show the sum of the point predictions using pre-election opinion polling and proportional swing in the three nations gives 333 seats to the Conservatives, a small majority. After comparing this with the prominent predictions made before the 2017 election, almost all of which predicted a substantial majority of seats for the Conservative Party, we claim that our methodology deserves much wider discussion. For example the combined

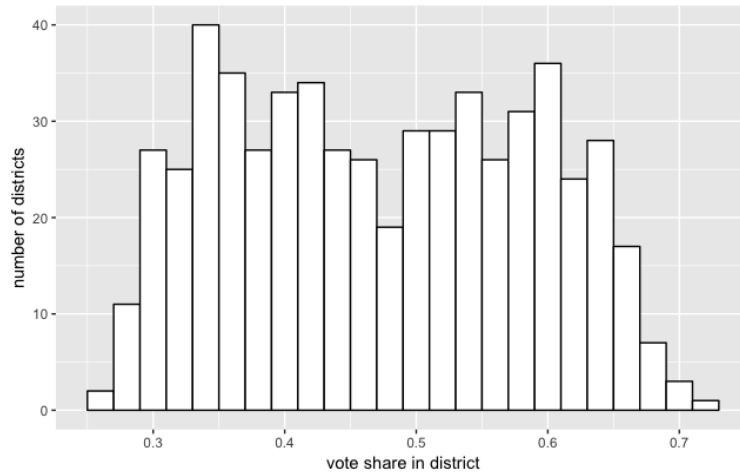
forecast from 2018-06-08 on ElectionsEtc site [] has an average Conservative seat tally of 358 which is outside the maximum we obtain by combining the maximum results in each nation. If we instead use the upper quartile in each nation we obtain an upper estimate of 340. Note that betting markets predicted 371 seats, a gross overestimate, and a probability of overall Conservative majority of 0.83.

The Conservatives' achievement in England of 296 seats is very close to the minimum over our simulations using pre-election polling. Clearly our forecasts are reasonably sensitive to opinion polling errors. When the exact exit poll is used, our method yields a prediction of 324 seats, closer to the exact result, as we would expect.

## 5. EXTENSIONS

One possible extension to the model is to remove the symmetry in the way imitation occurs. For example, some districts may be more influential than others. Or, the sphere of influence of a district may extend only locally – to other districts geographically or demographically close to it – rather than to all other districts as assumed in the present paper. In general, an influence structure could be modelled by a graph whose nodes are districts, with imitation probabilities on the (directed) edges. Something like this is probably needed to generate a model of the “vanishing marginals” phenomenon observed most strongly in the United States (visible in Figure 2), in which the districts self-organize (or are exogenously organized) into blocs favouring one party or the other, with very few competitive marginal districts. For models including third parties, another possibility is a regionally-based party with supporters who are less likely than most to imitate voters in other districts, or from other parties.

Another possible extension, which we also leave for later, is to consider more complicated preferences. For example, the model works with an arbitrary number of “colors”,

FIGURE 21. A simulation with  $N = 571$ ,  $m = 2$ ,  $p = 0.1$ 

each of which represented a candidate under plurality voting in this article. Instead, each color could be a linear preference order. In that case, however, it is less reasonable to assume that each change from a given color to another is equally likely, and a model that incorporates information about the structure of the preferences may be more useful. For example, a transposition of two adjacent elements of the preference order may be more likely than reversing the order entirely. More generally, we could model sincere preferences directly, as distinct from possibly strategic voting behavior.

## 6. CONCLUSION

Our results above demonstrate the flexibility and power of the model — with relatively few natural parameters we are able to reproduce a wide range of observed behaviour. In particular with 2 parties, symmetric initial conditions, and  $p = 0.1$  we can obtain rather realistic-looking data for FPP elections (compare the results shown in this paper with the seat-vote mappings and other data shown in [GT79], for example, and compare Figure 21 with Figure 1. This allows for a variety of applications to the analysis of electoral systems using plurality ballots. This includes systems based on multi-member districts in addition to the “First Past the Post” setup considered in the present paper. For example, we plan a thorough theoretical test of the tradeoff between decisiveness and proportionality of such



systems, as described using real election data by Carey & Hix [CH11]. Clearly, naive models that take no account of the district structure would be useless for such an undertaking. Another use of simulated data generated via the method of this paper is to the analysis of the vote-seat mapping when multi-member districts are involved. Taagepera [Taa86] generalized the basic cube law formula to the multi-member district case, and one could carry out a best fit analysis similar to that in Section 4.1 for that situation.

The fact that the model based on an imitation process that is widely used in related social science areas gives us some confidence that it can in fact model real situations reasonably well. The small number of parameters in the model means we avoid the risk of over-fitting. Note that we did not take great pains to optimize the parameter  $p$  in our model, instead estimating it based on our understanding of electoral swing models and vote-seat mappings.

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