Adaptive Mixed Finite Element Method for Elliptic problems with concentrated source terms

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Abstract

An adaptive mixed finite element method using the Lagrange multiplier technique are used to solve elliptic problems with Dirac delta source terms. The problem arises in the use of Chow-Anderssen linear functional methodology to recover coefficients locally in parameter estimation of elliptic equation from pointwise measurement. **In this article, we use a posteriori error estimator based on averaging technique as refinement indicators to produce a cycle of mesh adaptation, which are experimentally shown to capture singularity phenomena. Our result shows that adaptive refinement process is successfully refine elements around the center of the source terms and show that the global error estimation is better than uniform refinement process.**

1 Introduction

In the study of steady-state flow of groundwater in a confined aquifer Ω , the governing equation is the following elliptic partial differential equation ([6],[7])

$$-\nabla \cdot (\alpha \nabla u) = f \text{ in } \Omega \quad (1)$$

where $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$, is a bounded domain, $\alpha : \Omega \rightarrow \mathbb{R}$ denotes the transmissivity coefficient, $u : \Omega \rightarrow \mathbb{R}$ is pressure and f is the source term. When the source term f and the coefficient α are given, appropriate boundary information about the solution u is required so that the corresponding forward (direct) problem of determining u in Ω has a unique solution. Conversely, the recovery of information about the coefficient α , when u and f are given, is an inverse problem with α being sensitive to perturbations in u and f . It is the recovery of α from measurements of u and f , through the utilization of equation (1), that is called the "*aquifer parameter identification problem*".

Anderssen & Lamichhane [1] examine the numerical performance of ansatz for the recovery of aquifer transmissivity for observational data. A particular case of our interest is in the case of f is a point source function.

In order to recover coefficient α at single point, we solve a Poisson problem with delta dirac right hand side function and homogenous Dirichlet boundary condition:

$$\begin{cases} -\Delta u = \delta_{x_0} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygon domain and δ_{x_0} is delta Dirac function at point x_0 inside region Ω . Note that, the problem (2) arises in diverse fields such as the electric field generated by point charge, transport equations for efficient discharge in aquatic media, etc.

Due to the singular characteristics of the solution of the problem (2), mesh refinements around the source point x_0 are required to improve the quality of the approximation. Thus, an adaptive schemes based on a posteriori error indicators should be used. In this paper, we use a posteriori error control based on the averaging technique for the Poisson problem [2]. Furthermore, we restrict our study for two dimensional case, but this technique can be extended to three dimensional case by selecting corresponding adaptive method.

2 Adaptive Method

The adaptive procedure consists of solving the problem (2) on the mesh to obtain a solution of the mixed finite element method (**SOLVE**). A posteriori error estimates the global error η and local errors associated with each element η_T (**ESTIMATE**). We mark the element based on marking strategy for refinement (**MARK**). The procedure ended with mesh refinement using bisection method [3] of the marked element (**REFINE**). In short, adaptive finite element methods typically consist of successive loops of the sequence

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

2.1 Estimate

In this paper, mixed finite element method with Lagrange multiplier technique based on [2] used to obtain flux p_h and solution u_h . The a posteriori error estimates global error η and local errors for each element $\eta_K, K \in T$

$$\eta^2 = \sum_{K \in T} \eta_K^2$$

2.2 Mark

Then we compare global error η and error tolerance ε . If $\eta \leq \varepsilon$, then we are done, since the required tolerance has been achieved. If $\eta > \varepsilon$, we select subset $M \subset T$ which contains the elements where the local errors is relatively large. In our problem, we expect that elements close to x_0 to be marked for refinement, since this area is where the error will be the most significant. We ordered elements based on their local error and then selecting few first of them to construct a set M satisfies the following criteria: given a marking parameter θ , where $0 < \theta < 1$, find a subset $M \subset T$ such that

$$\sum_{K \in M} \eta_K^2 \geq \theta \eta^2$$

2.3 Refine

Mesh refinement is done using the vertex bisection method [3]. This method chooses a vertex of each marked element $K \in T$ as the peak of the element and creates an additional node at the edge opposite (known as the base of K). The refinement done by connecting the peak node to the additional node. If the neighboring element $\tilde{K}$, is not refined, then our additional node will be a hanging node. Therefore, we also need to refine the element $\tilde{K}$ which shares the same base with K . This type of refinement is chosen assuming that the support of regularized delta Dirac function is equally of the same distance from the triangle edges, and the inclusion triangle always assumed of isosceles triangle. Other cases needs careful analysis related to the error estimator such as studied in [4] for CR-basis.

3 Numerical Experiment

In this paper, we have done a numerical experiment to assess the performance of an adaptive mesh-refinement strategy based on a posteriori error control. To simulate a delta dirac function, we use

$$f_{x_0}(x) = \frac{1}{|B(x_0, a)|} \chi_{B(x_0, a)}(x)$$

where $B(x_0, a)$ is a ball centered at x_0 with radius a and $\chi_{B(x_0, a)}$ is the characteristic function of the ball $B(x_0, a)$. As we progress $a \rightarrow 0$, this function will behave similar to a dirac delta function.

We initiate the process with a very coarse mesh. At each step, our refinement method creates a new mesh better adapted to the solution of problem (2). To view a concentrated force function f_{x_0} on point x_0 , we initialize a longer than the edge of element containing the point x_0 . After several iterations, we set smaller values of a and continue the process.

The problem is solved in the square $\Omega := (-1, 1) \times (-1, 1)$, with $x_0 = (0, 0)$. We choose Dirichlet boundary conditions and initialize $a = 0.4$ and then set new values for a to 0.2; 0.1; 0.05 consecutively after three iterations. Then, we set $a = 0.01$ until error tolerance reached, which we set $\varepsilon = 0.01$. For this problem, we set our refinement parameter to be $\theta = 0.6$. Figure below show mesh grid and displacement for our initial iteration:

Figure 1: $a = 0.4$, $\theta = 0.6$, iter = 1, d.o.f. = 32

Figure below show some of the successive both refined meshes and displacements created in the process guided by η . We show the first iteration of different values of a , where $a \in [0.2, 0.1, 0.05, 0.01]$ with number of iteration and the number of degree of freedom (d.o.f.) of each respective mesh and displacement.

(a) Mesh grid

(b) Displacement

Figure 2: $a = 0.2$, $\theta = 0.6$, iter = 4, d.o.f. = 162

(a) Mesh grid

(b) Displacement

Figure 3: $a = 0.1$, $\theta = 0.6$, iter = 7, d.o.f. = 354

(a) Mesh grid

(b) Displacement

Figure 4: $a = 0.05$, $\theta = 0.6$, iter = 10, d.o.f. = 706

From these several figures, we can see that our method successfully refine the mesh grid around x_0 . The progression of a simulates the Dirac delta function as we expected. We can see that the mesh grid creates a "ring" around x_0 that slowly goes closer to the center because of our choices of approximation

Figure 5: $a = 0.01$, $\theta = 0.6$, iter = 13, d.o.f. = 1457

function. Figure 6 shows the final iteration when the global error is just below the error tolerance. We can see that the mesh around x_0 is finely refined. We can expect that with the progression of $a \rightarrow 0$, the function will behave similar to Dirac delta function.

Figure 6: $a = 0.01$, $\theta = 0.6$, iter = 24, d.o.f. = 26790

The following figure shows the error curve for our adaptive method compared to uniform refinement process. We could see that using our adaptive method, low global error is achieved in far lower number of elements than uniform refinement. This means that our adaptive method is better than uniform refinement in terms of computational cost.

Figure 7: Error curve

4 Conclusion

We have applied a posteriori error control to estimate mixed finite element approximation for the Poisson problem with delta Dirac function source terms. Adaptive algorithm is introduced to refine elements guided by the error estimator. Our result shows that adaptive refinement process is successfully refine elements around the center of the source terms. We also show that the global error estimation is better than uniform refinement process. Furthermore, we could extend our adaptive refinement method for another problems and by some modification for three dimensional case.

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References

- [1] Anderssen, R.S., Lamichhane, B.P., Piecewise Constant Aquifer Parameter Identification Recovery, MODSIM 2011, 19th International Congress on Modelling and Simulation. Modelling and Simulation Society of Australia and New Zealand, December 2011, pp. 364-370.

- [2] Bahriawati, C., Carstensen, C., Three Matlab Implementations of the Lowest-Order Raviart-Thomas MFEM with A Posteriori Error Control, Computational Methods in Applied Mathematics (2005), no. 4, pp. 333-361.
- [3] Chen, L., Zhang, C-S., AFEM.at.matlab: A Matlab Package of Adaptive Finite Element Methods, University of Maryland, 2006.
- [4] ShaoHong, D., XiaoPing, X., Residual-based a posteriori error estimates of non-conforming finite element method for elliptic problems with Dirac delta source terms, Science in China Series A: Mathematics (Aug. 2008), Vol. 51, No. 8, pp. 1440-1460.
- [5] Gagnon, M., An Adaptive Mixed Finite Element Method using Lagrange Multiplier Technique, A Project Report submitted to the Faculty of the Worcester Polytechnic Institute, May 2009.
- [6] Bear, J. (1972). The Dynamics of Fluids in Porous Media. Elsevier, Amsterdam.
- [7] Yeh, W. (1986). Review of parameter-identification procedures in groundwater hydrology - the inverse problem. Water Res. Research 22, 95-108.