The "hit" problem of five variables in the generic degree and its application

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(Dedicated to Professor James F. Peters)

Abstract

Let $P_s := \mathbb{F}_2[x_1, x_2, \dots, x_s]$ be the graded polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in s variables $x_1, x_2, \dots, x_s$, each of degree one. This algebra is considered as a graded module over the mod-2 Steenrod algebra, $\mathcal{A}$. We are interested in the "*hit*" problem of finding a minimal set of generators for $\mathcal{A}$ -module P_s . This problem is unresolved for every $s \geq 5$. In this paper, we study the hit problem of five variables in a generic degree, from which we investigate Singer's conjecture [Math. Z. 202 (1989), 493-523] for the transfer homomorphism of rank 5 in degrees given. This gives an efficient method to study the algebraic transfer and it is different from the ones of Singer.

Keywords:

Steenrod algebra; Peterson hit problem; Algebraic transfer; Steenrod squares; Invariant theory

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1. Introduction and main results

Let $\mathcal{A}$ denote the mod-2 Steenrod algebra. This algebra is defined to be the graded algebra over the prime field $\mathbb{F}_2$, generated by the Steenrod squares Sq^i , in grading $i \geq 0$, subject to the Adem

relations and $Sq^0 = 1$ (see Steenrod and Epstein [43]). From a topological point of view, the Steenrod algebra is the algebra of stable cohomology operations for ordinary cohomology H^* over $\mathbb{F}_2$. Let us denote by $P_s := \mathbb{F}_2[x_1, x_2, \dots, x_s] = \bigoplus_{d \geq 0} (P_s)_d$ the polynomial algebra over $\mathbb{F}_2$, which can be regarded as a graded left module over $\mathcal{A}$. The grading is by the degree of the homogeneous terms $(P_s)_d$ of degree d in s variables with the degree of each x_i being 1. The algebra P_s arises as the cohomology with $\mathbb{F}_2$ -coefficients of the elementary abelian 2-group $\mathbb{F}_2^s$ of rank s . As well known, the Steenrod algebra acts by the composition of linear operators on P_s and the action of the Steenrod squares $Sq^i : (P_s)_d \rightarrow (P_s)_{d+i}$ is determined by the Cartan formula and its elementary properties (see also [43], [56]).

A homogeneous element $f \in (P_s)_d$ in $\mathcal{A}$ -module P_s is *hit* if there is a finite sum $f = \sum_{j>0} Sq^j(f_j)$, where the homogeneous elements $f_j \in P_s$ have grading strictly less than d . Let us denote by $QP_s := \mathbb{F}_2 \otimes_{\mathcal{A}} P_s$ the quotient of the left $\mathcal{A}$ -module P_s by the hit elements in $\mathcal{A}^+ \cdot P_s$, where $\mathcal{A}^+$ denotes the augmentation ideal of $\mathcal{A}$ and $\mathbb{F}_2$ is viewed as a right $\mathcal{A}$ -module concentrated in grading 0. Then, QP_s is a graded vector space over $\mathbb{F}_2$ and a basis for QP_s lifts to a minimal generating set for P_s as a module over $\mathcal{A}$. It should be noted that this space is also considered as a form modular representation of GL_s over $\mathbb{F}_2$. We are interested in the *hit problem* of finding a monomial basis of QP_s in each s and degree $n \geq 0$. This is the same as the problem of determining of a minimal set of generators for $\mathcal{A}$ -module P_s . Peterson [23], Wood [55], Singer [41], and Priddy [37] laid the first foundation for the study of the hit problem and pointed out its relationship to several classical problems in the homotopy theory such as cobordism theory of manifolds, modular representation theory of the general linear group, and stable homotopy type of classifying spaces of finite groups. Later, many researchers have been interested in discovering these hit problems (see the works of Boardman [2], Bruner-Hà-Hung [4], Crabb and Hubbuck [5], Hung [11], Kameko [12], Silverman [39], Silverman and Singer [40], Walker and Wood [52, 53, 54], the present author [24]-[32], Sum [44]-[50] and others), but it was a well known unresolved problem for $s \geq 5$.

For a non-negative integer d , denote by $(QP_s)_d$ the subspace of QP_s consisting of all the classes represented by the homogeneous polynomials in $(P_s)_d$. One of the extremely useful tools for studying the hit problem is Kameko's homomorphism [12]: $\widetilde{Sq}_*^0 = (\widetilde{Sq}_*^0)_{(s, 2d+s)} : (QP_s)_{2d+s} \rightarrow (QP_s)_d$, which is induced by the $\mathbb{F}_2$ -linear map $\psi : P_s \rightarrow P_s$, given by

$$\psi(x_1^{n_1} x_2^{n_2} \dots x_s^{n_s}) = \begin{cases} x_1^{\frac{n_1-1}{2}} x_2^{\frac{n_2-1}{2}} \dots x_s^{\frac{n_s-1}{2}} & \text{if } n_1, n_2, \dots, n_s \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

for any monomial $x_1^{n_1} x_2^{n_2} \dots x_s^{n_s} \in P_s$. It is straightforward to see that ψ is not an $\mathcal{A}$ -homomorphism, but $\psi Sq^{2i} = Sq^i \psi$ and $\psi Sq^{2i+1} = 0$ for any $i \geq 0$. The structure of QP_s was systematically studied for the cases $s \leq 4$ (see [2, 12, 23, 47]). Specifically, by the works of Wood [55] and Sum [50], it is sufficient to determine QP_s at each degree d of the form:

$$d = r(2^t - 1) + 2^t m, \quad (1.1)$$

whenever r, t, m are non-negative integers satisfying $1 \leq r = \mu(d) \leq s$ and $\mu(m) < r$ (see Subsection 2.2). Here, $\mu(n)$ denotes the smallest number k such that n can be written in the form $\sum_{1 \leq i \leq k} (2^{u_i} - 1)$, where $u_i > 0$. For $r = s - 1$, the problem was investigated by Crabb and Hubbuck [5], Nam [21], Repka and Selick [38], Mothebe [19], Sum [45, 47] and by us [24, 25]. For $r = s = 5$, it is known by Sum [46, 50] with $m = 5, 10$ and by the present writer [26]-[32] with $m = 6, 8, 18, 22, 42$. So far there is no a completely answer for the case $s = 5$. It should be noted that solving special cases plays an important role in finding general solutions to the problem. By these reasons, our motivation to write up the present paper is to continue studying the hit problem in the 5-variable case. More specifically, we will explicitly describe QP_5 at degrees in (1.1) when $r = s - 2 = 3$ and $m = 1$.

We begin our work with the following, which is one of our main results.

Theorem 1.1. *Let $d = 3(2^t - 1) + 2^t$ with t a positive integer. The dimension of the $\mathbb{F}_2$ -vector space $(QP_5)_d$ is determined by the following table:*

$d = 3(2^t - 1) + 2^t$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t \geq 6$
$\dim(QP_5)_d$	46	250	645	945	1115	1116

Noting that if $t = 0$ then $d = 3(2^t - 1) + 2^t = 1$, and we can easily claim that $(QP_5)_1$ has dimension 5. The theorem has been proven by Sum [46, 50] for $t = 1$ and by Moetele-Mothebe [20] for $t = 2$. However, in this article, we prove the case $t = 2$ in another way (see Subsection 3.2). Now, it should be noticed that Kameko's map $\widetilde{Sq}_*^0 = (\widetilde{Sq}_*^0)_{(5,d)} : (QP_5)_d \longrightarrow (QP_5)_{\frac{d-5}{2}}$ is an epimorphism of $\mathbb{F}_2$ -vector spaces. From this and our previous results [24, 25], in order to prove the theorem, we only need to determine the kernel of $(\widetilde{Sq}_*^0)_{(5,d)}$. The calculations are quite long and complicated. Our method is based on the Kameko homomorphism and our recent results in [24, 25].

Let $GL_s := GL(s, \mathbb{F}_2)$ be the general linear group of rank s over $\mathbb{F}_2$. As it is known, this GL_s acts regularly on P_s by matrix substitution. Further, the two actions of $\mathcal{A}$ and GL_s upon P_s commute with each other; hence there is an inherited action of GL_s on QP_s . This shows that the problem we are interested in is one of the useful tools for studying the modular representation of GL_s over $\mathbb{F}_2$. Moreover, it is applied to describe the $\mathbb{F}_2$ -cohomology of the Steenrod algebra, $\text{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ through Singer's transfer homomorphism [41]. Computing explicitly this cohomology is one of the problems of great importance in stable homotopy theory. But despite intensive investigation for nearly half a century, its structure remains elusive. The Singer transfer [41] is a linear transformation $\varphi_s : \text{Tor}_{s,s+*}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_s)_*^{GL_s}$, from the s -th homology group of $\mathcal{A}$ to the space of GL_s -invariants of $(QP_s)_*$. Dually, let $PH_*(\mathbb{F}_2^s)$ be the subspace of $H_*(\mathbb{F}_2^s)$ consisting of all elements that are annihilated by all positive degree Steenrod squares, then there is an induced action of GL_s on $PH_*(\mathbb{F}_2^s)$, and we have an $\mathbb{F}_2$ -linear map from the coinvariant elements of $PH_*(\mathbb{F}_2^s)$ to mod-2 cohomology group $\text{Ext}_{\mathcal{A}}^{s,s+*}(\mathbb{F}_2, \mathbb{F}_2)$ of the Steenrod algebra, which is induced over the E_2 -term of the Adams spectral sequence by the geometrical transfer map $\Sigma^\infty(B(\mathbb{F}_2^s)_+) \longrightarrow \Sigma^\infty(S^0)$ in stable homotopy theory (see also Mitchell [18]). These transfers can be played a key role in the study of the Kervaire invariant one problem (see Browder [3], Mahowald [15], Minami [16, 17]). The Kervaire invariant was first introduced by Browder's work [3], where he shows that the classes $h_j^2 \in \text{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ are the permanent cycles in the classical Adams spectral sequence at the prime 2, if and only if smooth framed manifolds of Kervaire invariant one exist only in dimensions $2^{j+1} - 2$. The hypothetical element in stable $2^{j+1} - 2$ -stem, $\pi_{2^{j+1}-2}^S$, represented by such a framed manifold is denoted by θ_j . These θ_j are known to exist for $j \leq 5$ (see also Lin-Mahowald [13]). Many open issues in Algebraic and Differential Topology depend on knowing whether or not the Kervaire invariant one elements θ_j for $j \geq 6$. In 2016, Hill, Hopkins, and Ravenel [10] indicated that these elements do not exist for $j \geq 7$. So far the case $j = 6$ is still no answer. Thus, smooth framed manifolds of Kervaire invariant one therefore exist only in dimensions 2, 6, 14, 30, 62, and possibly 126.

Going back to the Singer transfer, in [41], Singer claimed that φ_s is an isomorphism for $s = 1, 2$ and the "total" dual algebraic transfer $\bigoplus_{s \geq 0} \varphi_s^*$ is an algebra homomorphism. Afterwards, Boardman [2], with additional calculations by Kameko [12], showed that φ_3 is also an isomorphism. Additionally, in higher rank, Singer states that the transfer homomorphism is not a monomorphism in bidegree (5, 14) and made the following conjecture.

Conjecture 1.2. *The homological transfer φ_s is an epimorphism for any $s > 0$.*

This prediction is known to be true for $s \leq 3$. After more than 3 decades, we have recently proved in [36] that it is also true for $s = 4$. The general case is still an open problem. Recently, some authors have been studied the conjecture for $s = 4, 5$ (see Bruner-Hà-Hung [4], Hưng [11], Chơn-Hà [7, 8], Hà [9], Nam [22], the present author [26], [28]-[35], Sum [46, 48, 49, 50] and others). In the present work, by using techniques of the hit problem of five variables, we investigate Conjecture 1.2 in

bidegree $(5, d + 5)$, where d is determined as in Theorem 1.1. This gives an efficient method to study the Singer transfer and it is different from that of Singer [41]. More explicitly, based on this approach, we obtain the following.

Theorem 1.3. *For each positive integer t , the space of GL_5 -invariants $(QP_5)_{3(2^t-1)+2^t}^{GL_5}$ is trivial.*

It is known that Kameko's map $\widetilde{Sq}_*^0 = (\widetilde{Sq}_*^0)_{(5, 3(2^t-1)+2^t)} : (QP_5)_{3(2^t-1)+2^t} \longrightarrow (QP_5)_{2^{t+1}-4}$ is an epimorphism of GL_5 -modules. On the other side, following [48], $(QP_5)_{2^{t+1}-4}^{GL_5}$ is trivial, for all $t > 0$. So, to prove this theorem, we only need to determine the invariant $(\text{Ker}((\widetilde{Sq}_*^0)_{(5, 3(2^t-1)+2^t)}))^{GL_5}$. The computations are based on a monomial basis of $(QP_5)_{3(2^t-1)+2^t}$, which are given in the proof of Theorem 1.1.

By using calculations by Chen [6], Lin [14] and Tangora [51] and passing to the dual, deduce that $\text{Tor}_{5, 2^{t+2}+2}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for all $t > 0$. This together with Theorem 1.3 imply that the homological transfer $\varphi_5 : \text{Tor}_{5, 2^{t+2}+2}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow (QP_5)_{3(2^t-1)+2^t}^{GL_5}$ is an isomorphism. As a consequence, we have immediately the following.

Corollary 1.4. *Conjecture 1.2 is true for the rank 5 and the degree $3(2^t-1)+2^t$ with t an arbitrary positive integer.*

The paper contains five sections and is organized as follows: Background is provided in Section 2. The proofs of Theorem 1.1 and 1.3 will be presented in Sections 3 and 4, respectively. All the admissible monomials of degree d in Theorem 1.1 are described in the Appendix.

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2. Background

In this section, we provide some known basis concepts, which might be useful in the next sections.

2.1. The weight vector of a monomial

Let us denote by $\alpha_j(d)$ the j -th coefficients in dyadic expansion of a non-negative integer d . Then, we have $d = \alpha_0(d)2^0 + \alpha_1(d)2^1 + \dots + \alpha_j(d)2^j + \dots$, for $\alpha_j(d) \in \{0, 1\}$ with $j \geq 0$. For a monomial $u = x_1^{a_1} x_2^{a_2} \dots x_s^{a_s} \in P_s$, we define two sequences associated with u by

$$\begin{aligned}\omega(u) &= (\omega_1(u), \omega_2(u), \dots, \omega_j(u), \dots), \\ \sigma(u) &= (a_1, a_2, \dots, a_s),\end{aligned}$$

where $\omega_j(u) := \sum_{1 \leq i \leq s} \alpha_{j-1}(a_i) \leq s$, for all j . The sequences $\omega(u)$ and $\sigma(u)$ are called the *weight vector* and the *exponent vector* of the monomial u , respectively.

Let $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ be a sequence of non-negative integers. The sequence ω is called a weight vector if $\omega_j = 0$ for $j \gg 0$. By convention, the sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

2.2. The admissible monomial

For a weight vector ω , we define $\deg \omega = \sum_{i \geq 1} 2^{i-1} \omega_i$. Let us denote by $P_s(\omega)$ the subspace of P_s spanned by all monomials $x \in P_s$ such that $\deg(x) = \deg(\omega)$, $\omega(x) \leq \omega$, and by $P_s^-(\omega)$ the subspace of P_s spanned by all monomials $x \in P_s$ such that $\deg(x) = \deg(\omega)$, $\omega(x) < \omega$.

Definition 2.2.1 (see [12], [47]). Let ω be a weight vector and let f, g be two homogeneous polynomials of the same degree in P_s .

- (i) $f \equiv g$, if and only if $(f + g) \in \mathcal{A}^+ \cdot P_s$. When $f \equiv 0$, then $f \in \mathcal{A}^+ \cdot P_s$.
- (ii) $f \equiv_\omega g$ if and only if $f, g \in P_s(\omega)$ and $(f + g) \in ((\mathcal{A}^+ \cdot P_s \cap P_s(\omega)) + P_s^-(\omega))$. Specifically, f is said to be ω -hit, if $f \equiv_\omega 0$.

The Definition 2.2.1(ii) is a modification of a definition in Sum [47, Definition 2.3]. It can be easily seen that the binary relations $\equiv$ and $\equiv_\omega$ on P_s are equivalence ones.

Definition 2.2.2 (see [12]). Let x and y be two monomials of the same degree in P_s . We say that $x < y$ if and only if one of the following holds:

- (i) $\omega(x) < \omega(y)$;
- (ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Our key instrument here is the following notion.

Definition 2.2.3 (see [12], [47]). We say that a monomial x in P_s is *inadmissible*, if there exist monomials $y_1, y_2, \dots, y_m$ such that $y_t < x$ for $1 \leq t \leq m$ and $x \equiv \sum_{1 \leq t \leq m} y_t$. Then, x is said to be *admissible*, if it is not inadmissible.

Thus, the above definitions indicated that the set of all the admissible monomials of degree d in P_s is a minimal set of $\mathcal{A}$ -generators for P_s in degree d . Therefore, $(QP_s)_d$ is an $\mathbb{F}_2$ -vector space with a basis consisting of all the classes represented by the admissible monomials of degree d in P_s .

Definition 2.2.4 (see [12], [47]). A monomial x in P_s is said to be strictly inadmissible if and only if there exists monomials $y_1, y_2, \dots, y_m$ such that $y_t < x$ for $1 \leq t \leq m$ and

$$x = \sum_{1 \leq t \leq m} y_t + \sum_{1 \leq j \leq 2^r - 1} Sq^j(h_j),$$

where $r = \max\{i \in \mathbb{Z} : \omega_i(x) > 0\}$ and suitable polynomials $h_j \in P_s$.

Noticing that if x is strictly inadmissible, then it is inadmissible. However, in general, the converse is not true. For instance, the monomial $x = x_1 x_2^2 x_3^2 x_4^2 x_5^2 x_6$ in P_6 is inadmissible, but it is not strictly inadmissible.

Theorem 2.2.5 (see [12], [47]). Let x, y and u be monomials in P_s such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_t(u) \neq 0$ and $\omega_i(u) = 0$ for $i > t > 0$.

- (i) If u is inadmissible, then xu^{2^r} is also inadmissible.
- (ii) If u is strictly inadmissible, then uy^{2^t} is also strictly inadmissible.

From now on, we use the following notations.

$$\begin{aligned} P_s^0 &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_s^{a_s} \in P_s \mid a_1 a_2 \dots a_s = 0\} \rangle, \\ P_s^+ &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_s^{a_s} \in P_s \mid a_1 a_2 \dots a_s > 0\} \rangle. \end{aligned}$$

Then P_s^0 and P_s^+ are the $\mathcal{A}$ -submodules of P_s . Furthermore, we have a direct summand decomposition of the $\mathbb{F}_2$ -vector spaces: $QP_s = QP_s^0 \oplus QP_s^+$, where $QP_s^0 := \mathbb{F}_2 \otimes_{\mathcal{A}} P_s^0$ and $QP_s^+ := \mathbb{F}_2 \otimes_{\mathcal{A}} P_s^+$.

Let $QP_s(\omega)$ be the quotient of $P_s(\omega)$ by the equivalence relation $\equiv_\omega$. According to Walker and Wood [54], we have a filtration of QP_s :

$$\{0\} \subseteq \dots \subseteq P_s^-(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s^-(\omega)) \subseteq P_s(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s(\omega)) \subseteq \dots \subseteq P_s/(\mathcal{A}^+ \cdot P_s) = QP_s.$$

It should be noticed that this is not only a filtration of QP_s as a vector space, but also as a GL_s -module. The inclusion of $P_s^-(\omega)$ into $P_s(\omega)$ induces the monomorphism

$$P_s^-(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s^-(\omega)) \rightarrow P_s(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s(\omega))$$

and the sequence is short exact:

$$0 \rightarrow P_s^-(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s^-(\omega)) \rightarrow P_s(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s(\omega)) \rightarrow P_s(\omega)/(((\mathcal{A}^+ \cdot P_s) \cap P_s(\omega)) + P_s^-(\omega)) \rightarrow 0.$$

Combining this and Definition 2.2.1(ii), one gets

$$QP_s(\omega) = P_s(\omega)/((\mathcal{A}^+ \cdot P_s \cap P_s(\omega)) + P_s^-(\omega)) \cong (P_s(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s(\omega)))/(P_s^-(\omega)/((\mathcal{A}^+ \cdot P_s) \cap P_s^-(\omega))).$$

Following [50], $QP_s(\omega)$ is also an GL_s -module, further this isomorphism is also an isomorphism of GL_s -modules. Combining this and the above filtration of QP_s , we have immediately

$$\dim((QP_s)_d) = \sum_{\deg(\omega)=d} \dim(QP_s(\omega)), \quad \dim((QP_s)_d^{GL_s}) \leq \sum_{\deg(\omega)=d} \dim(QP_s(\omega)^{GL_s}).$$

Throughout this paper, we will always denote the set of all admissible monomials of degree d in P_s by $\mathcal{B}_s(d)$. We put

$$\mathcal{B}_s^0(d) := \mathcal{B}_s(d) \cap (P_s^0)_d, \quad \mathcal{B}_s^+(d) := \mathcal{B}_s(d) \cap (P_s^+)_d.$$

Then, if ω is a weight vector of degree d , then we put

$$\mathcal{B}_s(\omega) := \mathcal{B}_s(d) \cap P_s(\omega), \quad \mathcal{B}_s^0(\omega) := \mathcal{B}_s(\omega) \cap (P_s^0)_d, \quad \mathcal{B}_s^+(\omega) := \mathcal{B}_s(\omega) \cap (P_s^+)_d.$$

Next, for a polynomial $f \in P_s$, we denote by $[f]$ the classes in QP_s represented by f . If ω is a weight vector and $f \in P_s(\omega)$, then denote by $[f]_\omega$ the classes in $QP_s(\omega)$ represented by f . For a subset $\mathcal{B} \subset P_s$, we denote $[\mathcal{B}] = \{[f] : f \in \mathcal{B}\}$. If $\mathcal{B} \subset P_s(\omega)$, then we set $[\mathcal{B}]_\omega = \{[f]_\omega : f \in \mathcal{B}\}$. Thus, we see that $[\mathcal{B}_s(\omega)]_\omega$, $[\mathcal{B}_s^0(\omega)]_\omega$ and $[\mathcal{B}_s^+(\omega)]_\omega$ are respectively the bases of the $\mathbb{F}_2$ -vector spaces $QP_s(\omega)$, $QP_s^0(\omega) := QP_s(\omega) \cap (QP_s^0)_d$ and $QP_s^+(\omega) := QP_s(\omega) \cap (QP_s^+)_d$.

For a natural number d , let $\mu(d) = \min\{r \in \mathbb{N} : d = \sum_{1 \leq i \leq r} (2^{u_i} - 1), u_i > 0\}$. So, by Wood [55], $(QP_s)_d = 0$ if $\mu(d) > s$, and by Sum [50], $\mu(d) = r \leq s$ if and only if there exists uniquely a sequence of integers $u_1 > u_2 > \dots > u_{r-1} \geq u_r > 0$ such that $d = \sum_{1 \leq i \leq r} (2^{u_i} - 1)$. From these, we only need to study $(QP_s)_d$ in degrees d satisfying $1 \leq \mu(d) = r \leq s$; moreover d is of the form $r(2^t - 1) + 2^t \cdot m$, whenever t, m are non-negative integers such that $\mu(m) < r$. Indeed, if $\mu(d) = r = 1$, then d has the form $d = 2^u - 1 = 1 \cdot (2^u - 1) + 2^u \cdot 0 = r \cdot (2^u - 1) + 2^u \cdot m$, where $u > 0$ and $\mu(m = 0) = 0 < 1$. If $\mu(d) = r \geq 2$, then by Sum [50], d has the form $\sum_{1 \leq i \leq r} (2^{u_i} - 1)$, where $u_1 > u_2 > \dots > u_{r-1} \geq u_r > 0$. Consider $m = \sum_{1 \leq j \leq r-1} (2^{u_j - u_r} - 1)$. According to Sum [50], we see that $\mu(m) = r - 2 < r$ if $u_{r-1} = u_r$ and $\mu(m) = r - 1 < r$ if $u_{r-1} > u_r$. Further, $d = r(2^t - 1) + 2^t \cdot m$ with $t = u_r$.

The following homomorphisms will be used throughout the article. For $1 \leq i \leq s$, define the homomorphism $\rho_i : P_{s-1} \rightarrow P_s$ of algebras by substituting

$$\rho_i(x_j) = \begin{cases} x_j & \text{if } 1 \leq j < i, \\ x_{j+1} & \text{if } i \leq j < s. \end{cases}$$

The maps ρ_i are also the monomorphism of $\mathcal{A}$ -modules. We consider the set

$$\mathcal{N}_s := \{(i; I) \mid I = (i_1, i_2, \dots, i_r), 1 \leq i < i_1 < i_2 < \dots < i_r \leq s, 0 \leq r < s\},$$

where by convention, $I = \emptyset$, if $r = 0$. Let $r = \ell(I)$ be the length of I .

Definition 2.2.6. Let $(i; I) \in \mathcal{N}_s$, $r = \ell(I)$, and let u be an integer with $1 \leq u \leq r$. A monomial $x = x_1^{a_1} x_2^{a_2} \dots x_{s-1}^{a_{s-1}} \in P_{s-1}$ is said to be u -compatible with $(i; I)$ if all of the following hold:

- (i) $a_{i_1-1} = a_{i_2-1} = \dots = a_{i_{u-1}-1} = 2^r - 1$,
- (ii) $a_{i_u-1} > 2^r - 1$,
- (iii) $\alpha_{r-t}(a_{i_u-1}) = 1, \forall t, 1 \leq t \leq u$,
- (iv) $\alpha_{r-t}(a_{i_t-1}) = 1, \forall t, u < t \leq r$.

Clearly, a monomial x can be u -compatible with a given $(i; I) \in \mathcal{N}_s$ for at most one value of u . By convention, x is 1-compatible with $(i; \emptyset)$.

Definition 2.2.7. Consider $(i; I) \in \mathcal{N}_s$, $0 < r < s$ and put $x_{(I,u)} = x_{i_u}^{2^{r-1}+2^{r-2}+\dots+2^{r-u}} \prod_{u < t \leq r} x_{i_t}^{2^{r-t}}$, for $1 \leq u \leq r$, $x_{(\emptyset,1)} = 1$. For a monomial $x \in P_{s-1}$, we define the monomial $\phi_{(i;I)}(x)$ in P_s by setting

$$\phi_{(i;I)}(x) = \begin{cases} \frac{x_i^{2^r-1} \rho_i(x)}{x_{(I,u)}} & \text{if there exist } u \text{ such that } x \text{ is } u\text{-compatible with } (i; I), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have a linear transformation $\phi_{(i;I)} : P_{s-1} \rightarrow P_s$. It is easily seen that $\phi_{(i;\emptyset)} = \rho_i$ and that $\phi_{(i;I)}$ is not an $\mathcal{A}$ -homomorphism. Moreover, if $\phi_{(i;I)}(x) \neq 0$, then $\omega(\phi_{(i;I)}(x)) = \omega(x)$.

For a subset $U \subset P_{s-1}$, we denote by

$$\begin{aligned} \Phi^0(U) &= \bigcup_{1 \leq i \leq s} \phi_{(i;\emptyset)}(U) = \bigcup_{1 \leq i \leq s} \rho_i(U), \\ \Phi^+(U) &= \bigcup_{(i;I) \in \mathcal{N}_s, 0 < \ell(I) < s} \phi_{(i;I)}(U) \setminus P_s^0. \end{aligned}$$

Then, it is easy to see that if U is a minimal set of generators for $\mathcal{A}$ -module P_{s-1} in degree d , then $\Phi^0(U)$ is also a minimal set of generators for $\mathcal{A}$ -module P_s^0 in degree d .

Definition 2.2.8. For any $(i; I) \in \mathcal{N}_s$, we define the homomorphism $p_{(i;I)} : P_s \rightarrow P_{s-1}$ of $\mathbb{F}_2$ -algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j < i, \\ \sum_{k \in I} x_{k-1} & \text{if } j = i, \\ x_{j-1} & \text{if } i < j \leq s. \end{cases}$$

It should be noticed that $p_{(i;I)}$ is also a homomorphism of $\mathcal{A}$ -modules. In particular, we have $p_{(i;\emptyset)}(x_i) = 0$ for $1 \leq i \leq s$ and $p_{(i;I)}(\rho_i(u)) = u$ for any $u \in P_{s-1}$. Moreover, the next technicality is important.

Lemma 2.2.9 (see [24]). *If x is a monomial in P_s , then $p_{(i;I)}(x) \in P_{s-1}(\omega(x))$.*

The lemma indicates that if ω is a weight vector and $x \in P_s(\omega)$, then $p_{(i;I)}(x) \in P_{s-1}(\omega)$. Furthermore, $p_{(i;I)}$ passes to a homomorphism from $QP_s(\omega)$ to $QP_{s-1}(\omega)$.

In Section 3, we use Lemma 2.2.9 and the results in Sum [47] to prove a certain subset of QP_s is linearly independent. More precisely, let $\mathcal{B}$ be a finite subset of P_s consisting of some monomials of degree d . Denote by $|\mathcal{B}|$ the cardinal of $\mathcal{B}$. To prove the set $[\mathcal{B}]$ is linearly independent in $(QP_s)_d$, we denote the elements of $\mathcal{B}$ by $b_{d,k}$, $1 \leq k \leq n = |\mathcal{B}|$ and assume that there is a linear relation $\mathcal{S} = \sum_{k=1}^n \gamma_k b_{d,k} \equiv 0$, with $\gamma_k \in \mathbb{F}_2$ for all k , $1 \leq k \leq n$. For $(i; I) \in \mathcal{N}_s$, we explicitly compute $p_{(i;I)}(\mathcal{S})$ in terms of the admissible monomials in $P_{s-1} \pmod{\mathcal{A}^+ \cdot P_{s-1}}$. Computing from some relations $p_{(i;I)}(\mathcal{S}) \equiv 0$ with $(i; I) \in \mathcal{N}_s$, we obtain $\gamma_k = 0$ for all k .

3. $\mathcal{A}$ -generators for P_5 in degree $3(2^t - 1) + 2^t$

The goal of this section is to prove Theorem 1.1. More precisely, we explicitly determine all admissible monomials of degree $3(2^t - 1) + 2^t$ in P_5 . We first recall a result of Singer [42] on the hit monomials in P_s .

3.1. Singer's criterion on the hit monomials

Definition 3.1.1. A monomial $z = x_1^{b_1} x_2^{b_2} \dots x_s^{b_s}$ in P_s is called a spike if $b_i = 2^{t_i} - 1$ for t_i a non-negative integer and $i = 1, 2, \dots, s$. If z is a spike with $t_1 > t_2 > \dots > t_{r-1} \geq t_r > 0$ and $t_j = 0$ for $j > r$, then it is called a minimal spike.

Lemma 3.1.2. *All the spikes in P_s are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \dots)$ is weakly decreasing and $\omega_1 \leq s$, then there is a spike $\mathcal{Z}$ in P_s such that $\omega(\mathcal{Z}) = \omega$.*

Proof. Let $\mathcal{Z} = \prod_{1 \leq i \leq s} x_i^{2^{d_i}-1}$ be a spike monomial in P_s . By way of contradiction, suppose that $\mathcal{Z}$ is inadmissible. Then, there exist monomials $y_1, y_2, \dots, y_m \in P_s$ such that $y_t < \mathcal{Z}$ for all t and $\mathcal{Z} = \sum_{1 \leq t \leq m} y_t + \sum_{j>0} Sq^j(w_j)$, with suitable monomial $w_j \in P_s$. This implies that there is a monomial $u \in P_s$ with $\deg u < \deg \mathcal{Z}$ such that $Sq^n(u) = \mathcal{Z} + f$, where $n > 0$ and $f = \sum_i X_i \in P_s$, $\deg X_i = \deg \mathcal{Z}$. Suppose $u = \prod_{1 \leq \ell \leq k} x_\ell^{a_\ell}$. Since $\deg u < \deg \mathcal{Z}$, there is a number t , $1 \leq t \leq s$ such that $a_t < 2^{d_t} - 1$. According to Cartan's formula, we have

$$\mathcal{Z} + f = Sq^n(u) = \sum_{i \geq 0} Sq^i(x_t^{a_t}) Sq^{n-i}(x_1^{a_1} \dots x_{t-1}^{a_{t-1}} x_{t+1}^{a_{t+1}} \dots x_s^{a_s}).$$

Since $2^{d_t} - 1 - a_t > 0$ and $\mathcal{Z}$ is a term of $\sum_{i \geq 0} Sq^i(x_t^{a_t}) Sq^{n-i}(x_1^{a_1} \dots x_{t-1}^{a_{t-1}} x_{t+1}^{a_{t+1}} \dots x_s^{a_s})$, hence $\mathcal{Z}$ is a term of $Sq^{2^{d_t}-1-a_t}(x_t^{a_t}) Sq^{n-(2^{d_t}-1-a_t)}(x_1^{a_1} \dots x_{t-1}^{a_{t-1}} x_{t+1}^{a_{t+1}} \dots x_s^{a_s})$. On the other hand, $2^{d_t} - 1 - a_t \leq 2^{d_t} - 1$, so there exist a number $\ell \geq 0$ such that $\alpha_\ell(2^{d_t} - 1 - a_t) = 1$. This implies $\alpha_\ell(a_t) = 0$. Since $\binom{a_t}{2^{d_t}-1-a_t}$ is even, $Sq^{2^{d_t}-1-a_t}(x_t^{a_t}) = 0$. So, one gets $\mathcal{Z} = 0$. This contradicts the fact that $\mathcal{Z} = \prod_{1 \leq i \leq s} x_i^{2^{d_i}-1} \neq 0$. Obviously, if $\mathcal{Z}$ is a spike, then its weight vector is weakly decreasing.

Now, assume that $\omega = (\omega_1, \omega_2, \dots)$ is weakly decreasing and $\omega_1 \leq s$. For $j \geq s$, we denote by a_j the non-negative integer such that

$$\alpha_i(a_j) = \begin{cases} 1, & \text{if } \omega_{i+1} \geq j, \\ 0, & \text{if } \omega_{i+1} < j. \end{cases}$$

Denote by d_j the greatest integer such that $\omega_{d_j} \geq j$. Since ω is weakly decreasing, $\alpha_i(a_j) = 1$ for $1 \leq i < d_j$ and $\alpha_i(a_j) = 0$ with $i \geq d_j$. So, $a_j = 2^{d_j} - 1$. On the other hand, we have $\omega_{i+1} \leq \omega_1 \leq s$ for all $i \geq 0$. Hence, if $j > s$, then $a_j = 0$. Therefore, $\mathcal{Z} = x_1^{a_1} x_2^{a_2} \dots x_s^{a_s}$ is a spike in P_s and $\omega(\mathcal{Z}) = \omega$. The lemma is proved. $\square$

In [42], Singer pointed out that if $\mu(d) \leq s$, then there exists uniquely a minimal spike of degree d in P_s . Further, we have the following.

Theorem 3.1.3 (see Singer [42]). *Let x be a monomial in P_s , where $\mu(\deg(x)) \leq s$. Suppose that z is the minimal spike with $\deg(z) = \deg(x)$. Then, if $\omega(x) < \omega(z)$, then x belongs to $\mathcal{A}^+ \cdot P_s$.*

3.2. Proof of Theorem 1.1

For convenience, we use the following notations:

$$\begin{aligned} \mathbb{N}_s &= \{1, 2, \dots, s\}, \\ X_{\mathbb{I}} &= X_{\{i_1, i_2, \dots, i_r\}} = \prod_{i \in \mathbb{N}_s \setminus \mathbb{I}} x_i, \quad \mathbb{I} = \{i_1, i_2, \dots, i_r\} \subseteq \mathbb{N}_s. \end{aligned}$$

Specifically, $X_{\mathbb{N}_s} = 1$, $X_{\emptyset} = x_1 x_2 \dots x_s$, and $X_{\{i\}} = x_1 \dots \hat{x}_i \dots x_s$, $i = 1, 2, \dots, s$. Let $x = x_1^{a_1} x_2^{a_2} \dots x_s^{a_s} \in P_s$ and let $\mathbb{I}_j(x) = \{i \in \mathbb{N}_s : \alpha_j(a_i) = 0\}$, for $j \geq 0$. Then, a careful but straightforward computation shows that $x = \prod_{j \geq 0} X_{\mathbb{I}_j(x)}^{2^j}$. Now, notice that Kameko's homomorphism $\widetilde{Sq}_*^0 = (\widetilde{Sq}_*^0)_{(5, 3(2^t-1)+2^t)} : (QP_5)_{3(2^t-1)+2^t} \longrightarrow (QP_5)_{2^{t+1}-4}$ is an epimorphism, and so, $(QP_5)_{3(2^t-1)+2^t} \cong \text{Ker}((\widetilde{Sq}_*^0)_{(5, 3(2^t-1)+2^t)}) \oplus (QP_5)_{2^{t+1}-4}$. Following our previous works [24, 25], the dimensions of $(QP_5)_{2^{t+1}-4}$ are determined as follows.

Theorem 3.2.1. *Let $n = 2^{t+1} - 4$ with t a positive integer. The dimension of the $\mathbb{F}_2$ -vector space $(QP_5)_n$ is determined by the following table:*

$n = 2^{t+1} - 4$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t \geq 6$
$\dim(QP_5)_n$	1	45	190	480	650	651

When $t = 1$, we see that $2^{t+1} - 4 = 0$ and $(QP_5)_0 \cong \mathbb{F}_2$. So, $\dim((QP_5)_0) = \dim(\mathbb{F}_2) = 1$. We now determine $\text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)})$. Put

$$\omega_{(5,t)} = (3, 3, \dots, 3, 1), \text{ and } \omega_{(5,t)}^* = (3, 3, \dots, 3) \text{ (} t \text{ times of 3)}.$$

Then, an important key in the proof of the theorem is the following lemma.

Lemma 3.2.2. *If x is an admissible monomial of degree $3(2^t - 1) + 2^t$ in P_5 and $[x]$ belongs to $\text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)})$ then $\omega(x) = \omega_{(5,t)}$.*

Proof. We prove the lemma by induction on t . By Sum [46, 50], we see that the lemma holds for $t = 1$.

Suppose $t > 1$ and the lemma holds for $1, 2, \dots, t-1$. Observe that the monomial $z = x_1^{2^{t+1}-1} x_2^{2^t-1} x_3^{2^t-1}$ is the minimal spike of degree $3(2^t - 1) + 2^t$ in P_5 and $\omega(z) = \omega_{(5,t)}$.

Since $3(2^t - 1) + 2^t$ is odd and $[x] \neq [0]$, one gets either $\omega_1(x) = 3$ or $\omega_1(x) = 5$. If $\omega_1(x) = 5$, then $x = X_\emptyset y^2$ with y a monomial of degree $2^{t+1} - 4$ in P_5 . Since x is admissible, by Theorem 2.2.5, y is admissible. Hence, $(\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)}([x]) = [y] \neq [0]$. This contradicts the fact that $[x] \in \text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)})$, so $\omega_1(x) = 3$. Then, we have $x = X_{\{i,j\}} u^2$ with $1 \leq i < j \leq 5$ and u an admissible monomial of degree $3(2^{t-1} - 1) + 2^{t-1}$ in P_5 . Now, the lemma follows from the inductive hypothesis. $\square$

Following Lemma 3.2.2, it may be concluded that

$$\text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)}) = (QP_5^0)_{3(2^t-1)+2^t} \oplus \mathbb{V},$$

where $\mathbb{V} = (\text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)})) \cap (QP_5^+)_{3(2^t-1)+2^t}$ is isomorphic to $QP_5^+(\omega_{(5,t)})$.

The case $t = 1$. Following [46, 50], we have $|\mathcal{B}_5^0(5)| = |\mathcal{B}_5^0(\omega_{(5,1)})| = 45$ and $|\mathcal{B}_5^+(\omega_{(5,1)})| = 0$. This together with Theorem 3.2.1 imply that $(QP_5)_5$ is 46-dimensional.

The case $t = 2$. It should be noted that for each $s \geq 1$, Moetele and Mothebe indicated in [20] that $\dim(QP_s)_{13} = \sum_{3 \leq j \leq 13} \binom{d}{j} C_j$, where C_j , $3 \leq j \leq 13$, are determined by the following table:

j	3	4	5	6	7	8	9	10	11	12	13
C_j	3	23	105	268	415	438	322	164	55	11	1

This implies that $(QP_5)_{13}$ has dimension 250. Their methods are based on Kameko's homomorphism [12] and a result on the admissible monomials [20, Theorem 2.7]. For this case, we will prove it in another way. More precisely, to compute $(QP_5)_{13}$, we determine explicitly all the admissible monomials of degree 13 in P_5 by using Lemma 2.2.9 and some homomorphisms in Subsection 2.2. Detailed calculations are performed as follows. By Sum [47], we have $|\mathcal{B}_4(13)| = |\mathcal{B}_4(\omega_{(5,2)})| = 35$. A simple computation shows that $\mathcal{B}_5^0(13) = \mathcal{B}_5^0(\omega_{(5,2)}) = \Phi^0(\mathcal{B}_4(13))$ is the set consisting of 145 admissible monomials, namely $q_{2,k}$, for $1 \leq k \leq 145$ (see Subsection 5.1), i.e., $(QP_5^0)_{13}$ is 145-dimensional. Next, we need to calculate $QP_5^+(\omega_{(5,2)})$. It is not difficult to show that $\Phi^+(\mathcal{B}_4(\omega_{(5,2)})) \cup \{x_1 x_2^2 x_3^4 x_4^3 x_5^3, x_1^3 x_2^4 x_3^3 x_4 x_5^2, x_1^3 x_2^4 x_3 x_4^3 x_5^2, x_1^3 x_2^4 x_3 x_4^2 x_5^3\}$ is the set consisting of 60 admissible monomials, namely $b_{13,k}$, for $1 \leq k \leq 60$ (see Subsection 5.1). The following assertion is of key importance in the proof.

Proposition 3.2.3. *The set $\{[b_{13,k}]_{\omega_{(5,2)}}, 1 \leq k \leq 60\}$ is a basis of $\mathbb{F}_2$ -vector space $QP_5^+(\omega_{(5,2)})$.*

We need some lemmas for the proof of the proposition. It is rather straightforward to prove the following directly, so we omit the details by leaving them to the interested reader for a direct check.

Lemma 3.2.4. *The following monomials are strictly inadmissible:*

- (i) $x_i^2 x_j^2 x_k x_\ell x_m^3$, $x_i^2 x_j x_k^2 x_\ell x_m^3$, $x_i^2 x_j x_k x_\ell^2 x_m^3$, $i < j < k < \ell$. Here (i, j, k, ℓ, m) is a permutation of $(1, 2, 3, 4, 5)$.
- (ii) $\rho_i(u)$, $1 \leq i \leq 5$, where u is one of the following monomials:

$$x_1^2 x_2 x_3^3 x_4^3, x_1^2 x_2^3 x_3 x_4^3, x_1^2 x_2^3 x_3^3 x_4, x_1^3 x_2^2 x_3 x_4^3, x_1^3 x_2^2 x_3^3 x_4, x_1^3 x_2^3 x_3^2 x_4, x_1^3 x_2^4 x_3^3 x_4.$$

The next lemma is important.

Lemma 3.2.5. *The following monomials are strictly inadmissible:*

$$\begin{aligned} & x_1 x_2^2 x_3^2 x_4 x_5^7, \quad x_1 x_2^2 x_3^2 x_4^3 x_5^5, \quad x_1 x_2^2 x_3^2 x_4^5 x_5^3, \quad x_1 x_2^2 x_3^2 x_4^7 x_5, \quad x_1 x_2^2 x_3^3 x_4^2 x_5^5, \\ & x_1 x_2^2 x_3^3 x_4^6 x_5, \quad x_1 x_2^2 x_3^6 x_4 x_5^3, \quad x_1 x_2^2 x_3^6 x_4^3 x_5, \quad x_1 x_2^2 x_3^7 x_4^2 x_5, \quad x_1 x_2^3 x_3^2 x_4^2 x_5^5, \\ & x_1 x_2^3 x_3^2 x_4^6 x_5, \quad x_1 x_2^3 x_3^6 x_4^2 x_5, \quad x_1 x_2^6 x_3^2 x_4 x_5^3, \quad x_1 x_2^6 x_3^3 x_4^2 x_5, \quad x_1 x_2^6 x_3^3 x_4^2 x_5, \\ & x_1 x_2^7 x_3^2 x_4 x_5, \quad x_1^3 x_2 x_3^2 x_4^2 x_5^5, \quad x_1^3 x_2 x_3^2 x_4^6 x_5, \quad x_1^3 x_2 x_3^6 x_4^2 x_5, \quad x_1^3 x_2^5 x_3^2 x_4^2 x_5, \\ & x_1^7 x_2 x_3^2 x_4 x_5. \end{aligned}$$

Proof. We prove the lemma for $x = x_1 x_2^2 x_3^2 x_4^3 x_5^5$. The others can be proven by a similar computation. By a direct computation using the Cartan formula, we have

$$\begin{aligned} x &= x_1 x_2 x_3^2 x_4^3 x_5^6 + x_1 x_2^2 x_3 x_4^3 x_5^6 + Sq^1(x_1 x_2 x_3^2 x_4^3 x_5^5 + x_1 x_2 x_3^2 x_4^5 x_5^3 + x_1 x_2 x_3^4 x_4^3 x_5^3 + x_1 x_2^2 x_3 x_4^3 x_5^5 \\ &\quad + x_1 x_2^2 x_3 x_4^5 x_5^3 + x_1 x_2^4 x_3 x_4^3 x_5^3 + x_1^2 x_2 x_3 x_4^5 x_5^3 + x_1^4 x_2 x_3 x_4^3 x_5^3) \\ &\quad + Sq^2(x_1 x_2 x_3 x_4^3 x_5^5 + x_1 x_2^2 x_3^2 x_4^3 x_5^3 + x_1^2 x_2 x_3^2 x_4^3 x_5^3 + x_1^2 x_2^2 x_3 x_4^3 x_5^3) \\ &\quad + Sq^4(x_1 x_2 x_3 x_4^3 x_5^3) \pmod{(P_5^-(\omega_{(5,2)}))}. \end{aligned}$$

This equality implies that x is strictly inadmissible. $\square$

Proof of Proposition 3.2.3. Let x be an admissible monomial of degree 13 in P_5 and $\omega(x) = \omega_{(5,2)}$. Then, $x = X_{\{i,j\}} u^2$ with $1 \leq i < j \leq 5$ and u a monomial of degree 5 in P_5 . Since x is admissible, according to Theorem 2.2.5, we get $u \in \mathcal{B}_5(\omega_{(5,1)})$.

By a direct computation, we see that for all $y \in \mathcal{B}_5(\omega_{(5,1)})$, such that $X_{\{i,j\}} y^2 \neq b_{13,k}$, $\forall k$, $1 \leq k \leq 60$, there is a monomial w which is given in one of Lemmas 3.2.4, and 3.2.5 such that $X_{\{i,j\}} y^2 = wz^{2^r}$ with a monomial $z \in P_5$ and $r = \max\{m \in \mathbb{Z} : \omega_m(w) > 0\}$. By Theorem 2.2.5, $X_{\{i,j\}} y^2$ is inadmissible. Since $x = X_{\{i,j\}} u^2$ is admissible, one gets $x = b_{13,k}$ for suitable k .

We now prove the set $\{[b_{13,k}]_{\omega_{(5,2)}} : 1 \leq k \leq 60\}$ is linearly independent in $(QP_5)_{13}$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq k \leq 60} \gamma_k b_{13,k} \equiv_{\omega_{(5,2)}} 0, \quad (3.2.1)$$

where $\gamma_k \in \mathbb{F}_2$.

From a result in [47], $\dim QP_4^+(\omega_{(5,2)}) = 23$, with the basis $\{[u_i]_{\omega_{(5,2)}} : 1 \leq i \leq 23\}$, where

$$\begin{aligned} & u_1. x_1 x_2^2 x_3^3 x_4^7, \quad u_2. x_1 x_2^2 x_3^7 x_4^3, \quad u_3. x_1 x_2^3 x_3^2 x_4^7, \quad u_4. x_1 x_2^3 x_3^3 x_4^6, \quad u_5. x_1 x_2^3 x_3^6 x_4^3, \\ & u_6. x_1 x_2^3 x_3^7 x_4^2, \quad u_7. x_1 x_2^6 x_3^3 x_4^3, \quad u_8. x_1 x_2^7 x_3^2 x_4^3, \quad u_9. x_1 x_2^7 x_3^3 x_4^2, \quad u_{10}. x_1^3 x_2^2 x_3^2 x_4^7, \\ & u_{11}. x_1^3 x_2^2 x_3^3 x_4^6, \quad u_{12}. x_1^3 x_2^2 x_3^6 x_4^3, \quad u_{13}. x_1^3 x_2^2 x_3^7 x_4^2, \quad u_{14}. x_1^3 x_2^3 x_3^6 x_4^4, \quad u_{15}. x_1^3 x_2^3 x_3^3 x_4^4, \\ & u_{16}. x_1^3 x_2^3 x_3^4 x_4^3, \quad u_{17}. x_1^3 x_2^3 x_3^5 x_4^2, \quad u_{18}. x_1^3 x_2^5 x_3^2 x_4^3, \quad u_{19}. x_1^3 x_2^5 x_3^3 x_4^2, \quad u_{20}. x_1^3 x_2^7 x_3^3 x_4^2, \\ & u_{21}. x_1^7 x_2 x_3^3 x_4^3, \quad u_{22}. x_1^7 x_2 x_3^3 x_4^2, \quad u_{23}. x_1^7 x_2 x_3^3 x_4^2. \end{aligned}$$

We explicitly compute $p_{(i,l)}(\mathcal{S})$ in terms u_j , $1 \leq j \leq 23$. By a direct computation using Lemma 2.2.9, Theorem 3.1.3, and from the relations $p_{(i,j)}(\mathcal{S}) \equiv_{\omega_{(5,2)}} 0$ with either $i = 1, j = 2, 3$ or $i = 2, j = 3, 4$, one gets

$$\begin{cases} \gamma_k = 0, \quad k \in \mathbb{J}, \\ \gamma_{27} = \gamma_{55}, \quad \gamma_{14} = \gamma_{49} = \gamma_{58}, \quad \gamma_{13} = \gamma_{20} = \gamma_{43} = \gamma_{59}, \\ \gamma_{30} + \gamma_{35} + \gamma_{58} = 0, \quad \gamma_{44} + \gamma_{55} + \gamma_{56} + \gamma_{58} = \gamma_{47} + \gamma_{53} + \gamma_{54} + \gamma_{59} = 0. \end{cases}$$

Here $\mathbb{J} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 28, 29, 31, 38, 39, 40, 41, 42, 45, 46, 48, 50, 51, 57, 60\}$. Then, the relation (3.2.1) becomes

$$\mathcal{S} = \sum_{k \in \mathbb{N}_{60} \setminus \mathbb{J}} \gamma_k b_{13,k} \equiv_{\omega_{(5,2)}} 0. \quad (3.2.2)$$

Applying the homomorphisms $p_{(1;4)}, p_{(3;4)}, p_{(3;5)}, p_{(4;5)} : P_5 \rightarrow P_4$ to (3.2.2), we obtain $\gamma_k = 0$, $1 \leq k \leq 60$. This finishes the proof. $\square$

Since $\dim(QP_5^0)_{13} = 145$, from Theorem 3.2.1 and Proposition 3.2.3, one gets $\dim(QP_5)_{13} = 250$.

For $t \geq 3$, we have $|\mathcal{B}_4(3(2^t - 1) + 2^t)| = 45$ (see Sum [47]). By a simple computation, we get

$$\mathcal{B}_5^0(3(2^t - 1) + 2^t) = \mathcal{B}_5^0(\omega_{(5,t)}) = \Phi^0(\mathcal{B}_4(3(2^t - 1) + 2^t)) = \{q_{t,k} : 1 \leq k \leq 195\},$$

for all $t \geq 3$. Here, the monomials $q_{t,k}$, $1 \leq k \leq 195$, are listed in Subsection 5.1.

We now compute $QP_5^+(\omega_{(5,t)})$ for $t \geq 3$.

The case $t = 3$. We obtain the following.

Proposition 3.2.6. $QP_5^+(\omega_{(5,3)})$ is the $\mathbb{F}_2$ -vector space of dimension 260 with a basis consisting of all the classes represented by the monomials $b_{29,k}$, $1 \leq k \leq 260$, which are determined as in Subsection 5.2.

To prove the proposition, we need some results. The following is a corollary of a result in [47].

Lemma 3.2.7. If $\bar{x}$ is one of the following monomials then $\rho_i(\bar{x})$, $1 \leq i \leq 5$ are strictly inadmissible:

$$\begin{array}{cccccc} x_1^6 x_2 x_3^7 x_4^7, & x_1^6 x_2^7 x_3 x_4^7, & x_1^6 x_2^7 x_3^7 x_4, & x_1^7 x_2^6 x_3 x_4^7, & x_1^7 x_2^6 x_3^7 x_4, & x_1^7 x_2^7 x_3^6 x_4, \\ x_1^2 x_2^5 x_3^7 x_4^7, & x_1^2 x_2^7 x_3^5 x_4^7, & x_1^2 x_2^7 x_3^7 x_4^5, & x_1^7 x_2^2 x_3^5 x_4^7, & x_1^7 x_2^2 x_3^7 x_4^5, & x_1^7 x_2^7 x_3^2 x_4^5, \\ x_1^3 x_2^4 x_3^7 x_4^7, & x_1^3 x_2^7 x_3^4 x_4^7, & x_1^3 x_2^7 x_3^7 x_4^4, & x_1^7 x_2^3 x_3^4 x_4^7, & x_1^7 x_2^3 x_3^7 x_4^4, & x_1^7 x_2^7 x_3^3 x_4^4, \\ x_1^3 x_2^6 x_3^5 x_4^7, & x_1^3 x_2^7 x_3^6 x_4^5, & x_1^3 x_2^7 x_3^7 x_4^5, & x_1^6 x_2^3 x_3^5 x_4^7, & x_1^6 x_2^3 x_3^7 x_4^5, & x_1^6 x_2^7 x_3^3 x_4^5, \\ x_1^7 x_2^3 x_3^6 x_4^5, & x_1^7 x_2^6 x_3^3 x_4^5, & x_1^7 x_2^7 x_3^3 x_4^5. \end{array}$$

Lemma 3.2.8. The following monomials are strictly inadmissible:

(i) $x_k^2 x_\ell x_m^5 x_t^7 x_u^7$, $k < \ell$, $x_k^2 x_\ell x_m^4 x_t^7 x_u^7$, $x_k^2 x_\ell x_m^7 x_t^7 x_u^7$, $x_k^6 x_\ell x_m^2 x_t^5 x_u^7$, $x_k^6 x_\ell x_m^4 x_t^7 x_u^7$, $k < \ell < m$. Here (k, ℓ, m, t, u) is a permutation of $(1, 2, 3, 4, 5)$.

(ii) $x_1 x_2^3 x_3^6 x_4^6 x_5^5$, $x_1 x_2^6 x_3^3 x_4^6 x_5^5$, $x_1 x_2^6 x_3^6 x_4^3 x_5^5$, $x_1^3 x_2 x_3^6 x_4^6 x_5^5$,
 $x_1^3 x_2^5 x_3^6 x_4^6 x_5^5$, $x_1^3 x_2^6 x_3^3 x_4^6 x_5^5$, $x_1^3 x_2^6 x_3^6 x_4^3 x_5^5$, $x_1^3 x_2^6 x_3^6 x_4^5 x_5^5$.

Proof. The proof of the lemma is straightforward. $\square$

Lemma 3.2.9. If (i, j, k, ℓ, m) is a permutation of $(1, 2, 3, 4, 5)$ such that $i < j < k < \ell$ then the following monomials are strictly inadmissible:

$$\begin{array}{ccccc} x_i x_j^6 x_k^6 x_\ell x_m^7, & x_i^6 x_j x_k x_\ell^6 x_m^7, & x_i^6 x_j x_k^6 x_\ell x_m^7, & x_i^6 x_j^6 x_k x_\ell x_m^7, & x_i x_j^2 x_k^6 x_\ell^5 x_m^7, \\ x_i x_j^6 x_k^2 x_\ell^5 x_m^7, & x_i^3 x_j^4 x_k x_\ell^6 x_m^7, & x_i^3 x_j^4 x_k^6 x_\ell x_m^7, & x_i^3 x_j^6 x_k^4 x_\ell x_m^7, & x_i^3 x_j^6 x_k x_\ell^4 x_m^7, \\ x_i^6 x_j x_k^3 x_\ell^5 x_m^6, & x_i^6 x_j^3 x_k x_\ell^5 x_m^6, & x_i^6 x_j^3 x_k^5 x_\ell x_m^6, & x_i^2 x_j^2 x_k^5 x_\ell^5 x_m^7, & x_i^2 x_j^5 x_k^2 x_\ell^5 x_m^7, \\ x_i^3 x_j^4 x_k^3 x_\ell^4 x_m^7, & x_i^2 x_j^3 x_k^4 x_\ell^5 x_m^7, & x_i^2 x_j^3 x_k^5 x_\ell^4 x_m^7, & x_i^2 x_j^4 x_k^3 x_\ell^5 x_m^7, & x_i^2 x_j^5 x_k^3 x_\ell^4 x_m^7, \\ x_i^3 x_j^2 x_k^4 x_\ell^5 x_m^7, & x_i^3 x_j^2 x_k^5 x_\ell^4 x_m^7, & x_i^3 x_j^4 x_k^2 x_\ell^5 x_m^7, & x_i^2 x_j^3 x_k^5 x_\ell^5 x_m^6, & x_i^2 x_j^5 x_k^3 x_\ell^5 x_m^6, \\ x_i^3 x_j^2 x_k^5 x_\ell^5 x_m^6, & x_i^3 x_j^5 x_k^2 x_\ell^5 x_m^6, & x_i^3 x_j^5 x_k^4 x_\ell^5 x_m^6, & x_i^3 x_j^5 x_k^5 x_\ell^4 x_m^6, & x_i^3 x_j^5 x_k^5 x_\ell^5 x_m^6, \\ x_i^3 x_j^5 x_k^3 x_\ell^4 x_m^6. \end{array}$$

Proof. We prove the lemma for the monomial $x_i x_j^6 x_k^6 x_\ell x_m^7$. The others can be proved by a similar computation. By using the Cartan formula, we have

$$\begin{aligned} x_i x_j^6 x_k^6 x_\ell x_m^7 &= Sq^1(x_i^2 x_j^5 x_k^5 x_\ell x_m^7) + Sq^2(x_i x_j^5 x_k^5 x_\ell x_m^7 + x_i x_j^3 x_k^5 x_\ell x_m^9) \\ &\quad + x_i x_j^6 x_k^5 x_\ell^2 x_m^7 + x_i x_j^5 x_k^6 x_\ell^2 x_m^7 \pmod{(P_5^-(\omega_{(5,3)}^*))}. \end{aligned}$$

This relation implies that $x_i x_j^6 x_k^6 x_\ell x_m^7$ is strictly inadmissible. The lemma is proven. $\square$

Lemma 3.2.10. *The following monomials are strictly inadmissible:*

$$\begin{array}{ccccc} x_1 x_2^6 x_3^7 x_4^8 x_5^7, & x_1 x_2^7 x_3^6 x_4^8 x_5^7, & x_1 x_2^7 x_3^{10} x_4^4 x_5^7, & x_1 x_2^7 x_3^{10} x_4^7 x_5^4, & x_1^3 x_2^3 x_3^4 x_4^{12} x_5^7, \\ x_1^3 x_2^3 x_3^{12} x_4^4 x_5^7, & x_1^3 x_2^3 x_3^{12} x_4^7 x_5^4, & x_1^3 x_2^5 x_3^6 x_4^3 x_5^{12}, & x_1^3 x_2^5 x_3^6 x_4^{11} x_5^4, & x_1^3 x_2^5 x_3^8 x_4^6 x_5^7, \\ x_1^3 x_2^5 x_3^8 x_4^7 x_5^6, & x_1^3 x_2^5 x_3^9 x_4^6 x_5^6, & x_1^3 x_2^5 x_3^{14} x_4^3 x_5^4, & x_1^3 x_2^{12} x_3^3 x_4^5 x_5^6, & x_1^3 x_2^{13} x_3^6 x_4^3 x_5^4, \\ x_1^7 x_2 x_3^6 x_4^8 x_5^7, & x_1^7 x_2 x_3^{10} x_4^4 x_5^7, & x_1^7 x_2 x_3^{10} x_4^7 x_5^4, & x_1^7 x_2^8 x_3^3 x_4^5 x_5^6, & x_1^7 x_2^9 x_3^2 x_4^4 x_5^7, \\ x_1^7 x_2^9 x_3^2 x_4^7 x_5^4, & x_1^7 x_2^9 x_3^2 x_4^4 x_5^4. & & & \end{array}$$

Proof. We prove the lemma for the monomial $x = x_1^3 x_2^5 x_3^6 x_4^3 x_5^{12}$. The others can be proved by a similar computation. By a direct computation, we have

$$\begin{aligned} x &= x_1^2 x_2^3 x_3^5 x_4^6 x_5^{13} + x_1^2 x_2^3 x_3^5 x_4^{12} x_5^7 + x_1^2 x_2^5 x_3^5 x_4^6 x_5^{11} + x_1^2 x_2^5 x_3^5 x_4^{10} x_5^7 + x_1^2 x_2^5 x_3^9 x_4^6 x_5^7 \\ &\quad + x_1^2 x_2^{11} x_3^4 x_4^5 x_5^7 + x_1^2 x_2^{11} x_3^5 x_4^4 x_5^7 + x_1^2 x_2^{13} x_3^3 x_4^4 x_5^7 + x_1^2 x_2^{13} x_3^4 x_4^3 x_5^7 + x_1^3 x_2^3 x_3^4 x_4^6 x_5^{13} \\ &\quad + x_1^3 x_2^3 x_3^4 x_4^{12} x_5^7 + x_1^3 x_2^3 x_3^5 x_4^6 x_5^{12} + x_1^3 x_2^3 x_3^6 x_4^5 x_5^{12} + x_1^3 x_2^3 x_3^{12} x_4^4 x_5^7 + x_1^3 x_2^4 x_3^3 x_4^6 x_5^{13} \\ &\quad + x_1^3 x_2^4 x_3^3 x_4^{12} x_5^7 + x_1^3 x_2^4 x_3^5 x_4^6 x_5^{11} + x_1^3 x_2^4 x_3^5 x_4^{10} x_5^7 + x_1^3 x_2^4 x_3^6 x_4^3 x_5^{13} + x_1^3 x_2^4 x_3^6 x_4^9 x_5^7 \\ &\quad + x_1^3 x_2^4 x_3^{12} x_4^3 x_5^7 + x_1^3 x_2^5 x_3^2 x_4^5 x_5^{14} + x_1^3 x_2^5 x_3^2 x_4^6 x_5^{13} + x_1^3 x_2^5 x_3^3 x_4^4 x_5^{14} + x_1^3 x_2^5 x_3^5 x_4^6 x_5^{10} \\ &\quad + x_1^3 x_2^5 x_3^6 x_4^2 x_5^{13} + x_1^5 x_2^5 x_3^5 x_4^5 x_5^9 + Sq^1(h_1) + Sq^2(h_2) + Sq^4(h_4) \pmod{(P_5^-(\omega_{(5,3)}))}, \text{ where} \\ h_1 &= x_1^3 x_2^3 x_3^5 x_4^6 x_5^{11} + x_1^3 x_2^3 x_3^5 x_4^{10} x_5^7 + x_1^3 x_2^3 x_3^9 x_4^2 x_5^{11} + x_1^3 x_2^3 x_3^9 x_4^3 x_5^{10} + x_1^3 x_2^3 x_3^9 x_4^6 x_5^7 \\ &\quad + x_1^3 x_2^5 x_3^2 x_4^5 x_5^{13} + x_1^3 x_2^5 x_3^3 x_4^4 x_5^{13} + x_1^3 x_2^5 x_3^4 x_4^5 x_5^{11} + x_1^3 x_2^5 x_3^4 x_4^9 x_5^7 + x_1^3 x_2^5 x_3^9 x_4^4 x_5^7 \\ &\quad + x_1^3 x_2^6 x_3^5 x_4^5 x_5^9 + x_1^3 x_2^9 x_3^2 x_4^3 x_5^{11} + x_1^3 x_2^{11} x_3^3 x_4^4 x_5^7 + x_1^3 x_2^{11} x_3^4 x_4^3 x_5^7 + x_1^{10} x_2^2 x_3^3 x_4^3 x_5^9 \\ &\quad + x_1^{10} x_2^3 x_3^3 x_4^5 x_5^7 + x_1^{10} x_2^3 x_3^5 x_4^3 x_5^7 + x_1^{10} x_2^5 x_3^3 x_4^3 x_5^7, \\ h_2 &= x_1^2 x_2^3 x_3^5 x_4^6 x_5^{11} + x_1^2 x_2^3 x_3^5 x_4^{10} x_5^7 + x_1^2 x_2^3 x_3^9 x_4^6 x_5^7 + x_1^2 x_2^{11} x_3^3 x_4^4 x_5^7 + x_1^2 x_2^{11} x_3^4 x_4^3 x_5^7 \\ &\quad + x_1^3 x_2^3 x_3^4 x_4^6 x_5^{11} + x_1^3 x_2^3 x_3^4 x_4^{10} x_5^7 + x_1^3 x_2^3 x_3^6 x_4^4 x_5^{11} + x_1^3 x_2^3 x_3^6 x_4^8 x_5^7 + x_1^3 x_2^3 x_3^8 x_4^6 x_5^7 \\ &\quad + x_1^3 x_2^3 x_3^{10} x_4^4 x_5^7 + x_1^3 x_2^4 x_3^3 x_4^6 x_5^{11} + x_1^3 x_2^4 x_3^3 x_4^{10} x_5^7 + x_1^3 x_2^4 x_3^6 x_4^3 x_5^{11} + x_1^3 x_2^4 x_3^{10} x_4^3 x_5^7 \\ &\quad + x_1^3 x_2^5 x_3^5 x_4^5 x_5^9 + x_1^3 x_2^6 x_3^2 x_4^5 x_5^{11} + x_1^3 x_2^6 x_3^2 x_4^9 x_5^7 + x_1^3 x_2^6 x_3^3 x_4^4 x_5^{11} + x_1^3 x_2^6 x_3^3 x_4^8 x_5^7 \\ &\quad + x_1^3 x_2^8 x_3^3 x_4^6 x_5^7 + x_1^3 x_2^8 x_3^6 x_4^3 x_5^7 + x_1^3 x_2^{10} x_3^2 x_4^5 x_5^7 + x_1^3 x_2^{10} x_3^3 x_4^4 x_5^7 + x_1^5 x_2^3 x_3^6 x_4^2 x_5^{11} \\ &\quad + x_1^5 x_2^3 x_3^6 x_4^3 x_5^{10} + x_1^5 x_2^3 x_3^6 x_4^6 x_5^7 + x_1^5 x_2^{10} x_3^2 x_4^3 x_5^7 + x_1^9 x_2^3 x_3^3 x_4^3 x_5^9 + x_1^9 x_2^3 x_3^3 x_4^5 x_5^7 \\ &\quad + x_1^9 x_2^3 x_3^5 x_4^3 x_5^7 + x_1^9 x_2^5 x_3^3 x_4^3 x_5^7 + x_1^9 x_2^6 x_3^2 x_4^3 x_5^7, \\ h_4 &= x_1^3 x_2^3 x_3^6 x_4^2 x_5^{11} + x_1^3 x_2^3 x_3^6 x_4^3 x_5^{10} + x_1^3 x_2^3 x_3^6 x_4^6 x_5^7 + x_1^3 x_2^4 x_3^5 x_4^6 x_5^7 + x_1^3 x_2^4 x_3^6 x_4^5 x_5^7 \\ &\quad + x_1^3 x_2^6 x_3^5 x_4^4 x_5^7 + x_1^3 x_2^{10} x_3^2 x_4^3 x_5^7 + x_1^5 x_2^3 x_3^4 x_4^6 x_5^7 + x_1^5 x_2^3 x_3^6 x_4^4 x_5^7 + x_1^5 x_2^4 x_3^3 x_4^6 x_5^7 \\ &\quad + x_1^5 x_2^4 x_3^6 x_4^3 x_5^7 + x_1^5 x_2^6 x_3^2 x_4^5 x_5^7 + x_1^5 x_2^6 x_3^3 x_4^4 x_5^7. \end{aligned}$$

Now, let $g = x_1 x_2 x_3 x_4 x_5^2$ and $h = x_1 x_2 x_3 x_4 x_5$. Then, we have

$$u := x_1^5 x_2^5 x_3^5 x_4^5 x_5^9 = h \cdot g^4 = h Sq^{12}(g^2).$$

On the other hand, by the formula χ -trick (see [54]), one gets

$$u = g^2 \chi(Sq^{12})(h) + \sum_{1 \leq i \leq 12} Sq^i(g^2 \chi(Sq^{12-i})(h)).$$

Since $\chi(Sq^{12}) = Sq^8Sq^4 + Sq^8Sq^3Sq^1$, by a direct computation, we conclude that

$$g^2\chi(Sq^{12})(h) \in P_5^-(\omega_{(5,3)}).$$

This implies that x is strictly inadmissible. The lemma follows. $\square$

Proof of Proposition 3.2.6. Let x be an admissible monomial in $(P_5^+)_{29}$ such that $\omega(x) = \omega_{(5,3)}$. Then $x = X_{\{i,j\}}y^2$ with $y \in \mathcal{B}_5(\omega_{(5,2)})$, and $1 \leq i < j \leq 5$.

Let $u \in \mathcal{B}_5(\omega_{(5,2)})$ such that $X_{\{i,j\}}u^2 \in P_5^+$. By a direct computation using Proposition 3.2.3, we see that if $X_{\{i,j\}}u^2 \neq b_{29,k}$, $\forall t$, $1 \leq k \leq 260$, then there is a monomial w which is given in one of Lemmas 3.2.7, 3.2.8, 3.2.9 and 3.2.10 such that $X_{\{i,j\}}u^2 = wu_1^{2^\ell}$ with suitable monomial $u_1 \in P_5$, and $\ell = \max\{r \in \mathbb{Z} : \omega_r(w) > 0\}$. By Theorem 2.2.5, $X_{\{i,j\}}u^2$ is inadmissible. Since $x = X_{\{i,j\}}y^2$ with $y \in \mathcal{B}_5(\omega_{(5,2)})$, and x is admissible, one can see that $x = b_{29,k}$ for some k , $1 \leq k \leq 260$. This implies $\mathcal{B}_5^+(\omega_{(5,3)}) \subseteq \{b_{29,k} : 1 \leq k \leq 260\}$.

Now, we prove the set $\{[b_{29,k}]_{\omega_{(5,3)}} : 1 \leq k \leq 260\}$ is linearly independent in $(QP_5)_{29}$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq k \leq 260} \gamma_k b_{29,k} \equiv_{\omega_{(5,3)}} 0,$$

where $\gamma_k \in \mathbb{F}_2$ for all k , $1 \leq k \leq 260$.

From Theorem 3.1.3, for $(i; I) \in \mathcal{N}_5$, we explicitly compute $p_{(i;I)}(\mathcal{S})$ in terms of a given minimal set of $\mathcal{A}$ -generators in $P_4 \pmod{(\mathcal{A}^+ \cdot P_4)}$. Based on the relations $p_{(i;I)}(\mathcal{S}) \equiv_{\omega_{(5,3)}} 0$, $\forall (i; I) \in \mathcal{N}_5$, $\ell(I) > 0$, and some other simple computations, we get $\gamma_k = 0$, $k = 1, 2, \dots, 260$. The proof is completed. $\square$

Recall that $\dim(QP_5^0)_{29} = 195$. Combining this with Theorem 3.2.1 and Proposition 3.2.6, we obtain $\dim(QP_5)_{29} = 645$.

The case $t \geq 4$. We have the following.

Proposition 3.2.11. *There exist exactly 270 admissible monomials in P_5^+ such that their weight vectors are $\omega_{(5,t)}$, for every $t \geq 4$. Consequently $\dim(QP_5^+(\omega_{(5,t)})) = 270$ for $t \geq 4$.*

We prove the proposition by showing that

$$\mathcal{B}_5^+(\omega_{(5,t)}) = \{b_{3(2^t-1)+2^t,k} : 1 \leq k \leq 270\},$$

for all $t \geq 4$. Here, the monomials $b_{3(2^t-1)+2^t,k}$, $1 \leq k \leq 270$, are listed in Subsection 5.2.

Lemma 3.2.12. *If (i, j, k, ℓ, m) is a permutation of $(1, 2, 3, 4, 5)$ such that $i < j < k < \ell$ then the following monomials are strictly inadmissible:*

$$\begin{array}{cccc} x_i^6 x_j x_k x_\ell^6 x_m^7, & x_i x_j^2 x_k^6 x_\ell^5 x_m^7, & x_i x_j^6 x_k^2 x_\ell^5 x_m^7, & x_i^6 x_j x_k^2 x_\ell^5 x_m^7, \\ x_i x_j^6 x_k^3 x_\ell^4 x_m^7, & x_i^3 x_j^6 x_k x_\ell^4 x_m^7, & x_i^6 x_j x_k^3 x_\ell^4 x_m^7, & x_i^6 x_j^3 x_k x_\ell^4 x_m^7, \\ x_i^6 x_j x_k^3 x_\ell^5 x_m^6, & x_i^6 x_j^3 x_k^5 x_\ell x_m^6, & x_i^6 x_j^3 x_k x_\ell^5 x_m^6, & x_i^2 x_j^2 x_k^5 x_\ell^5 x_m^7, \\ x_i^2 x_j^5 x_k^2 x_\ell^5 x_m^7, & x_i^2 x_j^3 x_k^4 x_\ell^5 x_m^7, & x_i^2 x_j^3 x_k^5 x_\ell^4 x_m^7, & x_i^2 x_j^4 x_k^3 x_\ell^5 x_m^7, \\ x_i^2 x_j^5 x_k^3 x_\ell^4 x_m^7, & x_i^3 x_j^2 x_k^4 x_\ell^5 x_m^7, & x_i^3 x_j^2 x_k^5 x_\ell^4 x_m^7, & x_i^3 x_j^4 x_k^2 x_\ell^5 x_m^7, \\ x_i^2 x_j^5 x_k^3 x_\ell^5 x_m^6, & x_i^3 x_j^2 x_k^5 x_\ell^5 x_m^6, & x_i^3 x_j^4 x_k^3 x_\ell^4 x_m^7, & x_i x_j^4 x_k^{10} x_\ell^{12} x_m^{15}, \\ x_i^7 x_j x_k^{10} x_\ell^{12} x_m^{15}, & x_i^3 x_j^3 x_k^{12} x_\ell^{12} x_m^{15}, & x_i^3 x_j^5 x_k^8 x_\ell^{14} x_m^{15}, & x_i^3 x_j^5 x_k^{14} x_\ell^8 x_m^{15}, \\ x_i^7 x_j^7 x_k^8 x_\ell^8 x_m^{15}. & & & \end{array}$$

Proof. We prove the lemma for the monomial $x = x_i^7 x_j^7 x_k^8 x_\ell^8 x_m^{15}$. The others can be proved by a similar computation.

Applying the Cartan formula, we have

$$\begin{aligned}
x &= x_i^7 x_j^5 x_k^{10} x_\ell^8 x_m^{15} + x_i^5 x_j^{11} x_k^6 x_\ell^8 x_m^{15} + x_i^5 x_j^7 x_k^{10} x_\ell^8 x_m^{15} \\
&\quad + x_i^5 x_j^{11} x_k^{10} x_\ell^4 x_m^{15} + x_i^5 x_j^7 x_k^{10} x_\ell^8 x_m^{15} + x_i^5 x_j^{10} x_k^{11} x_\ell^4 x_m^{15} \\
&\quad + x_i^5 x_j^6 x_k^{11} x_\ell^8 x_m^{15} + x_i^5 x_j^3 x_k^{14} x_\ell^8 x_m^{15} + x_i^4 x_j^{11} x_k^{11} x_\ell^4 x_m^{15} \\
&\quad + x_i^4 x_j^7 x_k^{11} x_\ell^8 x_m^{15} + Sq^1(x_i^7 x_j^7 x_k^{11} x_\ell^4 x_m^{15} + x_i^7 x_j^5 x_k^{13} x_\ell^4 x_m^{15}) \\
&\quad + Sq^2(y) + Sq^4(z) + Sq^8(x_i^7 x_j^5 x_k^6 x_\ell^4 x_m^{15}) \pmod{P_5^-(\omega_{(5,4)}^*)}, \text{ where} \\
y &= x_i^7 x_j^7 x_k^6 x_\ell^8 x_m^{15} + x_i^7 x_j^3 x_k^6 x_\ell^4 x_m^{23} + x_i^7 x_j^7 x_k^{10} x_\ell^4 x_m^{15} + x_i^7 x_j^6 x_k^{11} x_\ell^4 x_m^{15} + x_i^7 x_j^3 x_k^{14} x_\ell^4 x_m^{15}, \\
z &= x_i^5 x_j^7 x_k^6 x_\ell^8 x_m^{15} + x_i^{11} x_j^5 x_k^6 x_\ell^4 x_m^{15} + x_i^5 x_j^7 x_k^{10} x_\ell^4 x_m^{15} \\
&\quad + x_i^4 x_j^7 x_k^{11} x_\ell^4 x_m^{15} + x_i^5 x_j^6 x_k^{11} x_\ell^4 x_m^{15} + x_i^5 x_j^3 x_k^{14} x_\ell^4 x_m^{15}.
\end{aligned}$$

The above equalities imply that x is strictly inadmissible. $\square$

The following lemma is proved by a direct computation.

Lemma 3.2.13. *The following monomials are strictly inadmissible:*

- (i) $x_i^6 x_j^6 x_k x_l x_m^7$, $x_i^2 x_j x_k^4 x_l^7 x_m^7$, $x_i^2 x_j^4 x_k x_l^7 x_m^7$, $x_i^2 x_j^5 x_k x_l^6 x_m^7$, $x_i^2 x_j x_k^5 x_l^6 x_m^7$, $x_i^3 x_j^4 x_k x_l^6 x_m^7$, whenever (i, j, k, l, m) is a permutation of $(1, 2, 3, 4, 5)$ such that $i < j < k$.
- (ii) $x_1^3 x_2^3 x_3^6 x_4^4 x_5^5$, $x_1^3 x_2^3 x_3^6 x_4^5 x_5^4$, $x_1^3 x_2^4 x_3^3 x_4^6 x_5^5$, $x_1^3 x_2^4 x_3^6 x_4^3 x_5^5$,
 $x_1^3 x_2^5 x_3^3 x_4^4 x_5^5$, $x_1^3 x_2^5 x_3^3 x_4^5 x_5^4$, $x_1^3 x_2^6 x_3^4 x_4^3 x_5^5$, $x_1^3 x_2^6 x_3^5 x_4^3 x_5^4$,
 $x_1^6 x_2^3 x_3^3 x_4^4 x_5^5$, $x_1^6 x_2^3 x_3^3 x_4^5 x_5^4$, $x_1^6 x_2^3 x_3^4 x_4^3 x_5^5$, $x_1^6 x_2^3 x_3^5 x_4^3 x_5^4$,
 $x_1^3 x_2^5 x_3^2 x_4^6 x_5^5$, $x_1^3 x_2^5 x_3^6 x_4^2 x_5^5$, $x_1^3 x_2^6 x_3^5 x_4^2 x_5^5$, $x_1^6 x_2^3 x_3^5 x_4^2 x_5^5$,
 $x_1^3 x_2^3 x_3^4 x_4^6 x_5^5$, $x_1^3 x_2^5 x_3^9 x_4^{14} x_5^{14}$, $x_1^3 x_2^5 x_3^{14} x_4^9 x_5^{14}$.

Lemma 3.2.14. *The following monomials are strictly inadmissible:*

- (i) $\rho_i(f)$, $1 \leq i \leq 5$, where f is one of the following monomials:

$$\begin{aligned}
&x_1^2 x_2^5 x_3^7 x_4^7, \quad x_1^2 x_2^7 x_3^5 x_4^7, \quad x_1^2 x_2^7 x_3^7 x_4^5, \quad x_1^7 x_2^2 x_3^5 x_4^7, \\
&x_1^7 x_2^2 x_3^7 x_4^5, \quad x_1^7 x_2^7 x_3^5 x_4^7, \quad x_1^3 x_2^4 x_3^7 x_4^7, \quad x_1^3 x_2^7 x_3^4 x_4^7, \\
&x_1^3 x_2^7 x_3^7 x_4^4, \quad x_1^7 x_2^3 x_3^7 x_4^7, \quad x_1^7 x_2^3 x_3^7 x_4^4, \quad x_1^7 x_2^7 x_3^3 x_4^4, \\
&x_1^3 x_2^6 x_3^5 x_4^7, \quad x_1^3 x_2^6 x_3^7 x_4^5, \quad x_1^3 x_2^7 x_3^5 x_4^5, \quad x_1^6 x_2^3 x_3^5 x_4^7, \\
&x_1^6 x_2^3 x_3^7 x_4^5, \quad x_1^6 x_2^7 x_3^3 x_4^5, \quad x_1^7 x_2^3 x_3^6 x_4^5, \quad x_1^7 x_2^6 x_3^3 x_4^5, \\
&x_1^6 x_2^7 x_3^7 x_4^4, \quad x_1^6 x_2^7 x_3^7 x_4^7, \quad x_1^6 x_2^7 x_3^7 x_4^4, \quad x_1^7 x_2^6 x_3^7 x_4^7, \\
&x_1^7 x_2^6 x_3^7 x_4^4, \quad x_1^7 x_2^7 x_3^6 x_4^4. \\
(ii) &x_1 x_2^{15} x_3^{15} x_4^{18} x_5^{12}, \quad x_1^{15} x_2 x_3^{15} x_4^{18} x_5^{12}, \quad x_1^{15} x_2^{15} x_3 x_4^{18} x_5^{12}, \quad x_1 x_2^{14} x_3^{15} x_4^{15} x_5^{16}, \\
&x_1 x_2^{15} x_3^{14} x_4^{15} x_5^{16}, \quad x_1^{15} x_2 x_3^{14} x_4^{15} x_5^{16}, \quad x_1 x_2^{15} x_3^{15} x_4^{14} x_5^{16}, \quad x_1^{15} x_2 x_3^{15} x_4^{14} x_5^{16}, \\
&x_1^{15} x_2^{15} x_3 x_4^{14} x_5^{16}, \quad x_1^{15} x_2^{15} x_3^{17} x_4^{12} x_5^{12}, \quad x_1^3 x_2^{13} x_3^{14} x_4^{19} x_5^{12}, \quad x_1^3 x_2^{13} x_3^{15} x_4^{18} x_5^{12}, \\
&x_1^3 x_2^{15} x_3^{13} x_4^{18} x_5^{12}, \quad x_1^{15} x_2^3 x_3^{13} x_4^{18} x_5^{12}, \quad x_1^3 x_2^{13} x_3^{14} x_4^{15} x_5^{16}, \quad x_1^3 x_2^{13} x_3^{15} x_4^{14} x_5^{16}, \\
&x_1^3 x_2^{15} x_3^{13} x_4^{14} x_5^{16}, \quad x_1^{15} x_2^3 x_3^{13} x_4^{14} x_5^{16}, \quad x_1^7 x_2^7 x_3^8 x_4^9 x_5^{30}, \quad x_1^7 x_2^7 x_3^9 x_4^8 x_5^{30}, \\
&x_1^7 x_2^9 x_3^9 x_4^{30} x_5^8, \quad x_1^7 x_2^9 x_3^9 x_4^{14} x_5^{24}, \quad x_1^7 x_2^{11} x_3^{13} x_4^{16} x_5^{14}, \quad x_1^7 x_2^{15} x_3^{17} x_4^{10} x_5^{12}, \\
&x_1^{15} x_2^7 x_3^{17} x_4^{10} x_5^{12}, \quad x_1^{15} x_2^7 x_3^{19} x_4^{10} x_5^{12}.
\end{aligned}$$

Proof. The first part of the lemma is an immediate corollary from a result in [47]. We prove the second part of this lemma for the monomials $x = x_1 x_2^{14} x_3^{15} x_4^{15} x_5^{16}$ and $y = x_1^7 x_2^7 x_3^9 x_4^8 x_5^{30}$. The others

can be proved by a similar computation. By a direct computation, one gets

$$\begin{aligned}
x &= x_1 x_2^{10} x_3^{12} x_4^{15} x_5^{23} + x_1 x_2^{10} x_3^{12} x_4^{23} x_5^{15} + x_1 x_2^{10} x_3^{14} x_4^{13} x_5^{23} + x_1 x_2^{10} x_3^{14} x_4^{21} x_5^{15} \\
&\quad + x_1 x_2^{10} x_3^{20} x_4^{15} x_5^{15} + x_1 x_2^{10} x_3^{22} x_4^{13} x_5^{15} + x_1 x_2^{12} x_3^{15} x_4^{10} x_5^{23} + x_1 x_2^{12} x_3^{15} x_4^{15} x_5^{18} \\
&\quad + x_1 x_2^{12} x_3^{18} x_4^{15} x_5^{15} + x_1 x_2^{12} x_3^{26} x_4^{7} x_5^{15} + x_1 x_2^{14} x_3^{12} x_4^{15} x_5^{19} + x_1 x_2^{14} x_3^{12} x_4^{19} x_5^{15} \\
&\quad + x_1 x_2^{14} x_3^{14} x_4^{13} x_5^{19} + x_1 x_2^{14} x_3^{14} x_4^{17} x_5^{15} + x_1 x_2^{14} x_3^{15} x_4^8 x_5^{23} \\
&\quad + Sq^1(f_1) + Sq^2(f_2) + Sq^4(f_4) + Sq^8(f_8) \pmod{(P_5^-(\omega_{(5,4)}))}, \text{ where} \\
f_1 &= x_1^2 x_2^{11} x_3^{15} x_4^{15} x_5^{17} + x_1^2 x_2^{11} x_3^{15} x_4^{17} x_5^{15} + x_1^2 x_2^{11} x_3^{17} x_4^{15} x_5^{15} + x_1^2 x_2^{11} x_3^{23} x_4^7 x_5^{17} \\
&\quad + x_1^2 x_2^{11} x_3^{23} x_4^9 x_5^{15} + x_1^2 x_2^{11} x_3^{25} x_4^7 x_5^{15} + x_1^2 x_2^{13} x_3^{15} x_4^{15} x_5^{15} + x_1^2 x_2^{13} x_3^{23} x_4^7 x_5^{15}, \\
f_2 &= x_1 x_2^{11} x_3^{15} x_4^{15} x_5^{17} + x_1 x_2^{11} x_3^{15} x_4^{17} x_5^{15} + x_1 x_2^{11} x_3^{17} x_4^{15} x_5^{15} + x_1 x_2^{11} x_3^{23} x_4^7 x_5^{17} \\
&\quad + x_1 x_2^{11} x_3^{23} x_4^9 x_5^{15} + x_1 x_2^{11} x_3^{25} x_4^7 x_5^{15} + x_1 x_2^{13} x_3^{15} x_4^{15} x_5^{15} + x_1 x_2^{13} x_3^{23} x_4^7 x_5^{15} \\
&\quad + x_1 x_2^{14} x_3^{18} x_4^{11} x_5^{15} + x_1 x_2^{22} x_3^{15} x_4^6 x_5^{15}, \\
f_4 &= x_1 x_2^{14} x_3^{12} x_4^{15} x_5^{15} + x_1 x_2^{14} x_3^{14} x_4^{13} x_5^{15} + x_1 x_2^{14} x_3^{20} x_4^7 x_5^{15} + x_1 x_2^{20} x_3^{15} x_4^6 x_5^{15}, \\
f_8 &= x_1 x_2^{10} x_3^{12} x_4^{15} x_5^{15} + x_1 x_2^{10} x_3^{14} x_4^{13} x_5^{15} + x_1 x_2^{12} x_3^{15} x_4^{10} x_5^{15} + x_1 x_2^{14} x_3^{15} x_4^8 x_5^{15}.
\end{aligned}$$

The above relations show that x is strictly inadmissible. By a similar computation, we obtain

$$\begin{aligned}
y &= x_1^4 x_2^{11} x_3^{12} x_4^{19} x_5^{15} + x_1^4 x_2^{11} x_3^{19} x_4^{12} x_5^{15} + x_1^4 x_2^{20} x_3^{11} x_4^{11} x_5^{15} + x_1^4 x_2^{24} x_3^7 x_4^{11} x_5^{15} \\
&\quad + x_1^5 x_2^3 x_3^{12} x_4^{26} x_5^{15} + x_1^5 x_2^3 x_3^{26} x_4^{12} x_5^{15} + x_1^5 x_2^7 x_3^8 x_4^{14} x_5^{27} + x_1^5 x_2^7 x_3^8 x_4^{26} x_5^{15} \\
&\quad + x_1^5 x_2^7 x_3^{10} x_4^{12} x_5^{27} + x_1^5 x_2^7 x_3^{10} x_4^{24} x_5^{15} + x_1^5 x_2^{10} x_3^{12} x_4^{19} x_5^{15} + x_1^5 x_2^{10} x_3^{19} x_4^{12} x_5^{15} \\
&\quad + x_1^5 x_2^{11} x_3^5 x_4^{10} x_5^{30} + x_1^5 x_2^{11} x_3^6 x_4^{12} x_5^{27} + x_1^5 x_2^{11} x_3^8 x_4^{14} x_5^{23} + x_1^5 x_2^{11} x_3^8 x_4^{22} x_5^{15} \\
&\quad + x_1^5 x_2^{11} x_3^9 x_4^6 x_5^{30} + x_1^5 x_2^{11} x_3^{10} x_4^{20} x_5^{15} + x_1^5 x_2^{11} x_3^{16} x_4^{14} x_5^{15} + x_1^5 x_2^{11} x_3^{18} x_4^{12} x_5^{15} \\
&\quad + x_1^5 x_2^{18} x_3^{11} x_4^{12} x_5^{15} + x_1^7 x_2^3 x_3^{12} x_4^{24} x_5^{15} + x_1^7 x_2^3 x_3^{24} x_4^{12} x_5^{15} + x_1^7 x_2^5 x_3^9 x_4^{10} x_5^{30} \\
&\quad + x_1^7 x_2^5 x_3^{10} x_4^{12} x_5^{27} + x_1^7 x_2^7 x_3^8 x_4^{12} x_5^{27} + x_1^7 x_2^7 x_3^8 x_4^{14} x_5^{25} + x_1^7 x_2^7 x_3^8 x_4^{24} x_5^{15} \\
&\quad + Sq^1(g_1) + Sq^2(g_2) + Sq^4(g_4) + Sq^8(g_8) \pmod{(P_5^-(\omega_{(5,4)}))}, \text{ where} \\
g_1 &= x_1^7 x_2^5 x_3^{12} x_4^{21} x_5^{15} + x_1^7 x_2^5 x_3^{21} x_4^{12} x_5^{15} + x_1^7 x_2^7 x_3^5 x_4^{12} x_5^{29} + x_1^7 x_2^7 x_3^{12} x_4^{19} x_5^{15} \\
&\quad + x_1^7 x_2^7 x_3^{19} x_4^{12} x_5^{15} + x_1^7 x_2^9 x_3^8 x_4^{13} x_5^{23} + x_1^7 x_2^9 x_3^8 x_4^{21} x_5^{15} + x_1^7 x_2^9 x_3^{16} x_4^{13} x_5^{15} \\
&\quad + x_1^7 x_2^{17} x_3^9 x_4^{12} x_5^{15} + x_1^7 x_2^{20} x_3^7 x_4^{11} x_5^{15}, \\
g_2 &= x_1^7 x_2^3 x_3^5 x_4^6 x_5^{38} + x_1^7 x_2^3 x_3^6 x_4^{20} x_5^{23} + x_1^7 x_2^3 x_3^{12} x_4^{22} x_5^{15} + x_1^7 x_2^3 x_3^{22} x_4^{12} x_5^{15} \\
&\quad + x_1^7 x_2^6 x_3^{12} x_4^{19} x_5^{15} + x_1^7 x_2^6 x_3^{19} x_4^{12} x_5^{15} + x_1^7 x_2^7 x_3^5 x_4^{10} x_5^{30} + x_1^7 x_2^7 x_3^6 x_4^{12} x_5^{27} \\
&\quad + x_1^7 x_2^7 x_3^8 x_4^{14} x_5^{23} + x_1^7 x_2^7 x_3^8 x_4^{22} x_5^{15} + x_1^7 x_2^7 x_3^9 x_4^6 x_5^{30} + x_1^7 x_2^7 x_3^{10} x_4^{20} x_5^{15} \\
&\quad + x_1^7 x_2^7 x_3^{16} x_4^{14} x_5^{15} + x_1^7 x_2^7 x_3^{18} x_4^{12} x_5^{15} + x_1^7 x_2^{18} x_3^7 x_4^{12} x_5^{15} + x_1^7 x_2^{18} x_3^8 x_4^{11} x_5^{15}, \\
g_4 &= x_1^4 x_2^7 x_3^{12} x_4^{19} x_5^{15} + x_1^4 x_2^7 x_3^{19} x_4^{12} x_5^{15} + x_1^4 x_2^{20} x_3^7 x_4^{11} x_5^{15} + x_1^5 x_2^3 x_3^{12} x_4^{22} x_5^{15} \\
&\quad + x_1^5 x_2^3 x_3^{22} x_4^{12} x_5^{15} + x_1^5 x_2^6 x_3^{12} x_4^{19} x_5^{15} + x_1^5 x_2^6 x_3^{19} x_4^{12} x_5^{15} + x_1^5 x_2^7 x_3^5 x_4^{10} x_5^{30} \\
&\quad + x_1^5 x_2^7 x_3^6 x_4^{12} x_5^{27} + x_1^5 x_2^7 x_3^8 x_4^{14} x_5^{23} + x_1^5 x_2^7 x_3^8 x_4^{22} x_5^{15} + x_1^5 x_2^7 x_3^9 x_4^6 x_5^{30} \\
&\quad + x_1^5 x_2^7 x_3^{10} x_4^{20} x_5^{15} + x_1^5 x_2^7 x_3^{16} x_4^{14} x_5^{15} + x_1^5 x_2^7 x_3^{18} x_4^{12} x_5^{15} + x_1^5 x_2^7 x_3^{18} x_4^{12} x_5^{15} \\
&\quad + x_1^{11} x_2^5 x_3^5 x_4^6 x_5^{30} + x_1^{11} x_2^5 x_3^6 x_4^{12} x_5^{23}, \\
g_8 &= x_1^7 x_2^5 x_3^5 x_4^6 x_5^{30} + x_1^7 x_2^5 x_3^6 x_4^{12} x_5^{23} + x_1^7 x_2^9 x_3^{10} x_4^{12} x_5^{15} + x_1^7 x_2^{10} x_3^8 x_4^{13} x_5^{15}.
\end{aligned}$$

By Definition 2.2.4, we see that y is strictly inadmissible. The lemma follows. $\square$

Lemma 3.2.15. *For any $t \geq 4$, the following monomials are strictly inadmissible:*

$$\begin{array}{lll}
x_1^3 x_2^{2t-4} x_3^3 x_4^{2t-3} x_5^{2t+1-2}, & x_1^3 x_2^{2t-3} x_3^{2t+1-2} x_4^3 x_5^{2t-4}, & x_1^3 x_2^{2t-4} x_3^3 x_4^{2t+1-3} x_5^{2t-2}, \\
x_1^3 x_2^{2t+1-3} x_3^{2t-2} x_4^3 x_5^{2t-4}, & x_1^3 x_2^{2t+1-4} x_3^3 x_4^{2t-3} x_5^{2t-2}, & x_1^3 x_2^{2t-3} x_3^{2t-2} x_4^3 x_5^{2t+1-4}, \\
x_1^3 x_2^5 x_3^{2t+1-2} x_4^{2t-5} x_5^{2t-4}, & x_1^3 x_2^5 x_3^{2t-2} x_4^{2t-5} x_5^{2t+1-4}, & x_1^3 x_2^5 x_3^{2t-2} x_4^{2t+1-5} x_5^{2t-4}, \\
x_1^3 x_2^5 x_3^{2t-2} x_4^{2t+1-7} x_5^{2t-2}, & x_1^3 x_2^5 x_3^{2t+1-7} x_4^{2t-2} x_5^{2t-2}, & x_1^3 x_2^{2t+1-3} x_3^6 x_4^{2t-5} x_5^{2t-4}, \\
x_1^3 x_2^{2t-3} x_3^6 x_4^{2t-5} x_5^{2t+1-4}, & x_1^3 x_2^{2t-3} x_3^6 x_4^{2t+1-5} x_5^{2t-4}, & x_1^3 x_2^{2t+1-3} x_3^7 x_4^{2t-6} x_5^{2t-4}, \\
x_1^3 x_2^{2t-3} x_3^7 x_4^{2t-6} x_5^{2t+1-4}, & x_1^3 x_2^7 x_3^{2t+1-5} x_4^{2t-4} x_5^{2t-4}, & x_1^3 x_2^7 x_3^{2t+1-5} x_4^{2t-4} x_5^{2t-4}, \\
x_1^3 x_2^{2t+1-5} x_3^3 x_4^{2t-4} x_5^{2t-4}, & x_1^3 x_2^{2t-3} x_3^7 x_4^{2t+1-6} x_5^{2t-4}, & x_1^3 x_2^7 x_3^{2t+1-7} x_4^{2t-2} x_5^{2t-4}, \\
x_1^7 x_2^3 x_3^{2t+1-7} x_4^{2t-2} x_5^{2t-4}, & x_1^3 x_2^{13} x_3^{2t+1-9} x_4^{2t-6} x_5^{2t-4}, & x_1^3 x_2^{13} x_3^{2t+1-10} x_4^{2t-5} x_5^{2t-4}, \\
x_1^7 x_2^{2t-5} x_3^5 x_4^{2t-8} x_5^{2t+1-2}, & x_1^7 x_2^{2t-5} x_3^5 x_4^{2t+1-2} x_5^{2t-8}, & x_1^7 x_2^{2t+1-5} x_3^5 x_4^{2t-8} x_5^{2t-2}, \\
x_1^7 x_2^{2t+1-5} x_3^5 x_4^{2t-2} x_5^{2t-8}, & x_1^7 x_2^{2t-5} x_3^5 x_4^{2t-2} x_5^{2t+1-8}, & x_1^7 x_2^{2t-5} x_3^5 x_4^{2t+1-8} x_5^{2t-2}, \\
x_1^7 x_2^7 x_3^{2t-7} x_4^{2t-6} x_5^{2t+1-4}, & x_1^7 x_2^9 x_3^{2t-9} x_4^{2t-6} x_5^{2t+1-4}, & x_1^7 x_2^7 x_3^{2t-7} x_4^{2t+1-6} x_5^{2t-4}, \\
x_1^7 x_2^9 x_3^{2t-9} x_4^{2t+1-6} x_5^{2t-4}, & x_1^7 x_2^7 x_3^{2t+1-7} x_4^{2t-6} x_5^{2t-4}, & x_1^7 x_2^{11} x_3^{2t+1-11} x_4^{2t-8} x_5^{2t-2}, \\
x_1^7 x_2^{11} x_3^{2t+1-11} x_4^{2t-2} x_5^{2t-8}, & x_1^7 x_2^9 x_3^{2t+1-9} x_4^{2t-6} x_5^{2t-4}. &
\end{array}$$

Proof. We prove the lemma for $u = x_1^7 x_2^9 x_3^{2t+1-9} x_4^{2t-6} x_5^{2t-4}$. The others can be proven by a similar computation. By a direct computation using the Cartan formula, we have

$$\begin{aligned}
u &= x_1^4 x_2^7 x_3^{2t+1-5} x_4^{2t-5} x_5^{2t-4} + x_1^4 x_2^{11} x_3^{2t+1-9} x_4^{2t-5} x_5^{2t-4} + x_1^5 x_2^7 x_3^{2t+1-5} x_4^{2t-6} x_5^{2t-4} \\
&\quad + x_1^5 x_2^{11} x_3^{2t+1-9} x_4^{2t-6} x_5^{2t-4} + x_1^7 x_2^7 x_3^{2t+1-8} x_4^{2t-5} x_5^{2t-4} + x_1^7 x_2^7 x_3^{2t+1-7} x_4^{2t-6} x_5^{2t-4} \\
&\quad + x_1^7 x_2^8 x_3^{2t+1-9} x_4^{2t-5} x_5^{2t-4} + Sq^1(x_1^7 x_2^7 x_3^{2t+1-9} x_4^{2t-5} x_5^{2t-4}) + Sq^2(x_1^7 x_2^7 x_3^{2t+1-9} x_4^{2t-6} x_5^{2t-4}) \\
&\quad + Sq^4(x_1^4 x_2^7 x_3^{2t+1-9} x_4^{2t-5} x_5^{2t-4} + x_1^5 x_2^7 x_3^{2t+1-9} x_4^{2t-6} x_5^{2t-4}) \pmod{(P_5^-(\omega_{(5,t)}))}.
\end{aligned}$$

This equality shows that u is strictly inadmissible. The lemma is proved. $\square$

The proof of the following lemma is analogous to the proof of Lemma 3.2.15.

Lemma 3.2.16. *For any $t \geq 5$, the following monomials are strictly inadmissible:*

$$\begin{array}{lll}
x_1^3 x_2^{13} x_3^{2t-10} x_4^{2t-5} x_5^{2t+1-4}, & x_1^3 x_2^{13} x_3^{2t-10} x_4^{2t+1-5} x_5^{2t-4}, & x_1^3 x_2^{13} x_3^{2t-9} x_4^{2t-6} x_5^{2t+1-4}, \\
x_1^3 x_2^{13} x_3^{2t-9} x_4^{2t+1-6} x_5^{2t-4}, & x_1^3 x_2^{15} x_3^{2t-11} x_4^{2t-6} x_5^{2t+1-4}, & x_1^{15} x_2^3 x_3^{2t-11} x_4^{2t-6} x_5^{2t+1-4}, \\
x_1^3 x_2^{15} x_3^{2t-11} x_4^{2t+1-6} x_5^{2t-4}, & x_1^{15} x_2^3 x_3^{2t-11} x_4^{2t+1-6} x_5^{2t-4}, & x_1^7 x_2^{11} x_3^{2t-11} x_4^{2t-8} x_5^{2t+1-2}, \\
x_1^7 x_2^{11} x_3^{2t-11} x_4^{2t+1-2} x_5^{2t-8}, & x_1^7 x_2^{11} x_3^{2t-11} x_4^{2t-2} x_5^{2t+1-8}, & x_1^7 x_2^{11} x_3^{2t-11} x_4^{2t+1-8} x_5^{2t-2}.
\end{array}$$

Proof of Proposition 3.2.11. Let x be an admissible monomial in $(P_5^+)_{3(2t-1)+2^t}$ such that $\omega(x) = \omega_{(5,t)}$ with $t \geq 4$. By induction on t , we see that $x \neq b_{3(2t-1)+2^t,k}$ for $k = 1, 2, \dots, 270$, then there is a monomial z , which is given in one of Lemmas 3.2.12, 3.2.13, 3.2.14, 3.2.15, and 3.2.16 such that $x = zy^{2^\alpha}$ for some monomial y and positive integer α . By Theorem 2.2.5, x is inadmissible. Hence $\mathcal{B}_5^+(\omega_{(5,t)}) \subseteq \{b_{3(2t-1)+2^t,k} : 1 \leq k \leq 270\}$ for every $t \geq 4$.

Now we prove that the classes $[b_{3(2t-1)+2^t,k}]_{\omega_{(5,t)}} \neq [0]_{\omega_{(5,t)}}$ with $1 \leq k \leq 270$, and $t \geq 4$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq k \leq 270} \gamma_k b_{3(2t-1)+2^t,k} \equiv_{\omega_{(5,t)}} 0,$$

where $\gamma_k \in \mathbb{F}_2$ for all k , $1 \leq k \leq 270$.

For $t \geq 4$, according to Sum [47], $\mathcal{B}_4^+(3(2^t - 1) + 2^t)$ is the set consisting of 33 monomials, namely:

$$\begin{aligned}
v_1. & x_1 x_2^{2^t-2} x_3^{2^t-1} x_4^{2^{t+1}-1}, & v_2. & x_1 x_2^{2^t-2} x_3^{2^{t+1}-1} x_4^{2^t-1}, & v_3. & x_1 x_2^{2^t-1} x_3^{2^t-2} x_4^{2^{t+1}-1}, \\
v_4. & x_1 x_2^{2^t-1} x_3^{2^{t+1}-1} x_4^{2^t-2}, & v_5. & x_1 x_2^{2^{t+1}-1} x_3^{2^t-2} x_4^{2^t-1}, & v_6. & x_1 x_2^{2^{t+1}-1} x_3^{2^t-1} x_4^{2^t-2}, \\
v_7. & x_1^{2^t-1} x_2 x_3^{2^t-2} x_4^{2^{t+1}-1}, & v_8. & x_1^{2^t-1} x_2 x_3^{2^{t+1}-1} x_4^{2^t-2}, & v_9. & x_1^{2^t-1} x_2^{2^{t+1}-1} x_3 x_4^{2^t-2}, \\
v_{10}. & x_1^{2^{t+1}-1} x_2 x_3^{2^t-2} x_4^{2^t-1}, & v_{11}. & x_1^{2^{t+1}-1} x_2 x_3^{2^t-1} x_4^{2^t-2}, & v_{12}. & x_1^{2^{t+1}-1} x_2^{2^t-1} x_3 x_4^{2^t-2}, \\
v_{13}. & x_1 x_2^{2^t-1} x_3^{2^t-1} x_4^{2^{t+1}-2}, & v_{14}. & x_1 x_2^{2^t-1} x_3^{2^{t+1}-2} x_4^{2^t-1}, & v_{15}. & x_1 x_2^{2^{t+1}-2} x_3^{2^t-1} x_4^{2^t-1}, \\
v_{16}. & x_1^{2^t-1} x_2 x_3^{2^t-1} x_4^{2^{t+1}-2}, & v_{17}. & x_1^{2^t-1} x_2 x_3^{2^{t+1}-2} x_4^{2^t-1}, & v_{18}. & x_1^{2^t-1} x_2^{2^t-1} x_3 x_4^{2^{t+1}-2}, \\
v_{19}. & x_1^3 x_2^{2^t-3} x_3^{2^t-2} x_4^{2^{t+1}-1}, & v_{20}. & x_1^3 x_2^{2^t-3} x_3^{2^{t+1}-1} x_4^{2^t-2}, & v_{21}. & x_1^3 x_2^{2^{t+1}-1} x_3^{2^t-3} x_4^{2^t-2}, \\
v_{22}. & x_1^{2^{t+1}-1} x_2^3 x_3^{2^t-3} x_4^{2^t-2}, & v_{23}. & x_1^3 x_2^{2^t-3} x_3^{2^t-1} x_4^{2^{t+1}-2}, & v_{24}. & x_1^3 x_2^{2^t-3} x_3^{2^{t+1}-2} x_4^{2^t-1}, \\
v_{25}. & x_1^3 x_2^{2^t-1} x_3^{2^t-3} x_4^{2^{t+1}-2}, & v_{26}. & x_1^{2^t-1} x_2^3 x_3^{2^t-3} x_4^{2^{t+1}-2}, & v_{27}. & x_1^3 x_2^{2^t-1} x_3^{2^{t+1}-3} x_4^{2^t-2}, \\
v_{28}. & x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-2} x_4^{2^t-1}, & v_{29}. & x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-1} x_4^{2^t-2}, & v_{30}. & x_1^{2^t-1} x_2^3 x_3^{2^{t+1}-3} x_4^{2^t-2}, \\
v_{31}. & x_1^7 x_2^{2^{t+1}-5} x_3^{2^t-3} x_4^{2^t-2}, & v_{32}. & x_1^7 x_2^{2^t-5} x_3^{2^t-3} x_4^{2^{t+1}-2}, & v_{33}. & x_1^7 x_2^{2^t-5} x_3^{2^{t+1}-3} x_4^{2^t-2}.
\end{aligned}$$

By a direct computation using Theorem 3.1.3, we express $p_{(i;I)}(\mathcal{S})$, in terms of v_j , $1 \leq j \leq 33$. Computing directly from the relations

$$p_{(i;I)}(\mathcal{S}) \equiv_{\omega_{(5,t)}} 0, \quad \forall (i;I) \in \mathcal{N}_5, \quad \ell(I) = 1,$$

we obtain $\gamma_k = 0$, $k = 1, 2, \dots, 270$. The proposition follows. $\square$

We knew that

$$\begin{aligned}
(QP_5)_{3(2^t-1)+2^t} &= \text{Ker}((\widetilde{Sq_*^0})_{(5,3(2^t-1)+2^t)}) \oplus (QP_5)_{2^{t+1}-4}, \\
\text{Ker}((\widetilde{Sq_*^0})_{(5,3(2^t-1)+2^t)}) &= (QP_5)_{3(2^t-1)+2^t} \oplus QP_5^+(\omega_{(5,t)}),
\end{aligned}$$

and $\dim(QP_5)_{3(2^t-1)+2^t} = 195$ for any $t \geq 3$. So, from Theorem 3.2.1 and Proposition 3.2.11, we get

$$\dim(QP_5)_{3(2^t-1)+2^t} = \begin{cases} 945 & \text{if } t = 4, \\ 1115 & \text{if } t = 5, \\ 1116 & \text{if } t \geq 6. \end{cases}$$

The proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by using the results in Section 3. Firstly, we need some notations and definitions for the proof of the theorem.

For $1 \leq i \leq s$, define the $\mathcal{A}$ -homomorphism $\tau_i : P_s \rightarrow P_s$, which is determined by $\tau_i(x_i) = x_{i+1}$, $\tau_i(x_{i+1}) = x_i$, $\tau_i(x_j) = x_j$ for $j \neq i, i+1$, $1 \leq i < s$ and $\tau_s(x_1) = x_1 + x_2$, $\tau_s(x_j) = x_j$ for $j > 1$. Observe that the general linear group GL_s is generated by the matrices associated with τ_i , $0 \leq i \leq s$ and the symmetric group $\Sigma_s \subset GL_s$ is generated by the ones associated with τ_i , $1 \leq i < s$. Hence, a class $[f]_\omega$ represented by a homogeneous polynomial $f \in P_s$ is an GL_s -invariant if and only if $\tau_i(f) \equiv_\omega f$ for $1 \leq i \leq s$. $[f]_\omega$ is an Σ_s -invariant if and only if $\tau_i(f) \equiv_\omega f$ for $1 \leq i < s$.

For the weight vector $\omega_{(5,t)} = (3, 3, \dots, 3, 1)$ (t times of 3), we have $\deg \omega_{(5,t)} = 3(2^t - 1) + 2^t$. For any monomials $g_1, g_2, \dots, g_m$ in $P_5(\omega_{(5,t)})$ and for a subgroup $G \subset GL_5$, we denote $G(g_1, \dots, g_m)$ the G -submodule of $QP_5(\omega_{(5,t)})$ generated by the set $\{[g_i]_{\omega_{(5,t)}} : 1 \leq i \leq m\}$. We see that $\omega_{(5,t)}$ is the weight vector of the minimal spike $x_1^{2^{t+1}-1} x_2^{2^t-1} x_3^{2^t-1} \in (P_5)_{3(2^t-1)+2^t}$; hence $[g_i]_{\omega_{(5,t)}} = [g_i]$ for all i , $1 \leq i \leq m$.

Recall that Kameko's map $\widetilde{Sq_*^0} = (\widetilde{Sq_*^0})_{(5,3(2^t-1)+2^t)} : (QP_5)_{3(2^t-1)+2^t} \rightarrow (QP_5)_{2^{t+1}-4}$ is an epimorphism of GL_5 -modules. Then, as mentioned in Section 1, we need only to determine $\text{Ker}((\widetilde{Sq_*^0})_{(5,3(2^t-1)+2^t)})^{GL_5}$.

From the results in Section 3, we get

$$\dim(\text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)})) = \dim(QP_5(\omega_{(5,t)})).$$

Note that $QP_5(\omega_{(5,t)}) = QP_5^0(\omega_{(5,t)}) \oplus QP_5^+(\omega_{(5,t)})$ and

$$\dim(\text{Ker}((\widetilde{Sq}_*^0)_{(5,3(2^t-1)+2^t)}))^{GL_5} \leq \dim(QP_5(\omega_{(5,t)}))^{GL_5}.$$

Now, for $t = 1$, $3(2^t - 1) + 2^t = 5$. From a result of Sum [46, 50], we have $QP_5(\omega_{(5,1)})^{GL_5} = 0$. We now compute $QP_5(\omega_{(5,t)})^{GL_5}$ for $t \geq 2$.

4.1. Computation of $QP_5(\omega_{(5,t)})^{\Sigma_5}$

As it is known, $\mathcal{B}_5^0(3(2^t - 1) + 2^t) = \mathcal{B}_5^0(\omega_{(5,t)})$ is the set consisting of the admissible monomials $q_{t,k}$ (see Subsection 5.1). By a simple computation, we see that the following subspaces are Σ_5 -submodules of $QP_5^0(\omega_{(5,t)})$:

For $t \geq 2$,

$$\begin{aligned}\Sigma_5(q_{t,1}) &= \langle [q_{t,k}]_{\omega_{(5,t)}} : 1 \leq k \leq 30 \rangle, \\ \Sigma_5(q_{t,31}) &= \langle [q_{t,k}]_{\omega_{(5,t)}} : 31 \leq k \leq 90 \rangle.\end{aligned}$$

For $t = 2$,

$$\mathcal{M}_1 = \langle [q_{2,k}]_{\omega_{(5,2)}} : 91 \leq k \leq 145 \rangle.$$

For $t \geq 3$,

$$\Sigma_5(q_{t,91}) = \langle [q_{t,k}]_{\omega_{(5,t)}} : 91 \leq k \leq 110 \rangle.$$

For $t = 3$,

$$\mathcal{M}_2 = \langle [q_{3,k}]_{\omega_{(5,3)}} : 111 \leq k \leq 195 \rangle.$$

For $t \geq 4$,

$$\mathcal{M}_{t,1}^* = \langle [q_{t,k}]_{\omega_{(5,t)}} : 111 \leq k \leq 195 \rangle.$$

Lemma 4.1.1. *Let t be an integer. Then, we obtain the following:*

For $t \geq 2$,

$$\begin{aligned}\Sigma_5(q_{t,1})^{\Sigma_5} &= \langle [p_{t,1} := q_{t,1} + q_{t,2} + \cdots + q_{t,30}]_{\omega_{(5,t)}} \rangle, \\ \Sigma_5(q_{t,31})^{\Sigma_5} &= \langle [p_{t,2} := q_{t,31} + q_{t,32} + \cdots + q_{t,90}]_{\omega_{(5,t)}} \rangle.\end{aligned}$$

For $t \geq 3$,

$$\Sigma_5(q_{t,91})^{\Sigma_5} = \langle [p_{t,3} := q_{t,91} + q_{t,92} + \cdots + q_{t,110}]_{\omega_{(5,t)}} \rangle.$$

Outline of the proof. The set $B := \{[q_{t,k}]_{\omega_{(5,t)}} : 1 \leq k \leq 30\}$ is a basis of $\Sigma_5(q_{t,1})$ for $t \geq 2$. The action of Σ_5 on QP_5 induces the one of it on B . Furthermore, this action is transitive. Hence, if $g \equiv_{\omega_{(5,t)}} \sum_{k=1}^{30} \gamma_k q_{t,k}$ with $\gamma_t \in \mathbb{F}_2$ and $[g]_{\omega_{(5,t)}} \in \Sigma_5(q_{t,1})^{\Sigma_5}$, then the relations $\tau_i(g) \equiv_{\omega_{(5,t)}} g$, $i = 1, 2, 3, 4$, imply $\gamma_k = \gamma_1$, $\forall k$, $1 \leq k \leq 30$. Hence, $\Sigma_5(q_{t,1})^{\Sigma_5} = \langle [p_{t,1}]_{\omega_{(5,t)}} \rangle$ with $p_{t,1} = \sum_{k=1}^{30} q_{t,k}$.

By a similar computation, we obtain $\Sigma_5(q_{t,31})^{\Sigma_5} = \langle [p_{t,2}]_{\omega_{(5,t)}} \rangle$ with $p_{t,2} = \sum_{k=31}^{90} q_{t,k}$, and $\Sigma_5(q_{t,91})^{\Sigma_5} = \langle [p_{t,3}]_{\omega_{(5,t)}} \rangle$ with $p_{t,3} = \sum_{k=91}^{110} q_{t,k}$. The lemma follows. $\square$

Lemma 4.1.2. *We have $\mathcal{M}_1^{\Sigma_5} = 0$ and $\dim(\mathcal{M}_2^{\Sigma_5}) = \dim((\mathcal{M}_{t,1}^*)^{\Sigma_5}) = 1$ for every $t \geq 4$.*

Proof. Using a result in Section 3, we have $\dim(\mathcal{M}_1) = 55$. Suppose $h \equiv_{\omega_{(5,t)}} \sum_{k=91}^{145} \gamma_k q_{2,k}$ with $\gamma_k \in \mathbb{F}_2$ and $[h]_{\omega_{(5,t)}} \in \mathcal{M}_1^{\Sigma_5}$. By computing $\tau_j(h) + h$ in terms of $q_{2,k}$, $91 \leq k \leq 145$ and using the relations $\tau_j(h) + h \equiv_{\omega_{(5,t)}} 0$, $1 \leq j \leq 4$, we obtain $\gamma_k = 0$ for all k , $91 \leq k \leq 145$.

Now suppose $h \equiv_{\omega_{(5,t)}} \sum_{k=111}^{195} \gamma_k q_{3,k}$ and $[h]_{\omega_{(5,t)}} \in \mathcal{M}_2^{\Sigma_5}$. We compute $\tau_j(h) + h$ in terms of $q_{3,k}$, $111 \leq k \leq 195$. By computing directly from the relations $\tau_j(h) + h \equiv_{\omega_{(5,t)}} 0$, $j = 1, 2, 3, 4$, one

gets $\gamma_k = 0$ for $111 \leq k \leq 135$ and $\gamma_k = \gamma_{136}$ for $136 \leq k \leq 195$. This implies $\mathcal{M}_2^{\Sigma_5} = \langle [p_{3,4}]_{\omega_{(5,t)}} \rangle$ with $p_{3,4} = \sum_{i=136}^{195} q_{3,i}$.

Similarly, we have $(\mathcal{M}_{t,1}^*)^{\Sigma_5} = \langle [q_{t,1}^*]_{\omega_{(5,t)}} \rangle$ with $q_{t,1}^* = \sum_{i=136}^{195} q_{t,i}$ for all $t \geq 4$. The lemma is proved. $\square$

We now denote by $b_{t,k} = b_{3(2^t-1)+2^t,k}$ the admissible monomials in $(P_5^+)_{3(2^t-1)+2^t}$ as given in Subsection 5.2.

The case $t = 2$. By a direct computation using the results in Section 3, we have the direct summand decomposition of the Σ_5 -modules:

$$QP_5^+(\omega_{(5,2)}) = \Sigma_5(b_{2,1}) \oplus \mathcal{M}_3,$$

where $\Sigma_5(b_{2,1}) = \langle [b_{2,k}]_{\omega_{(5,2)}} : 1 \leq k \leq 10 \rangle$ and $\mathcal{M}_3 = \langle [b_{2,k}]_{\omega_{(5,2)}} : 11 \leq k \leq 60 \rangle$.

The following lemma is proved by a direct computation.

Lemma 4.1.3. $\Sigma_5(b_{2,1})^{\Sigma_5} = 0$ and $\mathcal{M}_3^{\Sigma_5} = \langle [p_{2,4} := b_{2,31} + b_{2,32} + \cdots + b_{2,60}]_{\omega_{(5,2)}} \rangle$.

From Lemmas 4.1.1 and 4.1.3, we obtain the following.

Proposition 4.1.4. $QP_5(\omega_{(5,2)})^{\Sigma_5} = \langle [p_{2,1}]_{\omega_{(5,2)}}, [p_{2,2}]_{\omega_{(5,2)}}, [p_{2,4}]_{\omega_{(5,2)}} \rangle$.

The case $t = 3$. By a simple computation, we see that

$$\begin{aligned} \Sigma_5(b_{3,1}) &= \langle [b_{3,k}]_{\omega_{(5,3)}} : 1 \leq k \leq 45 \rangle, \\ \Sigma_5(b_{3,46}) &= \langle [b_{3,k}]_{\omega_{(5,3)}} : 46 \leq k \leq 65 \rangle, \end{aligned}$$

and $\mathcal{M}_4 = \langle [b_{3,k}]_{\omega_{(5,3)}} : 66 \leq k \leq 260 \rangle$ are Σ_5 -submodules of $QP_5^+(\omega_{(5,3)})$. Hence, we have a direct summand decomposition of Σ_5 -modules:

$$QP_5^+(\omega_{(5,3)}) = \Sigma_5(b_{t,1}) \oplus \Sigma_5(b_{t,46}) \oplus \mathcal{M}_4.$$

Lemma 4.1.5. We have $\dim(\Sigma_5(b_{3,1})^{\Sigma_5}) = \dim(\Sigma_5(b_{3,46})^{\Sigma_5}) = 1$ and $\mathcal{M}_4^{\Sigma_5} = 0$.

Proof. We have $\dim \Sigma_5(b_{3,1}) = 45$ and the set $\{[b_{3,k}]_{\omega_{(5,3)}} : 1 \leq k \leq 45\}$ is a basis of $\Sigma_5(b_{3,1})$. Suppose $f \equiv_{\omega_{(5,3)}} \sum_{k=1}^{45} \gamma_k b_{3,k}$ with $\gamma_k \in \mathbb{F}_2$ and $[f]_{\omega_{(5,3)}} \in \Sigma_5(b_{3,1})^{\Sigma_5}$. By computing $\tau_j(f) + f$ in terms of $b_{3,k}$, $1 \leq k \leq 45$ and using the relations $\tau_j(f) + f \equiv_{\omega_{(5,3)}} 0$, $j = 1, 2, 3, 4$, one gets

$$\begin{aligned} \gamma_1 &= \gamma_k, \quad \text{for } 2 \leq k \leq 10, \\ \gamma_k &= 0, \quad \text{for } 11 \leq k \leq 45. \end{aligned}$$

Hence $\Sigma_5(b_{3,1})^{\Sigma_5} = \langle [p_{3,5}]_{\omega_{(5,3)}} \rangle$ with $p_{3,5} = \sum_{k=1}^{10} b_{3,k}$. By a similar computation, we get $\Sigma_5(b_{3,46})^{\Sigma_5} = \langle [p_{3,6}]_{\omega_{(5,3)}} \rangle$ with $p_{3,6} = \sum_{k=46}^{65} b_{3,k}$ and $\mathcal{M}_4^{\Sigma_5} = 0$. The lemma follows. $\square$

Based on Lemmas 4.1.1, 4.1.2 and 4.1.5, one gets the following.

Proposition 4.1.6. $QP_5(\omega_{(5,3)})^{\Sigma_5} = \langle [p_{3,1}]_{\omega_{(5,3)}}, [p_{3,2}]_{\omega_{(5,3)}}, \dots, [p_{3,6}]_{\omega_{(5,3)}} \rangle$.

The case $t \geq 4$. We set

$$b_{t,1} = x_1 x_2 x_3^{2^t-2} x_4^{2^t-2} x_5^{2^{t+1}-1}, \quad b_{t,46} = x_1 x_2 x_3^{2^t-4} x_4^{2^t-1} x_5^{2^{t+1}-1}.$$

Computing from the results in Section 3, one gets

$$\dim(\Sigma_5(b_{t,1})) = 45, \quad \dim(\Sigma_5(b_{t,46})) = 20,$$

where

$$\begin{aligned} \Sigma_5(b_{t,1}) &= \langle [b_{t,k}]_{\omega_{(5,t)}} : 1 \leq k \leq 45 \rangle, \\ \Sigma_5(b_{t,46}) &= \langle [b_{t,k}]_{\omega_{(5,t)}} : 46 \leq k \leq 65 \rangle. \end{aligned}$$

Lemma 4.1.7. For any $t \geq 4$, $\Sigma_5(b_{t,46})^{\Sigma_5} = \langle [q_{t,2}^* := b_{t,46} + b_{t,47} + \cdots + b_{t,65}]_{\omega_{(5,t)}} \rangle$.

The proof of this lemma is straightforward.

For $t \geq 4$, by an easy computation, we have a direct summand decomposition of Σ_5 -modules:

$$QP_5^+(\omega_{(5,t)}) = \Sigma_5(b_{t,1}) \oplus \Sigma_5(b_{t,46}) \oplus \mathcal{M}_{t,2}^*,$$

where $\mathcal{M}_{t,2}^* = \langle [b_{t,k}]_{\omega_{(5,t)}} : 66 \leq k \leq 270 \rangle$.

Lemma 4.1.8. For any integer $t \geq 4$, we have

$$\Sigma_5(b_{t,1})^{\Sigma_5} = \langle [q_{t,3}^*]_{\omega_{(5,t)}}, [q_{t,4}^*]_{\omega_{(5,t)}} \rangle, \quad (\mathcal{M}_{t,2}^*)^{\Sigma_5} = 0,$$

with $q_{t,3}^* := \sum_{k=1}^{15} b_{t,k}$, $q_{t,4}^* := \sum_{k=16}^{45} b_{t,k}$.

Proof. Suppose $g \equiv_{\omega_{(5,t)}} \sum_{k=1}^{45} \gamma_k b_{t,k}$ with $\gamma_k \in \mathbb{F}_2$ and $[g]_{\omega_{(5,t)}} \in \Sigma_5(b_{t,1})^{\Sigma_5}$. By computing $\tau_i(g) + g$ in terms of $b_{t,k}$, $1 \leq i \leq 45$ and using the relations $\tau_i(g) + g \equiv_{\omega_{(5,t)}} 0$, $i = 1, 2, 3, 4$, we get

$$\begin{aligned} \gamma_1 &= \gamma_k, \quad \text{for } 2 \leq k \leq 15, \\ \gamma_{16} &= \gamma_k, \quad \text{for } 17 \leq k \leq 45. \end{aligned}$$

This implies $\Sigma_5(b_{t,1})^{\Sigma_5} = \langle [q_{t,3}^*], [q_{t,4}^*]_{\omega_{(5,t)}} \rangle$ with $q_{t,3}^* = \sum_{k=1}^{15} b_{t,k}$ and $q_{t,4}^* = \sum_{k=16}^{45} b_{t,k}$.

Now suppose $g \equiv_{\omega_{(5,t)}} \sum_{k=66}^{270} \gamma_k b_{t,k}$ and $[g]_{\omega_{(5,t)}} \in (\mathcal{M}_{t,2}^*)^{\Sigma_5}$. We compute $\tau_j(g) + g$ in terms of $b_{t,k}$, $66 \leq k \leq 270$. By computing directly from the relations $\tau_j(g) + g \equiv_{\omega_{(5,t)}} 0$, $j = 1, 2, 3, 4$, we obtain $\gamma_k = 0$ for $66 \leq k \leq 270$. The lemma is proved. $\square$

Combining this result and Lemmas 4.1.1, 4.1.2, 4.1.7, 4.1.8, one gets the following.

Proposition 4.1.9. For any $t \geq 4$,

$$QP_5(\omega_{(5,t)})^{\Sigma_5} = \langle [p_{t,1}]_{\omega_{(5,t)}}, [p_{t,2}]_{\omega_{(5,t)}}, [p_{t,3}]_{\omega_{(5,t)}}, [q_{t,1}^*]_{\omega_{(5,t)}}, [q_{t,2}^*]_{\omega_{(5,t)}}, [q_{t,3}^*]_{\omega_{(5,t)}}, [q_{t,4}^*]_{\omega_{(5,t)}} \rangle.$$

4.2. Computation of $QP_5(\omega_{(5,t)})^{GL_5}$

Let $g \in (P_5)_{3(2^t-1)+2^t}$ such that $[g]_{\omega_{(5,t)}} \in (QP_5)_{3(2^t-1)+2^t}^{GL_5}$. Then we have $[g]_{\omega_{(5,t)}} \in QP_5(\omega_{(5,t)})^{GL_5}$.

For $t = 1$, we have $3(2^1 - 1) + 2^1 = 5$ and $(QP_5)_5^{GL_5} = 0$ (see Sum [46, 50]). So, the theorem is true for $t = 1$.

For $t = 2$, from Proposition 4.1.4, we have $g \equiv_{\omega_{(5,2)}} \gamma_1 p_{2,1} + \gamma_2 p_{2,2} + \gamma_3 p_{2,3}$ with $\gamma_i \in \mathbb{F}_2$, $i = 1, 2, 3$. By computing $\tau_5(g) + g$ in terms of the admissible monomials, we get

$$\tau_5(g) + g \equiv_{\omega_{(5,2)}} (\gamma_1 + \gamma_2) x_1^3 x_2^3 x_4^3 x_5^4 + \gamma_2 x_1^3 x_2 x_3^4 x_4^3 x_5^2 + (\gamma_2 + \gamma_3) x_1 x_2^3 x_3^2 x_4 x_5^6 + \text{other terms} \equiv 0.$$

This relation implies $\gamma_i = 0$ for $1 \leq i \leq 3$. The theorem is proved for $t = 2$.

For $t = 3$, from Proposition 4.1.6, we have $g \equiv_{\omega_{(5,3)}} \sum_{j=1}^6 \beta_j p_{3,j}$ with $\beta_j \in \mathbb{F}_2$, $j = 1, 2, \dots, 6$. A direct computation shows that

$$\begin{aligned} \tau_5(g) + g &\equiv_{\omega_{(5,3)}} \beta_1 x_1^7 x_2^{15} x_5^7 + \beta_2 x_2 x_3^6 x_4^7 x_5^{15} + (\beta_2 + \beta_3) x_1 x_2^7 x_4^6 x_5^{15} + (\beta_2 + \beta_4) x_1 x_2 x_3^{14} x_4^7 x_5^6 \\ &\quad + \beta_5 x_1^3 x_2^{15} x_3^3 x_4^4 x_5^4 + \beta_6 x_1^7 x_2^7 x_3^9 x_4^2 x_5^4 + \text{other terms} \equiv 0. \end{aligned}$$

From this relation, we obtain $\beta_j = 0$, $1 \leq j \leq 6$. The theorem holds for $t = 3$.

For $t \geq 4$, by Proposition 4.1.9, we have

$$g \equiv_{\omega_{(5,t)}} \lambda_1 p_{t,1} + \lambda_2 p_{t,2} + \lambda_3 p_{t,3} + \sum_{j=1}^4 \lambda_{3+j} q_{t,j}^*,$$

where $\lambda_k \in \mathbb{F}_2$, $1 \leq k \leq 7$. By computing $\tau_5(g) + g$ in terms of the admissible monomials, we get

$$\begin{aligned} \tau_5(g) + g &\equiv_{\omega_{(5,t)}} \lambda_1 x_2^3 x_3^{2^t-3} x_4^{2^t-1} x_5^{2^{t+1}-2} + (\lambda_2 + \lambda_4) x_1^{2^t-1} x_3^3 x_4^{2^t-3} x_5^{2^{t+1}-2} \\ &\quad + \lambda_3 x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-1} x_4^{2^t-2} + \lambda_4 x_1^7 x_2^{2^{t+1}-5} x_4^{2^t-3} x_5^{2^t-2} \\ &\quad + (\lambda_1 + \lambda_2 + \lambda_5) x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^{t+1}-3} x_5^{2^t-2} \\ &\quad + \lambda_6 x_1^3 x_2^5 x_3^{2^{t+1}-6} x_4^{2^t-1} x_5^{2^t-4} + \lambda_7 x_1^7 x_2^{2^t-5} x_3^5 x_4^{2^t-6} x_5^{2^{t+1}-4} + \text{other terms} \equiv 0. \end{aligned}$$

The last equality implies $\lambda_k = 0$, $k = 1, 2, \dots, 7$. The theorem is completely proved.

5. Appendix

In this section, we list all the admissible monomials of degree $d = 3(2^t - 1) + 2^t$ in P_5 .

5.1. The admissible monomials of degree $3(2^t - 1) + 2^t$ in P_5^0

Recall that

$$\mathcal{B}_5^0(3(2^t - 1) + 2^t) = \mathcal{B}_5^0(\omega_{(5,t)}) = \Phi^0(\mathcal{B}_4(3(2^t - 1) + 2^t))$$

with $\omega_{(5,t)} = \underbrace{(3, 3, \dots, 3)}_{t \text{ times of } 3}, 1$. By Sum [47], we have

$$|\mathcal{B}_4(3(2^t - 1) + 2^t)| = \begin{cases} 15 & \text{if } t = 1, \\ 35 & \text{if } t = 2, \\ 45 & \text{if } t \geq 3. \end{cases}$$

Set $u_t = |\mathcal{B}_5^0(3(2^t - 1) + 2^t)|$ and $\mathcal{B}_5^0(3(2^t - 1) + 2^t) = \{q_{t,k} : 1 \leq k \leq u_t\}$. We have $u_1 = 45$, $u_2 = 145$, $u_t = 195$ for $t \geq 3$.

The admissible monomials $q_{t,k}$, $1 \leq k \leq u_t$, are determined as follows:

For $t \geq 1$,

1. $x_3^{2^t-1} x_4^{2^t-1} x_5^{2^{t+1}-1}$	2. $x_3^{2^t-1} x_4^{2^{t+1}-1} x_5^{2^t-1}$	3. $x_3^{2^{t+1}-1} x_4^{2^t-1} x_5^{2^t-1}$
4. $x_2^{2^t-1} x_4^{2^t-1} x_5^{2^{t+1}-1}$	5. $x_2^{2^t-1} x_4^{2^{t+1}-1} x_5^{2^t-1}$	6. $x_2^{2^t-1} x_3^{2^t-1} x_5^{2^{t+1}-1}$
7. $x_2^{2^t-1} x_3^{2^t-1} x_4^{2^{t+1}-1}$	8. $x_2^{2^t-1} x_3^{2^{t+1}-1} x_5^{2^t-1}$	9. $x_2^{2^t-1} x_3^{2^{t+1}-1} x_4^{2^t-1}$
10. $x_2^{2^{t+1}-1} x_4^{2^t-1} x_5^{2^t-1}$	11. $x_2^{2^{t+1}-1} x_3^{2^t-1} x_5^{2^t-1}$	12. $x_2^{2^{t+1}-1} x_3^{2^t-1} x_4^{2^t-1}$
13. $x_1^{2^t-1} x_4^{2^t-1} x_5^{2^{t+1}-1}$	14. $x_1^{2^t-1} x_4^{2^{t+1}-1} x_5^{2^t-1}$	15. $x_1^{2^t-1} x_3^{2^t-1} x_5^{2^{t+1}-1}$
16. $x_1^{2^t-1} x_3^{2^t-1} x_4^{2^{t+1}-1}$	17. $x_1^{2^t-1} x_3^{2^{t+1}-1} x_5^{2^t-1}$	18. $x_1^{2^t-1} x_3^{2^{t+1}-1} x_4^{2^t-1}$
19. $x_1^{2^t-1} x_2^{2^t-1} x_5^{2^{t+1}-1}$	20. $x_1^{2^t-1} x_2^{2^t-1} x_4^{2^{t+1}-1}$	21. $x_1^{2^t-1} x_2^{2^t-1} x_3^{2^{t+1}-1}$
22. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_5^{2^t-1}$	23. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_4^{2^t-1}$	24. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_3^{2^t-1}$
25. $x_1^{2^{t+1}-1} x_4^{2^t-1} x_5^{2^t-1}$	26. $x_1^{2^{t+1}-1} x_3^{2^t-1} x_5^{2^t-1}$	27. $x_1^{2^{t+1}-1} x_3^{2^t-1} x_4^{2^t-1}$
28. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_5^{2^t-1}$	29. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_4^{2^t-1}$	30. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_3^{2^t-1}$.

For $t = 1$,

31. $x_2 x_3 x_4 x_5^2$	32. $x_2 x_3 x_4^2 x_5$	33. $x_2 x_3^2 x_4 x_5$	34. $x_1 x_3 x_4 x_5^2$	35. $x_1 x_3 x_4^2 x_5$
36. $x_1 x_3^2 x_4 x_5$	37. $x_1 x_2 x_4 x_5^2$	38. $x_1 x_2 x_4^2 x_5$	39. $x_1 x_2 x_3 x_5^2$	40. $x_1 x_2 x_3 x_4^2$
41. $x_1 x_2 x_3^2 x_5$	42. $x_1 x_2 x_3^2 x_4$	43. $x_1 x_2^2 x_4 x_5$	44. $x_1 x_2^2 x_3 x_5$	45. $x_1 x_2^2 x_3 x_4$.

For $t \geq 2$,

31. $x_1 x_2^{2^t-2} x_3^{2^t-1} x_4^{2^{t+1}-1}$	32. $x_1 x_2^{2^t-2} x_3^{2^{t+1}-1} x_4^{2^t-1}$	33. $x_1 x_2^{2^t-1} x_3^{2^t-2} x_4^{2^{t+1}-1}$
34. $x_1 x_2^{2^t-1} x_3^{2^{t+1}-1} x_4^{2^t-2}$	35. $x_1 x_2^{2^{t+1}-1} x_3^{2^t-2} x_4^{2^t-1}$	36. $x_1 x_2^{2^{t+1}-1} x_3^{2^t-1} x_4^{2^t-2}$
37. $x_1^{2^t-1} x_2 x_3^{2^t-2} x_4^{2^{t+1}-1}$	38. $x_1^{2^t-1} x_2 x_3^{2^{t+1}-1} x_4^{2^t-2}$	39. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_3 x_4^{2^t-2}$
40. $x_1^{2^{t+1}-1} x_2 x_3^{2^t-2} x_4^{2^t-1}$	41. $x_1^{2^{t+1}-1} x_2 x_3^{2^t-1} x_4^{2^t-2}$	42. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_3 x_4^{2^t-2}$
43. $x_2^{2^t-2} x_3^{2^t-1} x_4^{2^{t+1}-1}$	44. $x_2^{2^t-2} x_3^{2^{t+1}-1} x_4^{2^t-1}$	45. $x_2^{2^t-1} x_3^{2^t-2} x_4^{2^{t+1}-1}$
46. $x_2^{2^t-1} x_3^{2^{t+1}-1} x_4^{2^t-2}$	47. $x_2^{2^{t+1}-1} x_3^{2^t-2} x_4^{2^t-1}$	48. $x_2^{2^{t+1}-1} x_3^{2^t-1} x_4^{2^t-2}$
49. $x_1^{2^t-2} x_2^{2^t-1} x_3^{2^{t+1}-1}$	50. $x_1^{2^t-2} x_2^{2^{t+1}-1} x_3^{2^t-1}$	51. $x_1^{2^t-2} x_2^{2^t-1} x_3^{2^{t+1}-1}$
52. $x_1^{2^t-2} x_2^{2^t-1} x_3^{2^{t+1}-1}$	53. $x_1^{2^t-2} x_2^{2^{t+1}-1} x_3^{2^t-1}$	54. $x_1^{2^t-2} x_2^{2^{t+1}-1} x_3^{2^t-1}$
55. $x_1^{2^t-1} x_2^{2^t-2} x_3^{2^{t+1}-1}$	56. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_3^{2^t-2}$	57. $x_1^{2^t-1} x_2^{2^t-2} x_3^{2^{t+1}-1}$
58. $x_1^{2^t-1} x_2^{2^t-2} x_3^{2^{t+1}-1}$	59. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_3^{2^t-2}$	60. $x_1^{2^t-1} x_2^{2^{t+1}-1} x_3^{2^t-2}$
61. $x_1^{2^{t+1}-1} x_2^{2^t-2} x_3^{2^t-1}$	62. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_3^{2^t-2}$	63. $x_1^{2^{t+1}-1} x_2^{2^t-2} x_3^{2^t-1}$
64. $x_1^{2^{t+1}-1} x_2^{2^t-2} x_3^{2^t-1}$	65. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_3^{2^t-2}$	66. $x_1^{2^{t+1}-1} x_2^{2^t-1} x_3^{2^t-2}$
67. $x_2 x_3^{2^t-2} x_4^{2^{t+1}-1}$	68. $x_2 x_3^{2^{t+1}-1} x_4^{2^t-2}$	69. $x_2^{2^{t+1}-1} x_3 x_4^{2^t-2}$
70. $x_1 x_3^{2^t-2} x_4^{2^{t+1}-1}$	71. $x_1 x_3^{2^{t+1}-1} x_4^{2^t-2}$	72. $x_1 x_2^{2^t-2} x_3^{2^{t+1}-1}$
73. $x_1 x_2^{2^t-2} x_3^{2^{t+1}-1}$	74. $x_1 x_2^{2^{t+1}-1} x_3^{2^t-2}$	75. $x_1 x_2^{2^{t+1}-1} x_3^{2^t-2}$
76. $x_1^{2^{t+1}-1} x_3 x_4^{2^t-2}$	77. $x_1^{2^{t+1}-1} x_2 x_4^{2^t-2}$	78. $x_1^{2^{t+1}-1} x_2 x_3^{2^t-2}$
79. $x_2 x_3^{2^t-2} x_4^{2^t-1}$	80. $x_2 x_3^{2^t-1} x_4^{2^t-2}$	81. $x_2^{2^t-1} x_3 x_4^{2^t-2}$
82. $x_1 x_3^{2^t-2} x_4^{2^t-1}$	83. $x_1 x_3^{2^t-1} x_4^{2^t-2}$	84. $x_1 x_2^{2^t-2} x_3^{2^t-1}$
85. $x_1 x_2^{2^t-2} x_3^{2^t-1}$	86. $x_1 x_2^{2^t-1} x_3^{2^t-2}$	87. $x_1 x_2^{2^t-1} x_3^{2^t-2}$
88. $x_1^{2^t-1} x_3 x_4^{2^t-2}$	89. $x_1^{2^t-1} x_2 x_4^{2^t-2}$	90. $x_1^{2^t-1} x_2 x_3^{2^t-2}$

For $t = 2$,

91. $x_2 x_3^3 x_4^3 x_5^6$	92. $x_2 x_3^3 x_4^3 x_5^3$	93. $x_2 x_3^6 x_4^3 x_5^3$	94. $x_2^3 x_3 x_4^3 x_5^6$	95. $x_2^3 x_3 x_4^6 x_5^3$
96. $x_2^3 x_3^3 x_4^3 x_5^6$	97. $x_1 x_3^3 x_4^3 x_5^6$	98. $x_1 x_3^3 x_4^6 x_5^3$	99. $x_1 x_3^6 x_4^3 x_5^3$	100. $x_1 x_2^3 x_4^3 x_5^6$
101. $x_1 x_2^3 x_4^6 x_5^3$	102. $x_1 x_2^3 x_3^3 x_5^6$	103. $x_1 x_2^3 x_3^3 x_4^6$	104. $x_1 x_2^3 x_3^6 x_5^3$	105. $x_1 x_2^3 x_3^6 x_4^3$
106. $x_1 x_2^6 x_4^3 x_5^3$	107. $x_1 x_2^6 x_3^3 x_5^3$	108. $x_1 x_2^6 x_3^3 x_4^3$	109. $x_1^3 x_3 x_4^3 x_5^6$	110. $x_1^3 x_3 x_4^6 x_5^3$
111. $x_1^3 x_3^3 x_4^6 x_5^3$	112. $x_1^3 x_2 x_4^6 x_5^3$	113. $x_1^3 x_2 x_4^6 x_5^3$	114. $x_1^3 x_2 x_3^3 x_5^6$	115. $x_1^3 x_2 x_3^6 x_4^3$
116. $x_1^3 x_2 x_3^6 x_5^3$	117. $x_1^3 x_2 x_3^6 x_4^3$	118. $x_1^3 x_2^3 x_4^6 x_5^3$	119. $x_1^3 x_2^3 x_3^6 x_5^3$	120. $x_1^3 x_2^3 x_3^6 x_4^3$
121. $x_2^3 x_3^3 x_4^5 x_5^2$	122. $x_2^3 x_3^5 x_4^3 x_5^3$	123. $x_2^3 x_3^5 x_4^3 x_5^2$	124. $x_1^3 x_3^3 x_4^5 x_5^2$	125. $x_1^3 x_3^5 x_4^3 x_5^3$
126. $x_1^3 x_3^3 x_4^5 x_5^2$	127. $x_1^3 x_2^3 x_4^5 x_5^2$	128. $x_1^3 x_2^3 x_3^5 x_5^2$	129. $x_1^3 x_2^3 x_3^5 x_4^2$	130. $x_1^3 x_2^5 x_4^3 x_5^3$
131. $x_1^3 x_2^5 x_4^3 x_5^3$	132. $x_1^3 x_2^5 x_3^3 x_5^3$	133. $x_1^3 x_2^5 x_3^3 x_4^3$	134. $x_1^3 x_2^5 x_3^3 x_5^2$	135. $x_1^3 x_2^5 x_3^3 x_4^2$
136. $x_2^3 x_3^3 x_4^3 x_5^4$	137. $x_2^3 x_3^3 x_4^4 x_5^3$	138. $x_1^3 x_3^3 x_4^4 x_5^3$	139. $x_1^3 x_3^3 x_4^4 x_5^3$	140. $x_1^3 x_2^3 x_4^4 x_5^3$
141. $x_1^3 x_2^3 x_4^4 x_5^3$	142. $x_1^3 x_2^3 x_3^4 x_5^3$	143. $x_1^3 x_2^3 x_3^4 x_4^3$	144. $x_1^3 x_2^3 x_3^4 x_5^3$	145. $x_1^3 x_2^3 x_3^4 x_4^3$

For $t \geq 3$,

91. $x_2^3 x_3^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-1}$	92. $x_2^3 x_3^{2^t-3} x_4^{2^{t+1}-1} x_5^{2^t-2}$	93. $x_2^3 x_3^{2^{t+1}-1} x_4^{2^t-3} x_5^{2^t-2}$
94. $x_2^{2^{t+1}-1} x_3^3 x_4^{2^t-3} x_5^{2^t-2}$	95. $x_1^3 x_3^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-1}$	96. $x_1^3 x_3^{2^t-3} x_4^{2^{t+1}-1} x_5^{2^t-2}$
97. $x_1^3 x_3^{2^{t+1}-1} x_4^{2^t-3} x_5^{2^t-2}$	98. $x_1^3 x_2^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-1}$	99. $x_1^3 x_2^{2^t-3} x_4^{2^{t+1}-1} x_5^{2^t-2}$
100. $x_1^3 x_2^{2^t-3} x_3^3 x_5^{2^{t+1}-1}$	101. $x_1^3 x_2^{2^t-3} x_3^3 x_4^{2^{t+1}-1}$	102. $x_1^3 x_2^{2^t-3} x_3^3 x_5^{2^t-2}$
103. $x_1^3 x_2^{2^t-3} x_3^{2^{t+1}-1} x_4^{2^t-2}$	104. $x_1^3 x_2^{2^{t+1}-1} x_4^{2^t-3} x_5^{2^t-2}$	105. $x_1^3 x_2^{2^{t+1}-1} x_3^3 x_5^{2^t-2}$
106. $x_1^3 x_2^{2^{t+1}-1} x_3^{2^t-3} x_4^{2^t-2}$	107. $x_1^{2^{t+1}-1} x_3^3 x_4^{2^t-3} x_5^{2^t-2}$	108. $x_1^{2^{t+1}-1} x_2^3 x_4^{2^t-3} x_5^{2^t-2}$
109. $x_1^{2^{t+1}-1} x_2^3 x_3^3 x_5^{2^t-2}$	110. $x_1^{2^{t+1}-1} x_2^3 x_3^3 x_4^{2^t-2}$	

For $t = 3$,

111. $x_2x_3^7x_4^7x_5^{14}$	112. $x_2^7x_3x_4^7x_5^{14}$	113. $x_1x_3^7x_4^7x_5^{14}$	114. $x_1x_2^7x_4^7x_5^{14}$	115. $x_1x_2^7x_3^7x_5^{14}$
116. $x_1x_2^7x_3^7x_4^{14}$	117. $x_1^7x_3x_4^7x_5^{14}$	118. $x_1^7x_2x_4^7x_5^{14}$	119. $x_1^7x_2x_3^7x_5^{14}$	120. $x_1^7x_2x_3^7x_4^{14}$
121. $x_3^3x_3^5x_4^7x_5^{14}$	122. $x_3^3x_3^5x_4^7x_5^{14}$	123. $x_1^3x_3^5x_4^7x_5^{14}$	124. $x_1^3x_3^5x_4^7x_5^{14}$	125. $x_1^3x_2^5x_4^7x_5^{14}$
126. $x_1^3x_2^5x_3^7x_5^{14}$	127. $x_1^3x_2^5x_3^7x_4^7$	128. $x_1^3x_2^5x_4^7x_5^{14}$	129. $x_1^3x_2^5x_3^7x_5^{14}$	130. $x_1^3x_2^5x_3^7x_4^{14}$
131. $x_2^3x_3^7x_4^{13}x_5^6$	132. $x_1^3x_3^7x_4^{13}x_5^6$	133. $x_1^3x_2^7x_4^{13}x_5^6$	134. $x_1^3x_2^7x_3^{13}x_5^6$	135. $x_1^3x_2^7x_3^{13}x_4^6$
136. $x_2x_3^7x_4^{14}x_5^7$	137. $x_2x_3^{14}x_4^7x_5^7$	138. $x_2^7x_3x_4^{14}x_5^7$	139. $x_2^7x_3x_4^{14}x_5^7$	140. $x_1x_3^7x_4^{14}x_5^7$
141. $x_1x_3^{14}x_4^7x_5^7$	142. $x_1x_2^7x_4^{14}x_5^7$	143. $x_1x_2^7x_3^{14}x_5^7$	144. $x_1x_2^7x_3^{14}x_4^7$	145. $x_1x_2^{14}x_4^7x_5^7$
146. $x_1x_2^{14}x_3^7x_5^7$	147. $x_1x_2^{14}x_3^7x_4^7$	148. $x_1^7x_3x_4^{14}x_5^7$	149. $x_1^7x_3x_4^{14}x_5^7$	150. $x_1^7x_2x_4^{14}x_5^7$
151. $x_1^7x_2x_3^{14}x_5^7$	152. $x_1^7x_2x_3^{14}x_4^7$	153. $x_1^7x_2^7x_4^{14}x_5^{14}$	154. $x_1^7x_2^7x_3^5x_5^{14}$	155. $x_1^7x_2^7x_3^5x_4^{14}$
156. $x_3^3x_3^5x_4^7x_5^{14}$	157. $x_2^7x_3^5x_4^7x_5^{14}$	158. $x_1^3x_3^5x_7^7x_5^{14}$	159. $x_1^3x_2^5x_4^7x_5^{14}$	160. $x_1^3x_2^5x_3^7x_5^{14}$
161. $x_1^3x_2^5x_3^7x_4^{14}$	162. $x_1^7x_3^5x_4^7x_5^{14}$	163. $x_1^7x_3^5x_4^7x_5^{14}$	164. $x_1^7x_3^5x_4^7x_5^{14}$	165. $x_1^7x_3^5x_4^7x_5^{14}$
166. $x_2^3x_3^{13}x_4^6x_5^7$	167. $x_2^3x_3^{13}x_4^6x_5^7$	168. $x_2^7x_3^3x_4^{13}x_5^6$	169. $x_1^3x_3^{13}x_4^6x_5^7$	170. $x_1^3x_3^{13}x_4^6x_5^7$
171. $x_1^3x_2^{13}x_4^6x_5^7$	172. $x_1^3x_2^{13}x_4^6x_5^7$	173. $x_1^3x_2^{13}x_3^6x_5^7$	174. $x_1^3x_2^{13}x_3^6x_4^7$	175. $x_1^3x_2^{13}x_3^6x_5^6$
176. $x_2^3x_2^{13}x_3^6x_4^6$	177. $x_1^7x_3^3x_4^{13}x_5^6$	178. $x_1^7x_3^3x_4^{13}x_5^6$	179. $x_1^7x_2^3x_3^{13}x_5^6$	180. $x_1^7x_2^3x_3^{13}x_4^6$
181. $x_2^7x_3^{11}x_4^5x_6^6$	182. $x_1^7x_3^{11}x_4^5x_6^6$	183. $x_1^7x_2^{11}x_4^5x_6^6$	184. $x_1^7x_2^{11}x_3^5x_6^6$	185. $x_1^7x_2^{11}x_3^5x_4^6$
186. $x_2^7x_3^7x_4^9x_5^6$	187. $x_1^7x_3^7x_4^9x_5^6$	188. $x_1^7x_2^7x_4^9x_5^6$	189. $x_1^7x_2^7x_3^9x_5^6$	190. $x_1^7x_2^7x_3^9x_4^6$
191. $x_2^7x_3^7x_4^8x_5^8$	192. $x_1^7x_3^7x_4^8x_5^8$	193. $x_1^7x_2^7x_4^8x_5^8$	194. $x_1^7x_2^7x_3^8x_5^8$	195. $x_1^7x_2^7x_3^8x_4^8$

For $t \geq 4$,

111. $x_2^3x_3^{2^t-3}x_4^{2^t-1}x_5^{2^{t+1}-2}$	112. $x_2^3x_3^{2^t-3}x_4^{2^{t+1}-2}x_5^{2^t-1}$	113. $x_2^3x_3^{2^t-1}x_4^{2^t-3}x_5^{2^{t+1}-2}$
114. $x_2^{2^t-1}x_3^3x_4^{2^t-3}x_5^{2^{t+1}-2}$	115. $x_1^3x_3^{2^t-3}x_4^{2^t-1}x_5^{2^{t+1}-2}$	116. $x_1^3x_3^{2^t-3}x_4^{2^{t+1}-2}x_5^{2^t-1}$
117. $x_1^3x_3^{2^t-1}x_4^{2^t-3}x_5^{2^{t+1}-2}$	118. $x_1^3x_2^{2^t-3}x_4^{2^t-1}x_5^{2^{t+1}-2}$	119. $x_1^3x_2^{2^t-3}x_4^{2^{t+1}-2}x_5^{2^t-1}$
120. $x_1^3x_2^{2^t-3}x_3^{2^t-1}x_5^{2^{t+1}-2}$	121. $x_1^3x_2^{2^t-3}x_3^{2^t-1}x_4^{2^{t+1}-2}$	122. $x_1^3x_2^{2^t-3}x_3^{2^{t+1}-2}x_5^{2^t-1}$
123. $x_1^3x_2^{2^t-3}x_3^{2^{t+1}-2}x_4^{2^t-1}$	124. $x_1^3x_2^{2^t-1}x_4^{2^t-3}x_5^{2^{t+1}-2}$	125. $x_1^3x_2^{2^t-1}x_3^{2^t-3}x_5^{2^{t+1}-2}$
126. $x_1^3x_2^{2^t-1}x_3^{2^t-3}x_4^{2^{t+1}-2}$	127. $x_1^2x_2^{2^t-1}x_3^3x_4^{2^t-3}x_5^{2^{t+1}-2}$	128. $x_1^2x_2^{2^t-1}x_3^3x_4^{2^t-3}x_5^{2^{t+1}-2}$
129. $x_1^{2^t-1}x_2^3x_3^{2^t-3}x_5^{2^{t+1}-2}$	130. $x_1^{2^t-1}x_2^3x_3^{2^t-3}x_4^{2^{t+1}-2}$	131. $x_2^7x_3^{2^t-5}x_4^{2^{t+1}-3}x_5^{2^t-2}$
132. $x_1^7x_3^{2^t-5}x_4^{2^{t+1}-3}x_5^{2^t-2}$	133. $x_1^7x_2^{2^t-5}x_4^{2^{t+1}-3}x_5^{2^t-2}$	134. $x_1^7x_2^{2^t-5}x_3^{2^{t+1}-3}x_5^{2^t-2}$
135. $x_1^7x_2^{2^t-5}x_3^{2^{t+1}-3}x_4^{2^t-2}$	136. $x_2^7x_3^{2^t-1}x_4^{2^t-1}x_5^{2^{t+1}-2}$	137. $x_2^7x_3^{2^t-1}x_4^{2^{t+1}-2}x_5^{2^t-1}$
138. $x_2x_3^{2^{t+1}-2}x_4^{2^t-1}x_5^{2^t-1}$	139. $x_2^{2^t-1}x_3x_4^{2^t-1}x_5^{2^{t+1}-2}$	140. $x_2^{2^t-1}x_3x_4^{2^{t+1}-2}x_5^{2^t-1}$
141. $x_2^{2^t-1}x_3^{2^t-1}x_4x_5^{2^{t+1}-2}$	142. $x_1x_3^{2^t-1}x_4^{2^t-1}x_5^{2^{t+1}-2}$	143. $x_1x_3^{2^t-1}x_4^{2^{t+1}-2}x_5^{2^t-1}$
144. $x_1x_3^{2^{t+1}-2}x_4^{2^t-1}x_5^{2^t-1}$	145. $x_1x_2^{2^t-1}x_4^{2^t-1}x_5^{2^{t+1}-2}$	146. $x_1x_2^{2^t-1}x_4^{2^{t+1}-2}x_5^{2^t-1}$
147. $x_1x_2^{2^t-1}x_3^{2^t-1}x_5^{2^{t+1}-2}$	148. $x_1x_2^{2^t-1}x_3^{2^t-1}x_4^{2^{t+1}-2}$	149. $x_1x_2^{2^t-1}x_3^{2^{t+1}-2}x_5^{2^t-1}$
150. $x_1x_2^{2^t-1}x_3^{2^{t+1}-2}x_4^{2^t-1}$	151. $x_1x_2^{2^{t+1}-2}x_4^{2^t-1}x_5^{2^t-1}$	152. $x_1x_2^{2^{t+1}-2}x_3^{2^t-1}x_5^{2^t-1}$
153. $x_1x_2^{2^{t+1}-2}x_3^{2^t-1}x_4^{2^t-1}$	154. $x_1^{2^t-1}x_3x_4^{2^t-1}x_5^{2^{t+1}-2}$	155. $x_1^{2^t-1}x_3x_4^{2^{t+1}-2}x_5^{2^t-1}$
156. $x_1^{2^t-1}x_3^{2^t-1}x_4x_5^{2^{t+1}-2}$	157. $x_1^{2^t-1}x_2x_4^{2^t-1}x_5^{2^{t+1}-2}$	158. $x_1^{2^t-1}x_2x_4^{2^{t+1}-2}x_5^{2^t-1}$
159. $x_1^{2^t-1}x_2x_3^{2^t-1}x_5^{2^{t+1}-2}$	160. $x_1^{2^t-1}x_2x_3^{2^t-1}x_4^{2^{t+1}-2}$	161. $x_1^{2^t-1}x_2x_3^{2^{t+1}-2}x_5^{2^t-1}$
162. $x_1^{2^t-1}x_2x_3^{2^{t+1}-2}x_4^{2^t-1}$	163. $x_1^{2^t-1}x_2^{2^t-1}x_4x_5^{2^{t+1}-2}$	164. $x_1^{2^t-1}x_2^{2^t-1}x_3x_5^{2^{t+1}-2}$
165. $x_1^{2^t-1}x_2^{2^t-1}x_3x_4^{2^{t+1}-2}$	166. $x_2^3x_3^{2^t-1}x_4^{2^{t+1}-3}x_5^{2^t-2}$	167. $x_2^3x_3^{2^{t+1}-3}x_4^{2^t-2}x_5^{2^t-1}$
168. $x_2^3x_3^{2^{t+1}-3}x_4^{2^t-1}x_5^{2^t-2}$	169. $x_2^{2^t-1}x_3^3x_4^{2^{t+1}-3}x_5^{2^t-2}$	170. $x_1^3x_3^{2^t-1}x_4^{2^{t+1}-3}x_5^{2^t-2}$
171. $x_1^3x_3^{2^{t+1}-3}x_4^{2^t-2}x_5^{2^t-1}$	172. $x_1^3x_3^{2^{t+1}-3}x_4^{2^t-1}x_5^{2^t-2}$	173. $x_1^3x_2^{2^t-1}x_4^{2^{t+1}-3}x_5^{2^t-2}$
174. $x_1^3x_2^{2^t-1}x_3^{2^{t+1}-3}x_5^{2^t-2}$	175. $x_1^3x_2^{2^t-1}x_3^{2^{t+1}-3}x_4^{2^t-2}$	176. $x_1^3x_2^{2^{t+1}-3}x_4^{2^t-2}x_5^{2^t-1}$

$$\begin{array}{lll}
177. x_1^3 x_2^{2^{t+1}-3} x_4^{2^t-1} x_5^{2^t-2} & 178. x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-2} x_5^{2^t-1} & 179. x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-2} x_4^{2^t-1} \\
180. x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-1} x_5^{2^t-2} & 181. x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-1} x_4^{2^t-2} & 182. x_1^{2^t-1} x_3^3 x_4^{2^{t+1}-3} x_5^{2^t-2} \\
183. x_1^{2^t-1} x_2^3 x_4^{2^{t+1}-3} x_5^{2^t-2} & 184. x_1^{2^t-1} x_2^3 x_3^{2^{t+1}-3} x_5^{2^t-2} & 185. x_1^{2^t-1} x_2^3 x_3^{2^{t+1}-3} x_4^{2^t-2} \\
186. x_1^7 x_2^{2^t-5} x_4^{2^t-3} x_5^{2^{t+1}-2} & 187. x_1^7 x_2^{2^t-5} x_4^{2^t-3} x_5^{2^{t+1}-2} & 188. x_1^7 x_2^{2^t-5} x_4^{2^t-3} x_5^{2^{t+1}-2} \\
189. x_1^7 x_2^{2^t-5} x_3^{2^t-3} x_5^{2^{t+1}-2} & 190. x_1^7 x_2^{2^t-5} x_3^{2^t-3} x_4^{2^{t+1}-2} & 191. x_1^7 x_2^{2^{t+1}-5} x_4^{2^t-3} x_5^{2^t-2} \\
192. x_1^7 x_2^{2^{t+1}-5} x_4^{2^t-3} x_5^{2^t-2} & 193. x_1^7 x_2^{2^{t+1}-5} x_4^{2^t-3} x_5^{2^t-2} & 194. x_1^7 x_2^{2^{t+1}-5} x_3^{2^t-3} x_5^{2^t-2} \\
195. x_1^7 x_2^{2^{t+1}-5} x_3^{2^t-3} x_4^{2^t-2} & &
\end{array}$$

5.2. The admissible monomials of degree $3(2^t - 1) + 2^t$ in P_5^+

As it is known,

$$\mathcal{B}_5(3(2^t - 1) + 2^t) = \mathcal{B}_5^0(3(2^t - 1) + 2^t) \bigcup \varphi(\mathcal{B}_5(2^{t+1} - 4)) \bigcup B,$$

where $B = (\mathcal{B}_5^+(3(2^t - 1) + 2^t) \cap \text{Ker}((\widetilde{Sq}_*)_{(5, 3(2^t-1)+2^t)})) = \mathcal{B}_5^+(\omega_{(5,t)})$, and $\varphi : P_5 \rightarrow P_5$, $\varphi(y) = x_1 x_2 x_3 x_4 x_5 y^2$, $\forall y \in P_5$. We have

$$|\mathcal{B}_5^0(3(2^t - 1) + 2^t)| = \begin{cases} 45 & \text{if } t = 1, \\ 145 & \text{if } t = 2, \\ 195 & \text{if } t \geq 3, \end{cases} \quad \text{and } |\mathcal{B}_5(2^{t+1} - 4)| = \begin{cases} 1 & \text{if } t = 1, \\ 45 & \text{if } t = 2, \\ 190 & \text{if } t = 3, \\ 480 & \text{if } t = 4, \\ 650 & \text{if } t = 5, \\ 651 & \text{if } t \geq 6. \end{cases}$$

Set $v_t = |\mathcal{B}_5^+(\omega_{(5,t)})|$ and $\mathcal{B}_5^+(\omega_{(5,t)}) = \{b_{3(2^t-1)+2^t,k} : 1 \leq k \leq v_t\}$. Then, $v_1 = 1$, $v_2 = 60$, $v_3 = 260$, $v_t = 270$ for $t \geq 4$.

For $t = 1$, according to Sum [46, 50], $\mathcal{B}_5^+(\omega_{(5,1)}) = \{b_{5,1} = x_1 x_2 x_3 x_4 x_5\}$.

For $t \geq 2$, the admissible monomials $b_{3(2^t-1)+2^t,k}$, $1 \leq k \leq v_t$, are determined as follows:

For $t = 2$,

$$\begin{array}{llll}
1. x_1 x_2 x_3^2 x_4^2 x_5^7 & 2. x_1 x_2 x_3^2 x_4^7 x_5^2 & 3. x_1 x_2 x_3^7 x_4^2 x_5^2 & 4. x_1 x_2^2 x_3 x_4^2 x_5^7 \\
5. x_1 x_2^2 x_3 x_4^7 x_5^2 & 6. x_1 x_2^2 x_3^7 x_4^2 x_5^2 & 7. x_1 x_2^7 x_3 x_4^2 x_5^2 & 8. x_1 x_2^7 x_3^2 x_4^2 x_5^2 \\
9. x_1^7 x_2 x_3 x_4^2 x_5^2 & 10. x_1^7 x_2 x_3^2 x_4^2 x_5^2 & 11. x_1 x_2 x_3^2 x_4^3 x_5^6 & 12. x_1 x_2 x_3^2 x_4^6 x_5^3 \\
13. x_1 x_2 x_3^3 x_4^2 x_5^6 & 14. x_1 x_2 x_3^3 x_4^6 x_5^2 & 15. x_1 x_2 x_3^6 x_4^2 x_5^3 & 16. x_1 x_2 x_3^6 x_4^3 x_5^2 \\
17. x_1 x_2^2 x_3 x_4^3 x_5^6 & 18. x_1 x_2^2 x_3 x_4^6 x_5^3 & 19. x_1 x_2^2 x_3^3 x_4^2 x_5^6 & 20. x_1 x_2^2 x_3^3 x_4^6 x_5^2 \\
21. x_1 x_2^2 x_3 x_4^6 x_5^2 & 22. x_1 x_2^2 x_3^2 x_4^6 x_5^2 & 23. x_1 x_2^2 x_3^6 x_4^2 x_5^2 & 24. x_1 x_2^2 x_3^6 x_4^3 x_5^2 \\
25. x_1 x_2^6 x_3 x_4^2 x_5^2 & 26. x_1 x_2^6 x_3^3 x_4^2 x_5^2 & 27. x_1^3 x_2 x_3 x_4^2 x_5^6 & 28. x_1^3 x_2 x_3 x_4^6 x_5^2 \\
29. x_1^3 x_2 x_3^2 x_4^2 x_5^6 & 30. x_1^3 x_2 x_3^6 x_4^2 x_5^2 & 31. x_1 x_2^2 x_3^3 x_4^5 x_5^2 & 32. x_1 x_2^2 x_3^5 x_4^2 x_5^3 \\
33. x_1 x_2^2 x_3^5 x_4^2 x_5^2 & 34. x_1 x_2^3 x_3^2 x_4^5 x_5^2 & 35. x_1 x_2^3 x_3^5 x_4^2 x_5^2 & 36. x_1^3 x_2 x_3^2 x_4^5 x_5^2 \\
37. x_1^3 x_2 x_3^5 x_4^2 x_5^2 & 38. x_1^3 x_2^5 x_3 x_4^2 x_5^2 & 39. x_1^3 x_2^5 x_3^2 x_4^2 x_5^2 & 40. x_1 x_2^2 x_3^3 x_4^4 x_5^3 \\
41. x_1 x_2^2 x_3^3 x_4^4 x_5^3 & 42. x_1 x_2^2 x_3^4 x_4^3 x_5^3 & 43. x_1 x_2^3 x_3^2 x_4^4 x_5^3 & 44. x_1 x_2^3 x_3^2 x_4^4 x_5^3 \\
45. x_1 x_2^3 x_3^4 x_4^2 x_5^2 & 46. x_1 x_2^3 x_3^4 x_4^2 x_5^2 & 47. x_1 x_2^3 x_3^4 x_4^2 x_5^2 & 48. x_1 x_2^3 x_3^4 x_4^2 x_5^2 \\
49. x_1^3 x_2 x_3^2 x_4^4 x_5^2 & 50. x_1^3 x_2 x_3^2 x_4^4 x_5^2 & 51. x_1^3 x_2 x_3^2 x_4^4 x_5^2 & 52. x_1^3 x_2 x_3^2 x_4^4 x_5^2 \\
53. x_1^3 x_2 x_3^4 x_4^2 x_5^2 & 54. x_1^3 x_2 x_3^4 x_4^2 x_5^2 & 55. x_1^3 x_2^3 x_3 x_4^2 x_5^2 & 56. x_1^3 x_2^3 x_3 x_4^2 x_5^2 \\
57. x_1^3 x_2^3 x_3^4 x_4^2 x_5^2 & 58. x_1^3 x_2^4 x_3 x_4^2 x_5^2 & 59. x_1^3 x_2^4 x_3 x_4^2 x_5^2 & 60. x_1^3 x_2^4 x_3 x_4^2 x_5^2.
\end{array}$$

For $t = 3$,

1. $x_1x_2x_3^6x_4^6x_5^{15}$
5. $x_1^3x_2^3x_3^4x_4^4x_5^{15}$
9. $x_1^{15}x_2x_3x_4^6x_5^6$
13. $x_1x_2^2x_3^{15}x_4^5x_5^6$
17. $x_1x_2^3x_3^6x_4^{15}x_5^4$
21. $x_1x_2^6x_3x_4^{15}x_5^6$
25. $x_1x_2^{15}x_3^6x_4^5x_5^4$
29. $x_1^3x_2x_3^6x_4^4x_5^{15}$
33. $x_1^3x_2^5x_3^2x_4^4x_5^{15}$
37. $x_1^3x_2^{15}x_3x_4^4x_5^4$
41. $x_1^{15}x_2x_3^3x_4^6x_5^4$
45. $x_1^{15}x_2^3x_3^5x_4^4x_5^4$
49. $x_1x_2^2x_3^7x_4^{15}x_5^4$
53. $x_1x_2^7x_3^2x_4^{15}x_5^4$
57. $x_1x_2^{15}x_3^7x_4^2x_5^4$
61. $x_1^7x_2^{15}x_3x_4^2x_5^4$
65. $x_1^{15}x_2^7x_3x_4^2x_5^4$
69. $x_1x_2x_3^7x_4^{14}x_5^6$
73. $x_1x_2^2x_3^5x_4^{14}x_5^7$
77. $x_1x_2^2x_3^7x_4^{13}x_5^6$
81. $x_1x_2^3x_3^4x_7^4x_5^{14}$
85. $x_1x_2^3x_3^6x_4^5x_5^{14}$
89. $x_1x_2^3x_3^7x_4^4x_5^{14}$
93. $x_1x_2^3x_3^{12}x_4^4x_5^7$
97. $x_1x_2^3x_3^{14}x_4^5x_5^6$
101. $x_1x_2^6x_3^3x_4^5x_5^{14}$
105. $x_1x_2^6x_3^7x_4^9x_5^6$
109. $x_1x_2^7x_3^2x_4^5x_5^{14}$
113. $x_1x_2^7x_3^3x_4^4x_5^{14}$
117. $x_1x_2^7x_3^6x_4^4x_5^{14}$
121. $x_1x_2^7x_3^6x_4^8x_5^8$
125. $x_1x_2^7x_3^{11}x_4^4x_5^6$
129. $x_1x_2^{14}x_3x_4^7x_5^6$
133. $x_1^3x_2x_3^4x_4^{14}x_5^7$
137. $x_1^3x_2x_3^6x_4^7x_5^{12}$
141. $x_1^3x_2x_3^6x_4^{12}x_5^6$
145. $x_1^3x_2x_3^{12}x_4^7x_5^6$
149. $x_1^3x_2x_3^{14}x_4^4x_5^4$
153. $x_1^3x_2^2x_3^5x_4^4x_5^{14}$
157. $x_1^3x_2^3x_3^4x_4^{12}x_5^6$
161. $x_1^3x_2^3x_3^{13}x_4^6x_5^4$
165. $x_1^3x_2^5x_3x_4^6x_5^{14}$
169. $x_1^3x_2^5x_3^2x_4^{12}x_5^7$
173. $x_1^3x_2^5x_3^3x_4^{12}x_5^6$
177. $x_1^3x_2^5x_3^6x_4^8x_5^8$
181. $x_1^3x_2^5x_3^7x_4^8x_5^6$
2. $x_1x_2x_3^6x_4^{15}x_5^6$
6. $x_1^3x_2^3x_3^4x_4^{15}x_5^4$
10. $x_1^{15}x_2^3x_3^3x_4^4x_5^4$
14. $x_1x_2^3x_3^3x_4^6x_5^{15}$
18. $x_1x_2^3x_3^{15}x_4^4x_5^6$
22. $x_1x_2^6x_3^{15}x_4^4x_5^6$
26. $x_1x_2^{15}x_3^6x_4^4x_5^6$
30. $x_1^3x_2x_3^6x_4^{15}x_5^4$
34. $x_1^3x_2^5x_3^2x_4^{15}x_5^4$
38. $x_1^3x_2^{15}x_3^5x_4^4x_5^4$
42. $x_1^{15}x_2x_3^6x_4^4x_5^6$
46. $x_1x_2^2x_3^4x_4^{15}x_5^{15}$
50. $x_1x_2^2x_3^{15}x_4^4x_5^7$
54. $x_1x_2^7x_3^{15}x_4^2x_5^4$
58. $x_1^7x_2x_3^2x_4^4x_5^{15}$
62. $x_1^{15}x_2x_3^2x_4^4x_5^7$
66. $x_1x_2x_3^6x_4^7x_5^{14}$
70. $x_1x_2x_3^{14}x_4^6x_5^7$
74. $x_1x_2^2x_3^7x_4^5x_5^{14}$
78. $x_1x_2^2x_3^{12}x_4^7x_5^7$
82. $x_1x_2^3x_3^4x_4^{14}x_5^7$
86. $x_1x_2^3x_3^6x_4^5x_5^{12}$
90. $x_1x_2^3x_3^7x_4^5x_5^{12}$
94. $x_1x_2^3x_3^{12}x_4^6x_5^6$
98. $x_1x_2^3x_3^{14}x_4^7x_5^4$
102. $x_1x_2^6x_3^3x_4^{13}x_5^6$
106. $x_1x_2^6x_3^{11}x_4^5x_5^6$
110. $x_1x_2^7x_3^2x_4^5x_5^{12}$
114. $x_1x_2^7x_3^3x_4^4x_5^{12}$
118. $x_1x_2^7x_3^6x_4^4x_5^8$
122. $x_1x_2^7x_3^7x_4^6x_5^6$
126. $x_1x_2^7x_3^{11}x_4^6x_5^4$
130. $x_1x_2^{14}x_3^3x_4^5x_5^6$
134. $x_1^3x_2x_3^5x_4^{14}x_5^{14}$
138. $x_1^3x_2x_3^6x_4^{12}x_5^7$
142. $x_1^3x_2x_3^{13}x_4^{12}x_5^6$
146. $x_1^3x_2x_3^{13}x_4^6x_5^6$
150. $x_1^3x_2^2x_3^4x_4^5x_5^{14}$
154. $x_1^3x_2^2x_3^5x_4^4x_5^{12}$
158. $x_1^3x_2^3x_3^7x_4^{12}x_5^4$
162. $x_1^3x_2^3x_3^{13}x_4^5x_5^{14}$
166. $x_1^3x_2^5x_3x_4^{14}x_5^6$
170. $x_1^3x_2^5x_3^2x_4^{13}x_5^6$
174. $x_1^3x_2^5x_3^3x_4^{14}x_5^4$
178. $x_1^3x_2^5x_3^6x_4^9x_5^8$
182. $x_1^3x_2^5x_3^7x_4^{10}x_5^4$
3. $x_1x_2x_3^{15}x_4^6x_5^6$
7. $x_1^3x_2^3x_3^{15}x_4^4x_5^4$
11. $x_1x_2^2x_3^5x_4^6x_5^{15}$
15. $x_1x_2^3x_3^4x_4^{15}x_5^6$
19. $x_1x_2^3x_3^{15}x_4^6x_5^4$
23. $x_1x_2^{15}x_3^2x_4^5x_5^6$
27. $x_1^3x_2x_3^4x_4^6x_5^{15}$
31. $x_1^3x_2x_3^{15}x_4^4x_5^6$
35. $x_1^3x_2^5x_3^{15}x_4^4x_5^4$
39. $x_1^{15}x_2x_3^2x_4^5x_5^6$
43. $x_1^{15}x_2^3x_3x_4^4x_5^6$
47. $x_1x_2^2x_3^4x_4^{15}x_5^7$
51. $x_1x_2^2x_3^{15}x_4^7x_5^4$
55. $x_1x_2^{15}x_3^2x_4^4x_5^7$
59. $x_1^7x_2x_3^2x_4^{15}x_5^4$
63. $x_1^{15}x_2x_3^2x_4^4x_5^4$
67. $x_1x_2x_3^6x_4^{14}x_5^7$
71. $x_1x_2x_3^{14}x_4^7x_5^6$
75. $x_1x_2^2x_3^7x_4^5x_5^{12}$
79. $x_1x_2^2x_3^{13}x_4^6x_5^7$
83. $x_1x_2^3x_3^5x_4^4x_5^{14}$
87. $x_1x_2^3x_3^6x_4^{12}x_5^7$
91. $x_1x_2^3x_3^7x_4^{12}x_5^6$
95. $x_1x_2^3x_3^{13}x_4^6x_5^6$
99. $x_1x_2^6x_3x_4^7x_5^{14}$
103. $x_1x_2^6x_3^7x_4^5x_5^{14}$
107. $x_1x_2^7x_3x_4^6x_5^{14}$
111. $x_1x_2^7x_3^2x_4^{12}x_5^7$
115. $x_1x_2^7x_3^3x_4^{12}x_5^6$
119. $x_1x_2^7x_3^6x_4^9x_5^6$
123. $x_1x_2^7x_3^7x_4^{10}x_5^4$
127. $x_1x_2^7x_3^{14}x_4^6x_5^6$
131. $x_1x_2^{14}x_3^7x_4^6x_5^6$
135. $x_1^3x_2x_3^5x_4^{14}x_5^6$
139. $x_1^3x_2x_3^6x_4^{13}x_5^6$
143. $x_1^3x_2x_3^7x_4^{14}x_5^4$
147. $x_1^3x_2x_3^{14}x_4^4x_5^7$
151. $x_1^3x_2^2x_3^4x_4^7x_5^{12}$
155. $x_1^3x_2^2x_3^5x_4^{12}x_5^6$
159. $x_1^3x_2^3x_3^{12}x_4^5x_5^6$
163. $x_1^3x_2^3x_3^{13}x_4^6x_5^6$
167. $x_1^3x_2^5x_3^2x_4^5x_5^{14}$
171. $x_1^3x_2^5x_3^3x_4^4x_5^{14}$
175. $x_1^3x_2^5x_3^6x_4^4x_5^{14}$
179. $x_1^3x_2^5x_3^7x_4^2x_5^{12}$
183. $x_1^3x_2^5x_3^{10}x_4^4x_5^7$
4. $x_1x_2^{15}x_3x_4^6x_5^6$
8. $x_1^3x_2^{15}x_3^3x_4^4x_5^4$
12. $x_1x_2^2x_3^5x_4^4x_5^{15}$
16. $x_1x_2^3x_3^6x_4^4x_5^{15}$
20. $x_1x_2^6x_3x_4^6x_5^{15}$
24. $x_1x_2^{15}x_3^3x_4^4x_5^6$
28. $x_1^3x_2x_3^4x_4^{15}x_5^6$
32. $x_1^3x_2x_3^{15}x_4^6x_5^4$
36. $x_1^3x_2^{15}x_3x_4^4x_5^6$
40. $x_1^{15}x_2x_3^3x_4^4x_5^6$
44. $x_1^{15}x_2^3x_3x_4^6x_5^4$
48. $x_1x_2^2x_3^7x_4^4x_5^{15}$
52. $x_1x_2^7x_3^2x_4^4x_5^{15}$
56. $x_1x_2^{15}x_3^2x_4^7x_5^4$
60. $x_1^7x_2x_3^{15}x_4^2x_5^4$
64. $x_1^{15}x_2x_3^2x_4^4x_5^4$
68. $x_1x_2x_3^7x_4^6x_5^{14}$
72. $x_1x_2^2x_3^5x_4^7x_5^{14}$
76. $x_1x_2^2x_3^7x_4^{12}x_5^7$
80. $x_1x_2^2x_3^{13}x_4^7x_5^6$
84. $x_1x_2^3x_3^5x_4^{14}x_5^6$
88. $x_1x_2^3x_3^6x_4^{13}x_5^6$
92. $x_1x_2^3x_3^7x_4^{14}x_5^4$
96. $x_1x_2^3x_3^{14}x_4^4x_5^7$
100. $x_1x_2^6x_3x_4^{14}x_5^7$
104. $x_1x_2^6x_3^7x_4^8x_5^8$
108. $x_1x_2^7x_3x_4^{14}x_5^6$
112. $x_1x_2^7x_3^2x_4^{13}x_5^6$
116. $x_1x_2^7x_3^3x_4^{14}x_5^4$
120. $x_1x_2^7x_3^6x_4^2x_5^{12}$
124. $x_1x_2^7x_3^{10}x_4^5x_5^6$
128. $x_1x_2^{14}x_3x_4^6x_5^7$
132. $x_1^3x_2x_3^4x_4^{14}x_5^{14}$
136. $x_1^3x_2x_3^6x_4^5x_5^{14}$
140. $x_1^3x_2x_3^7x_4^4x_5^{14}$
144. $x_1^3x_2x_3^{12}x_4^6x_5^7$
148. $x_1^3x_2x_3^{14}x_4^5x_5^6$
152. $x_1^3x_2^2x_3^4x_4^{13}x_5^6$
156. $x_1^3x_2^2x_3^5x_4^{14}x_5^4$
160. $x_1^3x_2^3x_3^{13}x_4^4x_5^6$
164. $x_1^3x_2^3x_3^{11}x_4^5x_5^6$
168. $x_1^3x_2^5x_3^2x_4^7x_5^{12}$
172. $x_1^3x_2^5x_3^3x_4^9x_5^{12}$
176. $x_1^3x_2^5x_3^6x_4^7x_5^8$
180. $x_1^3x_2^5x_3^7x_4^6x_5^8$
184. $x_1^3x_2^5x_3^{10}x_4^5x_5^6$

$$\begin{array}{lll}
185. x_1^3 x_2^5 x_3^{10} x_4^7 x_5^4 & 186. x_1^3 x_2^5 x_3^{11} x_4^4 x_5^6 & 187. x_1^3 x_2^5 x_3^{11} x_4^6 x_5^4 \\
189. x_1^3 x_2^7 x_3^4 x_4^{14} x_5^5 & 190. x_1^3 x_2^7 x_3^6 x_4^5 x_5^{12} & 191. x_1^3 x_2^7 x_3^{12} x_4^6 x_5^6 \\
193. x_1^3 x_2^7 x_3^3 x_4^4 x_5^{12} & 194. x_1^3 x_2^7 x_3^3 x_4^{12} x_5^4 & 195. x_1^3 x_2^7 x_3^5 x_4^2 x_5^{12} \\
197. x_1^3 x_2^7 x_3^5 x_4^8 x_5^6 & 198. x_1^3 x_2^7 x_3^5 x_4^{10} x_5^4 & 199. x_1^3 x_2^7 x_3^8 x_4^5 x_5^6 \\
201. x_1^3 x_2^9 x_3^6 x_4^4 x_5^4 & 202. x_1^3 x_2^9 x_3^{11} x_4^4 x_5^4 & 203. x_1^3 x_2^9 x_3^{13} x_4^2 x_5^4 \\
205. x_1^3 x_2^{13} x_3^2 x_4^4 x_5^7 & 206. x_1^3 x_2^{13} x_3^2 x_4^5 x_5^6 & 207. x_1^3 x_2^{13} x_3^2 x_4^7 x_5^4 \\
209. x_1^3 x_2^3 x_3^6 x_4^4 x_5^4 & 210. x_1^3 x_2^{13} x_3^6 x_4^4 x_5^6 & 211. x_1^3 x_2^{13} x_3^7 x_4^2 x_5^4 \\
213. x_1^7 x_2 x_3 x_4^{14} x_5^6 & 214. x_1^7 x_2 x_3^2 x_4^5 x_5^{14} & 215. x_1^7 x_2 x_3^2 x_4^7 x_5^{12} \\
217. x_1^7 x_2 x_3^2 x_4^{13} x_5^6 & 218. x_1^7 x_2 x_3^3 x_4^4 x_5^{14} & 219. x_1^7 x_2 x_3^3 x_4^6 x_5^{12} \\
221. x_1^7 x_2 x_3^3 x_4^{14} x_5^4 & 222. x_1^7 x_2 x_3^6 x_4^4 x_5^{14} & 223. x_1^7 x_2 x_3^6 x_4^7 x_5^8 \\
225. x_1^7 x_2 x_3^7 x_4^2 x_5^{12} & 226. x_1^7 x_2 x_3^7 x_4^6 x_5^8 & 227. x_1^7 x_2 x_3^7 x_4^8 x_5^6 \\
229. x_1^7 x_2 x_3^{10} x_4^5 x_5^6 & 230. x_1^7 x_2 x_3^{11} x_4^4 x_5^6 & 231. x_1^7 x_2 x_3^{11} x_4^6 x_5^4 \\
233. x_1^7 x_3^3 x_3^4 x_4^5 x_5^{14} & 234. x_1^7 x_3^3 x_3^4 x_4^5 x_5^{12} & 235. x_1^7 x_3^3 x_3^4 x_4^5 x_5^6 \\
237. x_1^7 x_3^3 x_3^4 x_4^5 x_5^{12} & 238. x_1^7 x_3^3 x_3^4 x_4^5 x_5^4 & 239. x_1^7 x_3^3 x_3^5 x_4^2 x_5^{12} \\
241. x_1^7 x_3^3 x_3^5 x_4^8 x_5^6 & 242. x_1^7 x_3^3 x_3^5 x_4^{10} x_5^4 & 243. x_1^7 x_3^3 x_3^8 x_4^5 x_5^6 \\
245. x_1^7 x_3^3 x_3^9 x_4^4 x_5^4 & 246. x_1^7 x_3^3 x_3^{11} x_4^4 x_5^4 & 247. x_1^7 x_3^3 x_3^{13} x_4^2 x_5^4 \\
249. x_1^7 x_3^7 x_3^4 x_4^8 x_5^6 & 250. x_1^7 x_3^7 x_3^4 x_4^8 x_5^4 & 251. x_1^7 x_3^7 x_3^4 x_4^{10} x_5^4 \\
253. x_1^7 x_3^9 x_3^2 x_4^5 x_5^4 & 254. x_1^7 x_3^9 x_3^2 x_4^5 x_5^6 & 255. x_1^7 x_3^9 x_3^3 x_4^4 x_5^6 \\
257. x_1^7 x_2^{11} x_3 x_4^4 x_5^6 & 258. x_1^7 x_2^{11} x_3 x_4^4 x_5^4 & 259. x_1^7 x_2^{11} x_3^3 x_4^4 x_5^4 \\
188. x_1^3 x_2^5 x_3^{14} x_4 x_5^6 & 192. x_1^3 x_2^7 x_3^{14} x_4^4 x_5^5 & 196. x_1^3 x_2^7 x_3^5 x_4^6 x_5^8 \\
200. x_1^3 x_2^7 x_3^9 x_4^4 x_5^6 & 204. x_1^3 x_2^{13} x_3 x_4^6 x_5^6 & 208. x_1^3 x_2^{13} x_3^3 x_4^4 x_5^6 \\
212. x_1^7 x_2 x_3 x_4^6 x_5^{14} & 216. x_1^7 x_2 x_3^2 x_4^{12} x_5^7 & 220. x_1^7 x_2 x_3^3 x_4^{12} x_5^6 \\
224. x_1^7 x_2 x_3^6 x_4^9 x_5^6 & 228. x_1^7 x_2 x_3^7 x_4^{10} x_5^4 & 232. x_1^7 x_2 x_3^{14} x_4 x_5^6 \\
236. x_1^7 x_3^3 x_3^4 x_4^5 x_5^4 & 240. x_1^7 x_3^3 x_3^5 x_4^6 x_5^8 & 244. x_1^7 x_3^3 x_3^9 x_4^4 x_5^6 \\
248. x_1^7 x_3^7 x_3^4 x_4^8 x_5^6 & 252. x_1^7 x_3^7 x_3^4 x_4^8 x_5^4 & 256. x_1^7 x_3^9 x_3^3 x_4^4 x_5^4 \\
260. x_1^7 x_2^{11} x_3^3 x_4^4 x_5^4.
\end{array}$$

For $t \geq 4$,

$$\begin{array}{lll}
1. x_1 x_2 x_3^{2^t-2} x_4^{2^t-2} x_5^{2^{t+1}-1} & 2. x_1 x_2 x_3^{2^t-2} x_4^{2^{t+1}-1} x_5^{2^t-2} & 3. x_1 x_2 x_3^{2^{t+1}-1} x_4^{2^t-2} x_5^{2^t-2} \\
4. x_1 x_2^{2^t-2} x_3 x_4^{2^t-2} x_5^{2^{t+1}-1} & 5. x_1 x_2^{2^t-2} x_3 x_4^{2^{t+1}-1} x_5^{2^t-2} & 6. x_1 x_2^{2^t-2} x_3^{2^{t+1}-1} x_4 x_5^{2^t-2} \\
7. x_1 x_2^{2^{t+1}-1} x_3 x_4^{2^t-2} x_5^{2^t-2} & 8. x_1 x_2^{2^{t+1}-1} x_3^{2^t-2} x_4 x_5^{2^t-2} & 9. x_1^{2^{t+1}-1} x_2 x_3 x_4^{2^t-2} x_5^{2^t-2} \\
10. x_1^{2^{t+1}-1} x_2 x_3^{2^t-2} x_4 x_5^{2^t-2} & 11. x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^t-4} x_5^{2^{t+1}-1} & 12. x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^{t+1}-1} x_5^{2^t-4} \\
13. x_1^3 x_2^5 x_3^{2^{t+1}-1} x_4^{2^t-6} x_5^{2^t-4} & 14. x_1^3 x_2^{2^{t+1}-1} x_3^5 x_4^{2^t-6} x_5^{2^t-4} & 15. x_1^{2^{t+1}-1} x_2^3 x_3^5 x_4^{2^t-6} x_5^{2^t-4} \\
16. x_1 x_2^2 x_3^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-1} & 17. x_1 x_2^2 x_3^{2^t-3} x_4^{2^t-1} x_5^{2^t-2} & 18. x_1 x_2^2 x_3^{2^{t+1}-1} x_4^{2^t-3} x_5^{2^t-2} \\
19. x_1 x_2^{2^{t+1}-1} x_3^2 x_4^{2^t-3} x_5^{2^t-2} & 20. x_1^{2^{t+1}-1} x_2 x_3^2 x_4^{2^t-3} x_5^{2^t-2} & 21. x_1 x_3^2 x_3^{2^t-4} x_4^{2^t-2} x_5^{2^{t+1}-1} \\
22. x_1 x_2^3 x_3^{2^t-4} x_4^{2^{t+1}-1} x_5^{2^t-2} & 23. x_1 x_2^3 x_3^{2^t-2} x_4^{2^t-4} x_5^{2^{t+1}-1} & 24. x_1 x_2^3 x_3^{2^t-2} x_4^{2^{t+1}-1} x_5^{2^t-4} \\
25. x_1 x_2^3 x_3^{2^{t+1}-1} x_4^{2^t-4} x_5^{2^t-2} & 26. x_1 x_2^3 x_3^{2^{t+1}-1} x_4^{2^t-2} x_5^{2^t-4} & 27. x_1 x_2^{2^{t+1}-1} x_3^3 x_4^{2^t-4} x_5^{2^t-2} \\
28. x_1 x_2^{2^{t+1}-1} x_3^3 x_4^{2^t-2} x_5^{2^t-4} & 29. x_1^3 x_2 x_3^{2^t-4} x_4^{2^t-2} x_5^{2^{t+1}-1} & 30. x_1^3 x_2 x_3^{2^t-4} x_4^{2^{t+1}-1} x_5^{2^t-2} \\
31. x_1^3 x_2 x_3^{2^t-2} x_4^{2^t-4} x_5^{2^{t+1}-1} & 32. x_1^3 x_2 x_3^{2^t-2} x_4^{2^{t+1}-1} x_5^{2^t-4} & 33. x_1^3 x_2 x_3^{2^{t+1}-1} x_4^{2^t-4} x_5^{2^t-2} \\
34. x_1^3 x_2 x_3^{2^{t+1}-1} x_4^{2^t-2} x_5^{2^t-4} & 35. x_1^3 x_2^{2^{t+1}-1} x_3 x_4^{2^t-4} x_5^{2^t-2} & 36. x_1^3 x_2^{2^{t+1}-1} x_3 x_4^{2^t-2} x_5^{2^t-4} \\
37. x_1^{2^{t+1}-1} x_2 x_3^3 x_4^{2^t-4} x_5^{2^t-2} & 38. x_1^{2^{t+1}-1} x_2 x_3^3 x_4^{2^t-2} x_5^{2^t-4} & 39. x_1^{2^{t+1}-1} x_2^3 x_3 x_4^{2^t-4} x_5^{2^t-2} \\
40. x_1^{2^{t+1}-1} x_3^3 x_3 x_4^{2^t-2} x_5^{2^t-4} & 41. x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^t-4} x_5^{2^{t+1}-1} & 42. x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^{t+1}-1} x_5^{2^t-4} \\
43. x_1^3 x_2^{2^t-3} x_3^{2^{t+1}-1} x_4^2 x_5^{2^t-4} & 44. x_1^3 x_2^{2^{t+1}-1} x_3^{2^t-3} x_4^2 x_5^{2^t-4} & 45. x_1^{2^{t+1}-1} x_3^3 x_3^{2^t-3} x_4^2 x_5^{2^t-4} \\
46. x_1 x_2^2 x_3^{2^t-4} x_4^{2^t-1} x_5^{2^{t+1}-1} & 47. x_1 x_2^2 x_3^{2^t-4} x_4^{2^{t+1}-1} x_5^{2^t-1} & 48. x_1 x_2^2 x_3^{2^t-1} x_4^{2^t-4} x_5^{2^{t+1}-1} \\
49. x_1 x_2^2 x_3^{2^t-1} x_4^{2^{t+1}-1} x_5^{2^t-4} & 50. x_1 x_2^2 x_3^{2^{t+1}-1} x_4^{2^t-4} x_5^{2^t-1} & 51. x_1 x_2^2 x_3^{2^{t+1}-1} x_4^{2^t-1} x_5^{2^t-4} \\
52. x_1 x_2^{2^t-1} x_3^2 x_4^{2^t-4} x_5^{2^{t+1}-1} & 53. x_1 x_2^{2^t-1} x_3^2 x_4^{2^{t+1}-1} x_5^{2^t-4} & 54. x_1 x_2^{2^t-1} x_3^{2^{t+1}-1} x_4^2 x_5^{2^t-4} \\
55. x_1 x_2^{2^{t+1}-1} x_3^2 x_4^{2^t-4} x_5^{2^t-1} & 56. x_1 x_2^{2^{t+1}-1} x_3^2 x_4^{2^t-1} x_5^{2^t-4} & 57. x_1 x_2^{2^{t+1}-1} x_3^3 x_4^{2^t-1} x_5^{2^t-4} \\
58. x_1^{2^t-1} x_2 x_3^2 x_4^{2^t-4} x_5^{2^{t+1}-1} & 59. x_1^{2^t-1} x_2 x_3^2 x_4^{2^{t+1}-1} x_5^{2^t-4} & 60. x_1^{2^t-1} x_2 x_3^{2^{t+1}-1} x_4^2 x_5^{2^t-4} \\
61. x_1^{2^t-1} x_2^{2^{t+1}-1} x_3 x_4^2 x_5^{2^t-4} & 62. x_1^{2^{t+1}-1} x_2 x_3^2 x_4^{2^t-4} x_5^{2^t-1} & 63. x_1^{2^{t+1}-1} x_2 x_3^2 x_4^{2^t-1} x_5^{2^t-4} \\
64. x_1^{2^{t+1}-1} x_2 x_3^{2^t-1} x_4^2 x_5^{2^t-4} & 65. x_1^{2^{t+1}-1} x_2^{2^t-1} x_3 x_4^2 x_5^{2^t-4} & 66. x_1 x_2 x_3^{2^t-2} x_4^{2^t-1} x_5^{2^{t+1}-2} \\
67. x_1 x_2 x_3^{2^t-2} x_4^{2^{t+1}-2} x_5^{2^t-1} & 68. x_1 x_2 x_3^{2^t-1} x_4^{2^t-2} x_5^{2^{t+1}-2} & 69. x_1 x_2 x_3^{2^t-1} x_4^{2^{t+1}-2} x_5^{2^t-2}
\end{array}$$

70. $x_1 x_2 x_3^{2^{t+1}-2} x_4^{2^t-2} x_5^{2^t-1}$
73. $x_1 x_2^{2^t-2} x_3 x_4^{2^{t+1}-2} x_5^{2^t-1}$
76. $x_1 x_2^{2^t-1} x_3 x_4^{2^{t+1}-2} x_5^{2^t-2}$
79. $x_1 x_2^{2^{t+1}-2} x_3 x_4^{2^t-2} x_5^{2^t-1}$
82. $x_1^{2^t-1} x_2 x_3 x_4^{2^t-2} x_5^{2^{t+1}-2}$
85. $x_1^{2^t-1} x_2 x_3^{2^{t+1}-2} x_4 x_5^{2^t-2}$
88. $x_1 x_2^2 x_3^{2^t-1} x_4^{2^t-3} x_5^{2^{t+1}-2}$
91. $x_1 x_2^2 x_3^{2^t-1} x_4^{2^{t+1}-3} x_5^{2^t-2}$
94. $x_1 x_2^{2^t-1} x_3 x_4^{2^{t+1}-3} x_5^{2^t-2}$
97. $x_1 x_2^2 x_3^{2^t-1} x_4^{2^{t+1}-4} x_5^{2^t-1}$
100. $x_1 x_2^{2^t-1} x_3 x_4^{2^{t+1}-4} x_5^{2^t-1}$
103. $x_1^{2^t-1} x_2 x_3 x_4^{2^{t+1}-4} x_5^{2^t-1}$
106. $x_1 x_2^3 x_3^{2^t-4} x_4^{2^t-1} x_5^{2^{t+1}-2}$
109. $x_1 x_2^3 x_3^{2^t-1} x_4^{2^{t+1}-2} x_5^{2^t-4}$
112. $x_1 x_2^{2^t-1} x_3 x_4^{2^t-4} x_5^{2^{t+1}-2}$
115. $x_1^3 x_2 x_3^{2^t-4} x_4^{2^{t+1}-2} x_5^{2^t-1}$
118. $x_1^3 x_2 x_3^{2^{t+1}-2} x_4^{2^t-4} x_5^{2^t-1}$
121. $x_1^3 x_2^{2^t-1} x_3 x_4^{2^{t+1}-2} x_5^{2^t-4}$
124. $x_1^{2^t-1} x_2^3 x_3 x_4^{2^t-4} x_5^{2^{t+1}-2}$
127. $x_1 x_2^3 x_3^{2^t-3} x_4^{2^{t+1}-2} x_5^{2^t-2}$
130. $x_1 x_2^{2^t-2} x_3 x_4^{2^t-3} x_5^{2^{t+1}-2}$
133. $x_1^3 x_2 x_3^{2^t-3} x_4^{2^{t+1}-2} x_5^{2^t-2}$
136. $x_1^3 x_2^{2^t-3} x_3 x_4^{2^t-2} x_5^{2^{t+1}-2}$
139. $x_1^3 x_2^{2^t-3} x_3^{2^{t+1}-2} x_4 x_5^{2^t-2}$
142. $x_1 x_2^{2^t-2} x_3 x_4^{2^{t+1}-3} x_5^{2^t-2}$
145. $x_1^3 x_2^{2^{t+1}-3} x_3 x_4^{2^t-2} x_5^{2^t-2}$
148. $x_1 x_2^3 x_3^{2^t-2} x_4^{2^{t+1}-4} x_5^{2^t-1}$
151. $x_1 x_2^3 x_3^{2^{t+1}-4} x_4^{2^t-2} x_5^{2^t-1}$
154. $x_1 x_2^{2^t-1} x_3 x_4^{2^{t+1}-4} x_5^{2^t-2}$
157. $x_1^3 x_2 x_3^{2^t-1} x_4^{2^t-2} x_5^{2^{t+1}-4}$
160. $x_1^3 x_2 x_3^{2^{t+1}-4} x_4^{2^t-1} x_5^{2^t-2}$
163. $x_1^{2^t-1} x_2 x_3 x_4^{2^t-2} x_5^{2^{t+1}-4}$
166. $x_1^{2^t-1} x_2^3 x_3 x_4^{2^{t+1}-4} x_5^{2^t-2}$
169. $x_1 x_2^6 x_3^{2^{t+1}-5} x_4^{2^t-3} x_5^{2^t-2}$
172. $x_1 x_2^7 x_3^{2^t-6} x_4^{2^{t+1}-3} x_5^{2^t-2}$
175. $x_1 x_2^7 x_3^{2^t-5} x_4^{2^{t+1}-2} x_5^{2^t-4}$
178. $x_1^7 x_2^{2^t-5} x_3 x_4^{2^t-4} x_5^{2^{t+1}-2}$
181. $x_1 x_2^7 x_3^{2^t-5} x_4^{2^{t+1}-4} x_5^{2^t-2}$
184. $x_1^7 x_2^{2^t-5} x_3 x_4^{2^t-2} x_5^{2^{t+1}-4}$
187. $x_1 x_2^7 x_3^{2^{t+1}-5} x_4^{2^t-2} x_5^{2^t-4}$
190. $x_1^7 x_2^{2^{t+1}-5} x_3 x_4^{2^t-4} x_5^{2^t-2}$
193. $x_1^7 x_2 x_3^{2^{t+1}-6} x_4^{2^t-3} x_5^{2^t-2}$
71. $x_1 x_2 x_3^{2^{t+1}-2} x_4^{2^t-1} x_5^{2^t-2}$
74. $x_1 x_2^{2^t-2} x_3^{2^t-1} x_4 x_5^{2^{t+1}-2}$
77. $x_1 x_2^{2^t-1} x_3^{2^t-2} x_4 x_5^{2^{t+1}-2}$
80. $x_1 x_2^{2^{t+1}-2} x_3 x_4^{2^t-1} x_5^{2^t-2}$
83. $x_1^{2^t-1} x_2 x_3 x_4^{2^{t+1}-2} x_5^{2^t-2}$
86. $x_1 x_2^2 x_3^{2^t-3} x_4^{2^t-1} x_5^{2^{t+1}-2}$
89. $x_1 x_2^{2^t-1} x_3^2 x_4^{2^t-3} x_5^{2^{t+1}-2}$
92. $x_1 x_2^2 x_3^{2^{t+1}-3} x_4^{2^t-2} x_5^{2^t-1}$
95. $x_1^{2^t-1} x_2 x_3^2 x_4^{2^{t+1}-3} x_5^{2^t-2}$
98. $x_1 x_2^2 x_3^{2^{t+1}-4} x_4^{2^t-1} x_5^{2^t-1}$
101. $x_1 x_2^{2^t-1} x_3^{2^t-1} x_4 x_5^{2^{t+1}-4}$
104. $x_1^{2^t-1} x_2 x_3^{2^t-1} x_4 x_5^{2^{t+1}-4}$
107. $x_1 x_2^3 x_3^{2^t-4} x_4^{2^{t+1}-2} x_5^{2^t-1}$
110. $x_1 x_2^3 x_3^{2^{t+1}-2} x_4^{2^t-4} x_5^{2^t-1}$
113. $x_1 x_2^{2^t-1} x_3 x_4^{2^{t+1}-2} x_5^{2^t-4}$
116. $x_1^3 x_2 x_3^{2^t-1} x_4^{2^t-4} x_5^{2^{t+1}-2}$
119. $x_1^3 x_2 x_3^{2^{t+1}-2} x_4^{2^t-1} x_5^{2^t-4}$
122. $x_1^{2^t-1} x_2 x_3 x_4^{2^t-4} x_5^{2^{t+1}-2}$
125. $x_1^{2^t-1} x_2^3 x_3 x_4^{2^{t+1}-2} x_5^{2^t-4}$
128. $x_1 x_2^3 x_3^{2^t-2} x_4^{2^t-3} x_5^{2^{t+1}-2}$
131. $x_1 x_2^{2^{t+1}-2} x_3 x_4^{2^t-3} x_5^{2^t-2}$
134. $x_1^3 x_2 x_3^{2^t-2} x_4^{2^t-3} x_5^{2^{t+1}-2}$
137. $x_1^3 x_2^{2^t-3} x_3 x_4^{2^{t+1}-2} x_5^{2^t-2}$
140. $x_1 x_2^3 x_3^{2^t-2} x_4^{2^{t+1}-3} x_5^{2^t-2}$
143. $x_1^3 x_2 x_3^{2^t-2} x_4^{2^{t+1}-3} x_5^{2^t-2}$
146. $x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-2} x_4 x_5^{2^t-2}$
149. $x_1 x_2^3 x_3^{2^t-1} x_4^{2^t-2} x_5^{2^{t+1}-4}$
152. $x_1 x_2^3 x_3^{2^{t+1}-4} x_4^{2^t-1} x_5^{2^t-2}$
155. $x_1^3 x_2 x_3^{2^t-2} x_4^{2^t-1} x_5^{2^{t+1}-4}$
158. $x_1^3 x_2 x_3^{2^t-1} x_4^{2^{t+1}-4} x_5^{2^t-2}$
161. $x_1^3 x_2^{2^t-1} x_3 x_4^{2^t-2} x_5^{2^{t+1}-4}$
164. $x_1^{2^t-1} x_2 x_3 x_4^{2^{t+1}-4} x_5^{2^t-2}$
167. $x_1 x_2^6 x_3^{2^t-5} x_4^{2^t-3} x_5^{2^{t+1}-2}$
170. $x_1 x_2^7 x_3^{2^t-6} x_4^{2^t-3} x_5^{2^{t+1}-2}$
173. $x_1^7 x_2 x_3^{2^t-6} x_4^{2^{t+1}-3} x_5^{2^t-2}$
176. $x_1^7 x_2 x_3^{2^t-5} x_4^{2^t-4} x_5^{2^{t+1}-2}$
179. $x_1^7 x_2^{2^t-5} x_3 x_4^{2^{t+1}-2} x_5^{2^t-4}$
182. $x_1^7 x_2 x_3^{2^t-5} x_4^{2^t-2} x_5^{2^{t+1}-4}$
185. $x_1^7 x_2^{2^t-5} x_3 x_4^{2^{t+1}-4} x_5^{2^t-2}$
188. $x_1^7 x_2 x_3^{2^{t+1}-5} x_4^{2^t-4} x_5^{2^t-2}$
191. $x_1^7 x_2^{2^{t+1}-5} x_3 x_4^{2^t-2} x_5^{2^t-4}$
194. $x_1^3 x_2^{2^t-1} x_3^{2^{t+1}-3} x_4 x_5^{2^t-4}$
72. $x_1 x_2^{2^t-2} x_3 x_4^{2^t-1} x_5^{2^{t+1}-2}$
75. $x_1 x_2^{2^t-1} x_3 x_4^{2^t-2} x_5^{2^{t+1}-2}$
78. $x_1 x_2^{2^t-1} x_3^{2^t-1} x_4 x_5^{2^t-2}$
81. $x_1 x_2^{2^{t+1}-2} x_3^{2^t-1} x_4 x_5^{2^t-2}$
84. $x_1^{2^t-1} x_2 x_3^{2^t-2} x_4 x_5^{2^{t+1}-2}$
87. $x_1 x_2^2 x_3^{2^t-3} x_4^{2^{t+1}-2} x_5^{2^t-1}$
90. $x_1^{2^t-1} x_2 x_3^2 x_4^{2^t-3} x_5^{2^{t+1}-2}$
93. $x_1 x_2^2 x_3^{2^{t+1}-3} x_4^{2^t-1} x_5^{2^t-2}$
96. $x_1 x_2^2 x_3^{2^t-1} x_4^{2^t-1} x_5^{2^{t+1}-4}$
99. $x_1 x_2^{2^t-1} x_3^2 x_4^{2^t-1} x_5^{2^{t+1}-4}$
102. $x_1^{2^t-1} x_2 x_3 x_4^{2^t-1} x_5^{2^{t+1}-4}$
105. $x_1^{2^t-1} x_2^{2^t-1} x_3 x_4^2 x_5^{2^{t+1}-4}$
108. $x_1 x_2^3 x_3^{2^t-1} x_4^{2^t-4} x_5^{2^{t+1}-2}$
111. $x_1 x_2^3 x_3^{2^{t+1}-2} x_4^{2^t-1} x_5^{2^t-4}$
114. $x_1^3 x_2 x_3^{2^t-4} x_4^{2^t-1} x_5^{2^{t+1}-2}$
117. $x_1^3 x_2 x_3^{2^t-1} x_4^{2^{t+1}-2} x_5^{2^t-4}$
120. $x_1^3 x_2^{2^t-1} x_3 x_4^{2^t-4} x_5^{2^{t+1}-2}$
123. $x_1^{2^t-1} x_2 x_3 x_4^{2^{t+1}-2} x_5^{2^t-4}$
126. $x_1 x_2^3 x_3^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-2}$
129. $x_1 x_2^3 x_3^{2^{t+1}-2} x_4^{2^t-3} x_5^{2^t-2}$
132. $x_1^3 x_2 x_3^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-2}$
135. $x_1^3 x_2 x_3^{2^{t+1}-2} x_4^{2^t-3} x_5^{2^t-2}$
138. $x_1^3 x_2^{2^t-3} x_3^{2^t-2} x_4 x_5^{2^{t+1}-2}$
141. $x_1 x_2^3 x_3^{2^{t+1}-3} x_4^{2^t-2} x_5^{2^t-2}$
144. $x_1^3 x_2 x_3^{2^{t+1}-3} x_4^{2^t-2} x_5^{2^t-2}$
147. $x_1 x_2^3 x_3^{2^t-2} x_4^{2^t-1} x_5^{2^{t+1}-4}$
150. $x_1 x_2^3 x_3^{2^t-1} x_4^{2^{t+1}-4} x_5^{2^t-2}$
153. $x_1 x_2^{2^t-1} x_3 x_4^{2^t-2} x_5^{2^{t+1}-4}$
156. $x_1^3 x_2 x_3^{2^t-2} x_4^{2^{t+1}-4} x_5^{2^t-1}$
159. $x_1^3 x_2 x_3^{2^{t+1}-4} x_4^{2^t-2} x_5^{2^t-1}$
162. $x_1^3 x_2^{2^t-1} x_3 x_4^{2^{t+1}-4} x_5^{2^t-2}$
165. $x_1^{2^t-1} x_2^3 x_3 x_4^{2^t-2} x_5^{2^{t+1}-4}$
168. $x_1 x_2^6 x_3^{2^t-5} x_4^{2^{t+1}-3} x_5^{2^t-2}$
171. $x_1^7 x_2 x_3^{2^t-6} x_4^{2^t-3} x_5^{2^{t+1}-2}$
174. $x_1 x_2^7 x_3^{2^t-5} x_4^{2^t-4} x_5^{2^{t+1}-2}$
177. $x_1^7 x_2 x_3^{2^t-5} x_4^{2^{t+1}-2} x_5^{2^t-4}$
180. $x_1 x_2^7 x_3^{2^t-5} x_4^{2^t-2} x_5^{2^{t+1}-4}$
183. $x_1^7 x_2 x_3^{2^t-5} x_4^{2^{t+1}-4} x_5^{2^t-2}$
186. $x_1 x_2^7 x_3^{2^t-5} x_4^{2^t-4} x_5^{2^t-2}$
189. $x_1^7 x_2 x_3^{2^{t+1}-5} x_4^{2^t-2} x_5^{2^t-4}$
192. $x_1 x_2^7 x_3^{2^{t+1}-6} x_4^{2^t-3} x_5^{2^t-2}$
195. $x_1^3 x_2^{2^t-1} x_3^2 x_4^{2^t-4} x_5^{2^t-1}$

196. $x_1^3 x_2^{2^{t+1}-3} x_3^2 x_4^{2^t-1} x_5^{2^t-4}$
199. $x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^t-3} x_5^{2^{t+1}-2}$
202. $x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^t-1} x_5^{2^{t+1}-4}$
205. $x_1^3 x_2^{2^t-1} x_3^2 x_4^{2^t-3} x_5^{2^{t+1}-4}$
208. $x_1^7 x_2^{2^t-5} x_3^{2^{t+1}-3} x_4^2 x_5^{2^t-4}$
211. $x_1^3 x_2^3 x_3^{2^t-3} x_4^{2^t-4} x_5^{2^{t+1}-2}$
214. $x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^{t+1}-2} x_5^{2^t-4}$
217. $x_1^3 x_2^3 x_3^{2^{t+1}-3} x_4^{2^t-2} x_5^{2^t-4}$
220. $x_1^3 x_2^3 x_3^{2^t-3} x_4^{2^t-2} x_5^{2^{t+1}-4}$
223. $x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^t-2} x_5^{2^{t+1}-4}$
226. $x_1^3 x_2^4 x_3^{2^t-5} x_4^{2^{t+1}-3} x_5^{2^t-2}$
229. $x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^{t+1}-3} x_5^{2^t-2}$
232. $x_1^3 x_2^5 x_3^{2^t-1} x_4^{2^t-6} x_5^{2^{t+1}-4}$
235. $x_1^3 x_2^5 x_3^{2^t-5} x_4^{2^t-4} x_5^{2^{t+1}-2}$
238. $x_1^3 x_2^5 x_3^{2^t-5} x_4^{2^{t+1}-4} x_5^{2^t-2}$
241. $x_1^3 x_2^5 x_3^{2^t-1} x_4^{2^{t+1}-6} x_5^{2^t-4}$
244. $x_1^3 x_2^{2^t-1} x_3^5 x_4^{2^{t+1}-6} x_5^{2^t-4}$
247. $x_1^3 x_2^7 x_3^{2^t-7} x_4^{2^t-4} x_5^{2^{t+1}-2}$
250. $x_1^7 x_2^3 x_3^{2^t-7} x_4^{2^{t+1}-2} x_5^{2^t-4}$
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262. $x_1^7 x_2^3 x_3^{2^{t+1}-7} x_4^{2^t-4} x_5^{2^t-2}$
265. $x_1^7 x_2^{2^t-5} x_3^5 x_4^{2^t-6} x_5^{2^{t+1}-4}$
268. $x_1^7 x_2^{2^{t+1}-5} x_3^5 x_4^{2^t-6} x_5^{2^t-4}$.
197. $x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-1} x_4^2 x_5^{2^t-4}$
200. $x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^{t+1}-3} x_5^{2^t-2}$
203. $x_1^3 x_2^{2^t-3} x_3^2 x_4^{2^{t+1}-4} x_5^{2^t-1}$
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212. $x_1^3 x_2^3 x_3^{2^t-3} x_4^{2^{t+1}-2} x_5^{2^t-4}$
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227. $x_1^3 x_2^4 x_3^{2^{t+1}-5} x_4^{2^t-3} x_5^{2^t-2}$
230. $x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^t-1} x_5^{2^{t+1}-4}$
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236. $x_1^3 x_2^5 x_3^{2^t-5} x_4^{2^{t+1}-2} x_5^{2^t-4}$
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245. $x_1^{2^t-1} x_2^3 x_3^5 x_4^{2^{t+1}-6} x_5^{2^t-4}$
248. $x_1^3 x_2^7 x_3^{2^t-7} x_4^{2^{t+1}-2} x_5^{2^t-4}$
251. $x_1^3 x_2^7 x_3^{2^t-7} x_4^{2^t-2} x_5^{2^{t+1}-4}$
254. $x_1^7 x_2^3 x_3^{2^t-7} x_4^{2^{t+1}-4} x_5^{2^t-2}$
257. $x_1^7 x_2^3 x_3^{2^t-3} x_4^{2^t-6} x_5^{2^{t+1}-4}$
260. $x_1^7 x_2^3 x_3^{2^t-3} x_4^{2^{t+1}-6} x_5^{2^t-4}$
263. $x_1^3 x_2^{15} x_3^{2^{t+1}-11} x_4^{2^t-6} x_5^{2^t-4}$
266. $x_1^7 x_2^{2^t-5} x_3^5 x_4^{2^{t+1}-6} x_5^{2^t-4}$
198. $x_1^{2^t-1} x_2^3 x_3^{2^{t+1}-3} x_4^2 x_5^{2^t-4}$
201. $x_1^3 x_2^{2^{t+1}-3} x_3^2 x_4^{2^t-3} x_5^{2^t-2}$
204. $x_1^3 x_2^{2^t-3} x_3^{2^t-1} x_4^2 x_5^{2^{t+1}-4}$
207. $x_1^7 x_2^{2^t-5} x_3^{2^t-3} x_4^2 x_5^{2^{t+1}-4}$
210. $x_1^3 x_2^3 x_3^{2^t-4} x_4^{2^t-3} x_5^{2^{t+1}-2}$
213. $x_1^3 x_2^{2^t-3} x_3^3 x_4^{2^t-4} x_5^{2^{t+1}-2}$
216. $x_1^3 x_2^3 x_3^{2^{t+1}-3} x_4^{2^t-4} x_5^{2^t-2}$
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222. $x_1^3 x_2^3 x_3^{2^{t+1}-4} x_4^{2^t-3} x_5^{2^t-2}$
225. $x_1^3 x_2^4 x_3^{2^t-5} x_4^{2^t-3} x_5^{2^{t+1}-2}$
228. $x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^t-3} x_5^{2^{t+1}-2}$
231. $x_1^3 x_2^5 x_3^{2^t-6} x_4^{2^{t+1}-4} x_5^{2^t-1}$
234. $x_1^{2^t-1} x_2^3 x_3^5 x_4^{2^t-6} x_5^{2^{t+1}-4}$
237. $x_1^3 x_2^5 x_3^{2^t-5} x_4^{2^t-2} x_5^{2^{t+1}-4}$
240. $x_1^3 x_2^5 x_3^{2^{t+1}-5} x_4^{2^t-2} x_5^{2^t-4}$
243. $x_1^3 x_2^5 x_3^{2^{t+1}-6} x_4^{2^t-1} x_5^{2^t-4}$
246. $x_1^3 x_2^5 x_3^{2^{t+1}-6} x_4^{2^t-3} x_5^{2^t-2}$
249. $x_1^7 x_2^3 x_3^{2^t-7} x_4^{2^t-4} x_5^{2^{t+1}-2}$
252. $x_1^3 x_2^7 x_3^{2^t-7} x_4^{2^{t+1}-4} x_5^{2^t-2}$
255. $x_1^3 x_2^7 x_3^{2^{t+1}-3} x_4^{2^t-6} x_5^{2^t-4}$
258. $x_1^7 x_2^3 x_3^{2^t-3} x_4^{2^t-6} x_5^{2^{t+1}-4}$
261. $x_1^3 x_2^7 x_3^{2^{t+1}-7} x_4^{2^t-4} x_5^{2^t-2}$
264. $x_1^{15} x_2^3 x_3^{2^{t+1}-11} x_4^{2^t-6} x_5^{2^t-4}$
267. $x_1^7 x_2^{11} x_3^{2^{t+1}-11} x_4^{2^t-6} x_5^{2^t-4}$

For $t = 4$,

$$269. x_1^7 x_2^{11} x_3^{13} x_4^{14} x_5^{16} \quad 270. x_1^7 x_2^{11} x_3^{13} x_4^{18} x_5^{12}.$$

For $t \geq 5$,

$$269. x_1^7 x_2^{11} x_3^{2^t-11} x_4^{2^t-6} x_5^{2^{t+1}-4} \quad 270. x_1^7 x_2^{11} x_3^{2^t-11} x_4^{2^{t+1}-6} x_5^{2^t-4}.$$

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