Snow machining and side-slipping in alpine skiing

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Abstract

A simple approximate theory of snow machining is developed and tested against the results of past laboratory experiments. It is also applied to side-slipping and traversing in alpine skiing, and yields realistic predictions which can be tested experimentally on real ski slopes. Eventually, the theory could be used in modelling of skiing turns involving the phase of skidding.

Keywords: alpine skiing, modelling, balance/stability, performance

Introduction

The interaction between skis and snow is one of the most important aspects of skiing, which has received a lot of attention from many researches. One of the most basic types of skiing manoeuvres is gliding (or sliding), where skis are completely flat on the snow. In this case, the forces acting on the ski from the snow reduce to the normal reaction force $\mathbf{N}$ and the friction force $\mathbf{F}_f$. The snow-slider friction has been a subject of detailed investigation in numerous studies as it is important not only for skiing but also for transportation in geographic areas with cold and snowy winters. As in many other applications, the friction force is usually described via the Coulomb model $\mathbf{F}_f = -\mu N \hat{\mathbf{m}}$, where μ is the kinetic coefficient of friction, and $\hat{\mathbf{m}}$ is the unit vector in the direction of relative motion. This law is not as simple as it looks because the coefficient of friction can depend on many factors and the physics behind it can be rather complex. Interested readers may find useful the reviews by Colbeck (1992), Lind and Sanders (1996), and Nachbauer, Kaps, Hasler, and Mössner (2016).

The most fundamental property of the Coulomb model is its isotropy – the magnitude of the friction force does not depend on the direction of relative motion. Moreover, the force is perfectly aligned with the relative velocity. Clearly, this implies that mechanical properties of interacting surfaces are isotropic. This, however, may not be true if the surfaces have a structure with clearly identified directions. In

such a case, sliding along and across such directions will yield different responses and this requires to replace the Coulomb's model with a more general tensor model (e.g. Michalowski & Mróz, 1978; Mróz & Stupkiewicz, 1994; Zmitrowicz, 1989). In fact, a typical ski base has a regular structure which involves small grooves aligned with ski's longitudinal axis. This structure is introduced for managing the meltwater produced at the snow-ski interface, and its purpose is to reduce the kinetic friction for sliding in the longitudinal direction. It is only natural to expect that for sliding in the transverse direction this structure will have an opposite effect.

Another source of anisotropy is the division of ski base into dry and meltwater lubricated areas. When ski slides in the longitudinal direction, the friction leads to heating of the ski base and the snow underneath it, leading to snow melting and hence lubrication of the ski base (Colbeck, 1992). At the tip of the ski, where it meets the cold snow for the first time, the temperature is at its lowest and the friction is still dry, but it is wet under the rest of the ski. For a 2m-long ski moving in the longitudinal direction at the speed of 10 m/s on snow at -10°C temperature, Colbeck (1992) estimated the length of the dry section to be just short of 1 cm. Although the coefficient of dry friction can be quite high, the small scale of the dry section ensures that the overall effect is weak. If however the ski slides in the transverse direction, then it meets cold snow all over its side, and the relevant geometric scale is not the ski length but its width. Hence, one would expect the overall friction coefficient to be noticeably higher.

The extent of the friction anisotropy in skiing is not known as it has not been explored experimentally. Moreover, the side-slipping in skiing can rarely be described as gliding – when it occurs, skis are normally not flat on the snow but put on edge, and remove a top layer of the snow. This process is similar to machining (or cutting) of metal and other materials, which is involved in many types of manufacturing. Naturally, the mechanics of machining has been a hot subject of both experimental and theoretical research, and a great deal of information has been accumulated in this area of mechanics. Like Coulomb friction, the machining of snow is accompanied by a braking force, but in the case of oblique machining a force normal to the direction of motion, called in skiing the turning force, appears as well.

One of the main goals for the theory of skiing is to describe the dynamics of skiing turns. The generic ski turn on compacted snow is hybrid in nature. It starts as side-slipping (skidding), where the turning force is likely to originate from the machining of snow, and finishes with carving where the turning force is a component of the reaction force originated at the curved platform carved in the snow by the ski. (At the two extremes of this generic turn, we find a purely skidded turn and a purely carved turn, which is initiated by rolling skis from one edge to another.) Hence, understanding of snow machining and its role in skiing turns is an important part of skiing science.

Lieu and Mote (1984) carried out an experimental study of orthogonal ice cutting at rake angles relevant to skiing (negative rake angles). Later they arrived with

empirical expressions for the ice-reaction forces as functions of the rake angle and ice cutting (penetration) depth, and used a scaled version of these functions to model ski-snow interaction in skiing turns (Lieu & Mote, 1985). A similar study of ice machining, including the oblique case, was carried out by Tada and Hirano (1999) who found a reasonably good agreement with the results of Lieu and Mote (1984). Tada and Hirano (2002) and Tada and Kobayashi (2005) carried out an experimental study of both orthogonal and oblique machining of compacted snow. In the orthogonal case the relative velocity is orthogonal to the ski longitudinal axis, whereas in the oblique case the angle between the two, the angle of attack, is below 90° . Their results could not be fitted by the functions of Lieu and Mote (1985), especially at small edge angles, so they come up with their own rather complicated empirical functions for the snow reaction forces and used them to model skidded skiing turns.

In contrast to these experimental studies, Brown (2009) proposed to use the theory of metal cutting developed by Merchant (1945b), for the case of continuous (type 2) chip. However, snow and ice are highly brittle materials and instead of continuous chip their machining normally results in a spray of ice particles.

In this paper, we describe an attempt to build a simple approximate model for machining of ice and snow, which could be easily incorporated into the theory of ski turns. Our main hypothesis is that we may neglect the friction between the ski and the spray of ice particles without significant loss of accuracy. We check the model against the available data of laboratory studies and apply it to a number of basic skiing manoeuvres which allow straightforward experimental tests of the model under the conditions of real ski slopes.

Methods. Machining of ice and snow

In orthogonal machining of metal, a tool with flat rake face and straight sharp edge is moving over the workpiece in the direction perpendicular to its edge with a fixed cutting depth. It separates from the workpiece its upper layer as a chip. The chip works as a link between the uncut metal and the tool. The total force applied to the tool by the chip and the workpiece is $\mathbf{F}_r = \mathbf{F}_n + \mathbf{F}_f + \mathbf{F}_{\text{edge}}$, where F_n is the normal force applied to the tool by the chip at the rake face and $\mathbf{F}_f$ is the friction force force emerging at this face (see figure 1) and $\mathbf{F}_{\text{edge}}$ is called the edge force (Guo & Chou, 2004; Stevenson, 1998; Wyen & K. Wegener, 2010). The edge-force is also referred to as a “zero-feed force”, as it remains finite in the limit of vanishing chip thickness. It emerges due to the elastic and plastic deformations of the workpiece at the very tip of the tool where its curvature becomes important. This force depends on the shape of the edge and is difficult to predict in the theory of machining. It increases with the edge radius and can be made rather small for very sharp tools.

The resultant force can also be split into the component normal to the uncut metal surface, $\mathbf{N}$ (known as the “thrust” or “feed” force) and the cutting force $\mathbf{F}_c$ aligned with this surface (Merchant, 1945a). If the edge force is negligibly small then

from the basic geometry of the problem it follows that

$$F_c = N \tan(\Psi - \tau), \quad (1)$$

where Ψ is the angle between the rake face of the cutting tool (in skiing this is the ski base) and the machined surface of the workpiece (in our case the snow surface), and $\tau = \arctan \mu$ is the friction angle (see figure 1). In contrast to the case with positive rake angle ($\alpha_r = \Psi - 90^\circ > 0$), the friction does not increase but decreases the cutting force. This force vanishes when $\Psi = \tau$ and even becomes negative for smaller edge angles. Presumably, this indicates that at such angles the chip is no longer formed and $\mathbf{F}_r$ reduces to $\mathbf{F}_{\text{edge}}$. Thus the edge force can only be ignored for sufficiently large edge angles.

There are few differences between machining of metals and machining of hard snow and ice. In the case of metals, μ can be very high. For example, for clean and dry steel surfaces $\mu = 0.5 - 0.8$, leading to the friction angle $\tau = 26^\circ - 39^\circ$. In contrast, ice and compacted snow are very slippery and characterised by substantially lower values of μ (e.g. Colbeck, 1992; Kietzig, Hatzikiriakos, & Englezos, 2010). Moreover, ice and snow are extremely brittle and easily disintegrate into individual grains under sufficiently high strain rate, with very low energy consumption (e.g. Jordaan & Timco, 1988; Schweizer, Michot, & Kirchner, 2004). As a result, instead of a continuous chip, a spray of pulverised snow is formed. This process of snow/ice crashing is likely to proceed in a quasi-periodic fashion: a snow crashing event is followed by an ejection of pulverised snow, contact of the ski edge with yet untouched snow, and then next crashing event. Due to its fluidity, the friction between the spray and the ski base is expected to be even lower than the friction between the ski base and uncrushed snow. This argument suggests that at sufficiently large angles equation (1) is well approximated by

$$F_c = N \tan(\Psi). \quad (2)$$

There are only very few laboratory studies into machining of ice and snow. Lieu and Mote (1984) experimented with orthogonal machining of ice using a flat-faced tool with sharp edge, positioned at negative rake angle. In their experiments they varied the machining speed from 0.01 to 1 m/s, the cutting depth from 0.04 to 1.2 mm, the edge angle from 30° to 75° . The total cutting force per unit edge length could reach 12 kN/m. They found that in all cases the resultant ice reaction force was almost normal to the rake face, in agreement with equation (2). At the lower end of the cutting depth range, the chip was more or less continuous, but at the higher end it was in the form of spray of small ice particles. In the latter case, Lieu and Mote (1984) observed a pronounced tool chatter and fluctuation of the cutting force about its mean value on the level of approximately 10%, as expected for quasi-periodic crashing of ice (Jordaan & Timco, 1988). Later, Tada and Hirano (1999) found a reasonable agreement with the results of Lieu and Mote (1984) in their own experiments in ice machining.

Figure 1. *Left panel:* Interaction of a ski with hard compacted snow as orthogonal machining at negative rake angle $\alpha = \Psi - 90^\circ$. The sketch shows a cross section of the ski as it skids sideways from right to left. *Right panel:* Separation of the total reaction force $\mathbf{F}_r$ into mutually orthogonal pairs $\{\mathbf{F}_n, \mathbf{F}_f\}$ and $\{\mathbf{F}_c, \mathbf{N}\}$ in the theory of orthogonal machining by Merchant (1945a), where the edge force $\mathbf{F}_{\text{edge}}$ is not included.

Figure 2. Dependence of the normalised cutting force F_c/N on the edge angle in orthogonal machining of snow. The dashed line shows the function $y = F(x)$ given by eq.3 and based on the experimental results by Tada and Hirano (2002). The solid line shows the function $y = \tan(x)$.

Tada and Hirano (2002) carried out similar experiments with orthogonal and oblique machining of compacted snow (of density 383 kg/m^3) at the edge angle $15^\circ \leq \Psi \leq 90^\circ$. Using nonlinear regression, they obtained empirical expressions for the components of the snow reaction force as functions of cutting depth D , the edge angle Ψ , and the obliqueness angle γ (the angle of attack in skiing terminology). Because the method of regression requires to specify the space of trial functions, it is most useful for verifying theoretical models. Otherwise, it is purely descriptive, and the quality of data representation depends to a large degree on luck and experience. Regardless, Tada and Hirano (2002) found that in the orthogonal case ($\gamma = 90^\circ$)

$$\frac{F_c}{N} = AF(\Psi), \quad \text{where} \quad F(\Psi) = \frac{\Psi^{3.993} \tan(\Psi)^{0.0153}}{\sin(\Psi)^{2.94}} \quad (3)$$

where $A = 1.7 \times 10^{-7}$. This value of the coefficient A is unrealistic and presumably results from a typographical error. In fact, it is in conflict with the plots of the force components presented in figure 6 of their paper, which suggest $A \simeq 1$. However, the function $F(\Psi)$ is very close to $\tan(\Psi)$ for $0^\circ < \Psi \lesssim 70^\circ$ (see figure 2) and hence in a very good agreement with what was found for ice cutting by Lieu and Mote (1984).

In oblique machining, the tool edge is no longer orthogonal to the relative velocity, and the orthogonal projection of the reaction force onto the workpiece surface is no longer aligned with the vector of relative velocity. From the kinematic viewpoint, the oblique case differs from the orthogonal case only by changing the direction of the relative motion between the rake face and the chip, which now has a component along the cutting edge. This motion does not effect the cutting process directly but only via the modification of the friction force. If the friction force is small, this difference should not have much of an effect on the process of machining – in a reference frame aligned with the cutting edge, the reaction force experienced by the tool should remain approximately the same. Hence we may write

$$\mathbf{F}_r = \mathbf{N} + \mathbf{F}_c, \quad (4)$$

where

$$\mathbf{F}_c = F_c \hat{\mathbf{n}}_s, \quad (5)$$

where F_c is the same as in the case of orthogonal machining and

$$\hat{\mathbf{n}}_s = \frac{(\hat{\mathbf{m}} \times \hat{\mathbf{s}}) \times \hat{\mathbf{s}}}{|\hat{\mathbf{m}} \times \hat{\mathbf{s}}|} \quad (6)$$

is the unit vector in the plane of the snow surface which is normal to the edge and points to the side opposite to the direction of motion (see figure 3). This is the same as in the theory of oblique machining by Lin and Oxley (1972) in the limit of vanishing friction between the tool and the chip. The same approximation was used in Lieu and Mote (1985).

Figure 3. Separation of the cutting force $\mathbf{F}_c$ into braking $\mathbf{F}_b$ and turning $\mathbf{F}_t$ components. $\hat{\mathbf{s}}$ is the unit vector aligned with the ski, $\hat{\mathbf{m}}$ is the unit vector in the direction of motion on the snow, $\hat{\mathbf{n}}_s$ is the unit vector orthogonal to $\hat{\mathbf{s}}$ as described by equation (6), and γ is the angle of attack.

In application to skiing, the component of $\mathbf{F}_c$ aligned with the relative velocity is responsible for braking and the component orthogonal to it is responsible for turning. Hence we denote these components as F_b and F_t respectively. It is easy to see that

$$F_b = F_c \sin \gamma \quad \text{and} \quad F_t = F_c \cos \gamma. \quad (7)$$

Thus, the cutting force contributes mostly to braking at large angles of attack and mostly to turning at small angles of attack. (In the theory of machining, $\mathbf{F}_b$ is still called the cutting force and $\mathbf{F}_t$ is called the radial force.)

In order to see how well or otherwise our simplified model of machining can describe the actual machining of snow, we used the results of the experimental study by Tada and Kobayashi (2005). (This paper is available only in Japanese, and we used the automated translation service provided by Google to figure out the basic details of this experiment.) It involved a snow specimen of density 550 kg/m^3 placed on a rotating table and machined with curved tools which ensured the same angle of attack at every position on the cutting edge. The main result of the study is an array of measured forces (N , F_b , and F_t), corresponding to preselected values of the cutting depth, edge angle, angle of attack, and cutting speed. These data were presented only in the form of plots, so we had to use graphic application GIMP to manually convert the data back into digital form and then to calculate the normalised braking and

Figure 4. Fitting of the experimental data by Tada and Kobayashi (2005). The filled circles are the experimental points for the normalised braking force F_b/N and the squares are the experimental points for the normalised turning force F_t/N . The dashed lines correspond to $F_c = \tan \Psi N$ and the solid lines correspond to $F_c = (\tan \Psi + 0.2)N$.

turning forces, F_b/N and F_t/N . Fortunately, the plots had very few experimental points, which made their identification easy.

First we compared the data to the model with vanishing edge force, where $F_c/N = \tan \Psi$. In figure 4, the corresponding theoretical curves are represented by dashed lines. One can see that the agreement is reasonable for $\Psi = 45^\circ$, but rather poor for $\Psi = 15^\circ$, indicating significant edge force. Visual inspection of the data for orthogonal machining suggests that the data points are shifted above the theoretical curve by about 0.2. Such a shift is a manifestation of the edge force (Guo & Chou, 2004; Wyen & K. Wegener, 2010). So next we tried $F_c/N = \tan \Psi + 0.2$. Using this modification we obtained the theoretical curves for the braking and turning forces shown in figure 4 as solid lines. One can see a significant improvement for the $\Psi = 15^\circ$ case and some improvement for the $\Psi = 45^\circ$ case as well.

Equation (7) predicts $F_b = F_t$ for $\gamma = 45^\circ$. However, the experimental data show that this equality of forces occurs at a somewhat lesser angle of attack. This could be a manifestation of the friction force. However, the effect is sufficiently small and at the current stage in the development of the theory of skiing it can be ignored as a secondary factor.

When it comes to applying the results of industrial machining to skiing, one has to take into account the following clear difference between the setups. In industrial machining, the cutting depth D is a control parameter and the reaction force acting on the tool is a response variable. In skiing, the cutting depth is also a response variable determined by the dynamics of the ski-skier system.

Figure 5. A skier side-slipping down the fall line. This is a screen capture from the video “Alltracks Ski Tips. Skiing with flow”, published on YouTube by Alltracks Academy (<https://www.youtube.com/watch?v=NoQsLdRD7nE>). (The image is used with permission from Alltracks Academy, West Tytherley, Hampshire, SP5 1LX, UK)

Results

Side-slipping down the fall line

As a first application to skiing, we consider one of its most basic manoeuvre, the side-slipping down the fall line. Since this manoeuvre requires the skis to be kept perpendicular to the fall line, it corresponds to the case of orthogonal machining (see figure 5). Assuming that the air resistance is small, we can write the equation of motion as

$$m\mathbf{a} = m\mathbf{g} + \mathbf{F}_r, \quad (8)$$

where m is the mass of the skier, $\mathbf{a}$ is their acceleration, $\mathbf{g}$ is the gravitational acceleration, and $\mathbf{F}_r$ is the snow reaction force due to its machining as given by equation (4).

In contrast to industrial machining, where the tool is firmly held in its place by fixing devices, a skier has to find a balanced position over their skis. This dictates their inclination to the ski slope and the edge angle. Hence before we solve the equation of motion, we have to deal with skier’s balance.

In a balanced position, the total torque acting on a mechanical body has to vanish (Landau & Lifshitz, 1969). Since both the gravity and inertial forces can be treated as applied at skier’s centre of mass (CM), this condition can be formulated as vanishing of the torque relative to the CM due to the snow reaction force. This force is applied partly at the skier’s downhill foot and partly at the skier’s uphill foot, which

Figure 6. Origin of the torque relative to the skier CM associated with the force F_n for negative (left panel) and positive (right panel) angulation $\Theta = \Psi - \Phi$.

makes the problem a bit complicated. In order to simplify the analysis, let us first assume that the force originates entirely at the downhill foot. In this case, the torque condition implies that the snow reaction force F_r is parallel to the line connecting this foot with the CM, the “CM inclination line”. Here we denote the angle between this line and the normal to the slope surface as Φ . This angle may differ from the ski edge angle Ψ , mostly because skiers can angulate their body at the hip and at the knees. Here we denote the difference as $\Theta = \Psi - \Phi$ and call it the angulation. Let us suppose that F_r is dominated by the force F_n , which is orthogonal to the ski base. Then for $\Theta < 0$, the snow reaction force has a component orthogonal to the CM inclination line and hence produces a torque. As one can see in the left panel of figure 6, this torque further increases the CM inclination angle Φ (via a clockwise rotation in this figure). In terms of skiing experiences, the ski slips from under the skier body, resulting in a loss of support and a fall. The edge force, F_{edge} always has a smaller inclination to the slope than F_n , and even it did not vanish in this case, it would not prevent the slippage but only accelerated it.

For $\Theta = 0$, F_r is aligned with the CM line, and we have a balance, but it is unstable. Any increase of Φ above Ψ , no matter how small, brings about a loss of balance.

For $\Theta > 0$, the configuration is more interesting. Now the torque due to F_n implies a counter-clockwise rotation about the CM, but such a rotation is not allowed by the uncut snow under the ski edge (see the right panel of figure 6). Hence the edge force F_{edge} is significant and in the balanced configuration its torque cancels the torque due to F_n . This implies that the resultant total snow reaction force F_r is still aligned with the CM inclination line. This configuration is stable as small perturbations of either Φ or Ψ do not change the sign of Θ and the edge force automatically adjusts to the perturbations to maintain vanishing torque. This enhanced stability must be the main reason why all proficient skiers use angulation, and hereafter we will assume $\Theta > 0$.

When both the uphill and the downhill skis are taken into consideration, the analysis of torque balance becomes a little bit more complicated because the total torque may vanish even when the torques originated at each ski do not, provided they have opposite directions. However, in this case, one of the skis will be slipping from under skier's CM, leading to undesired separation or collision between the skis. It is much safer to keep both skis at zero torque, which requires $\Phi < \Psi$ for each ski. From the geometry of the problem it follows that this condition is easier to satisfy for the uphill ski (see figure 7), which is even allowed to have a smaller edge angle. The total snow reaction force will have the inclination angle $\Phi_u < \Phi_{\text{total}} < \Phi_d$, where Φ_u and Φ_d are the inclination angles of the uphill and downhill ski respectively. When the loading of the downhill ski increases, $\Phi_{\text{total}} \rightarrow \Phi_d$.

The increase of the edge angle associated with positive angulation may result in ski edges digging deeper into the snow and “hooking”. That is they may stop crashing the snow, leading to termination of skidding. When this occurs, the skier may tumble over the hooked ski. In such a “hard braking”, the hooked ski plays the role of an affixed axis of rotation, and the torque calculations have to be done not relative to the CM but relative to the ski edge. The tumbling occurs when the angular velocity of the CM relative to the edge is high enough to move it over the vertical position against the gravity force. When the skier increases the CM inclination in anticipation of the problem and thus avoids tumbling, the torque relative to the downhill ski associated with the gravity force will be pushing the skier on the snow in the uphill direction. This is where the uphill ski becomes important for controlling the total torque and restoring the equilibrium. In extreme cases, the skier can push against the snow with their uphill hand as well.

Because skiers do load predominantly the downhill ski, we will ignore the contribution due to the uphill ski in the rest of the paper. This significantly simplifies the analysis.

Using Cartesian coordinates with the basis vectors $\hat{\mathbf{k}}$ (normal to the ski slope), $\hat{\mathbf{i}}$ (parallel to the fall line), and $\hat{\mathbf{j}}$ (perpendicular to the other two), we can write $\mathbf{a} = a\hat{\mathbf{i}}$, $\mathbf{g} = g \sin \alpha \hat{\mathbf{i}} - g \cos \alpha \hat{\mathbf{k}}$, $\mathbf{N} = N\hat{\mathbf{k}}$, and

$$\mathbf{F}_r = -N \tan \Phi \hat{\mathbf{i}} + N \hat{\mathbf{k}}. \quad (9)$$

Projecting equation (8) on the direction of $\hat{\mathbf{k}}$, we immediately find that

$$N = mg \cos \alpha, \quad (10)$$

whereas from its projection onto the plane of the ski slope it follows that

$$a = g \cos \alpha (\tan \alpha - \tan \Phi). \quad (11)$$

This result tells us that for sideslipping at constant speed the inclination angle has to satisfy the condition

$$\Phi = \alpha. \quad (12)$$

Figure 7. Geometry of the problem in the case where both the downhill (d) and the uphill (u) skis are loaded.

In order to speed up their descent skiers have to reduce their inclination, and in order to slow down (and finally stop) they have to increase it. This is in qualitative agreement with skiing practice. Moreover, the critical value of Φ also seems about right (c.f. figure 5). Unfortunately, we are not aware of any proper experimental study of this manoeuvre.

In the so-called “hockey stop” manoeuvre, skiers (skaters) quickly pivot and push forward their skis (skates). The idea is to achieve the angle of attack $\delta \approx 90^\circ$, where the cutting force is reduced to pure braking and to increase the CM inclination in order to increase the braking force further and to reach a balanced position at the same time. Equation (11) can be used to obtain crude estimates of the corresponding braking time and the distance to full stop as

$$T_{\text{br}} = V_0/a \quad \text{and} \quad L_{\text{br}} = \frac{V_0^2}{2a}$$

respectively, where V_0 is the initial speed. For $\alpha = 15^\circ$ and $V_0 = 10 \text{ m/s}$ and $\mu_\perp^0 = 0$, equation (11) predicts $a = 7.1 \text{ m/s}^2$, $T_{\text{br}} = 1.4 \text{ s}$, and $L_{\text{br}} = 7.0 \text{ m}$ if $\Phi = 45^\circ$ and $a = 14 \text{ m/s}^2$, $T_{\text{br}} = 0.7 \text{ s}$, and $L_{\text{br}} = 3.5 \text{ m}$ if $\Phi = 60^\circ$. These values seem quite realistic, though the author could not find any experimental study of the manoeuvre required for quantitative comparison.

Traversing

The second basic skiing manoeuvre, we consider in this paper, is the straight line skidded traverse. With zero sidecut skis, straight traverse is possible without any side-slipping. In this case, it is characterised by a rapid acceleration until the gravitational force in the direction of motion is balanced by the aerodynamic drag force. On shaped skis, such traverse becomes impossible as the skis run on edge and engage in a carving turn instead. If however the skis are put at a finite angle of attack

Figure 8. A skier traversing a ski slope at fixed angle of attack. This is a screen capture from the video “Alltracks Ski Tips. Skiing with flow”, published on YouTube by Alltracks Academy (<https://www.youtube.com/watch?v=NoQsLdRD7nE>).

and hence side-slip, the turning force acting perpendicular to the direction of motion can balance the perpendicular component of the gravity force in the slope plane. (This is very different to the effect of Coulomb’s friction where the friction force is aligned with velocity and hence cannot neutralise the gravitational pull towards the fall line.) One can find few demonstrations of skidded straight traverse on YouTube, including the one we used to produce the image presented in figure 8.

The skidded traverse is usually faster than the side-slipping down the fall line and the aerodynamic drag may become significant as a result. For this reason we write the equation of motion as

$$m\mathbf{a} = m\mathbf{g} + \mathbf{F}_r - F_d\hat{\mathbf{m}}, \quad (13)$$

where $-F_d\hat{\mathbf{m}}$ is the aerodynamics drag force. The snow reaction force $\mathbf{F}_r = \mathbf{F}_n + \mathbf{F}_{\text{edge}} + \mathbf{N}$ is essentially the same as in the case of fall-line slipping. It is orthogonal to the ski axis, and the only difference is that the skis are no longer required to be orthogonal to the fall line. Repeating the same argument as in the previous section, we conclude that the condition of vanishing torque due to this force can be satisfied when $\Psi \geq \Phi$, in which case

Figure 9. Key unit vectors and angles in the plane of the ski slope which emerge in the problem of traverse.

$$\mathbf{F}_r = N \tan \Phi \hat{\mathbf{n}}_s + N \hat{\mathbf{k}}. \quad (14)$$

In principle, the aerodynamic drag force can also contribute to the total torque about skier's CM. It is difficult to calculate this torque due to the complicated shape of human body. However, the application area of the drag force is spread around the CM and hence the effective lever must be significantly shorter compared to that of the snow reaction force. Moreover, the aerodynamic drag force varies like v^2 and is rather weak at lower speeds. For these reasons, we ignore its contribution.

Combining equations (13) with (14) we obtain

$$m\mathbf{a} = m\mathbf{g} + N \tan \Phi \hat{\mathbf{n}}_s + N \hat{\mathbf{k}} - F_d \hat{\mathbf{m}}. \quad (15)$$

Projecting this equation onto the direction of $\hat{\mathbf{k}}$ yields $N = mg \cos \alpha$, the same result as before. Projecting it onto the slope plane now yields

$$\mathbf{a}/g = \sin \alpha \hat{\mathbf{i}} + \cos \alpha \tan \Phi \hat{\mathbf{n}}_s - \eta \hat{\mathbf{m}}, \quad (16)$$

where $\eta = F_d/mg$. Let us denote the angle between the velocity vector and the fall line as β (the angle of traverse) and agree that the angle of attack $-90^\circ \leq \gamma \leq 90^\circ$ is positive in the rotational direction away from the fall-line and negative in the opposite direction (see figure 9). Hence $\hat{\mathbf{m}} = (\cos \beta, \sin \beta, 0)$ and $\hat{\mathbf{n}}_s = \text{sign}(\gamma) (-\sin(\gamma + \beta), \cos(\gamma + \beta), 0)$.

The skier trajectory is straight provided $\mathbf{a} = a\hat{\mathbf{m}}$ and hence normal to $\hat{\mathbf{n}} = (-\sin \beta, \cos \beta, 0)$. Hence, we can write this condition as

$$\sin \alpha (\hat{\mathbf{i}} \cdot \hat{\mathbf{n}}) + \cos \alpha \tan \Phi (\hat{\mathbf{n}}_s \cdot \hat{\mathbf{n}}) = 0, \quad (17)$$

or

$$\tan \alpha \sin \beta = \text{sign}(\gamma) \tan \Phi \cos \gamma. \quad (18)$$

Clearly, for $0^\circ < \alpha, \beta, \Phi < 90^\circ$, this equation can be satisfied only if $\gamma > 0$ and hence the skis are pivoted away from the fall line (see figure 9). Conversely, if skis are pivoted towards the fall line ($\gamma < 0$), the trajectory must be curved. In fact, this is exactly how skis are pivoted in a skidded skiing turn. For side slipping down the fall line we have $\beta = 0$, and equation (18) implies $\gamma = 90^\circ$. This justifies the assumption $\gamma = 90^\circ$ made in the previous section on the basis of observations.

In order to find the magnitude of the acceleration, we project equation (16) onto the direction of $\hat{\mathbf{m}}$ and obtain

$$a/g = \sin \alpha \cos \beta - |\sin \gamma| \tan \Phi \cos \alpha - \eta. \quad (19)$$

Consider the case where both the acceleration and the aerodynamic drag are negligibly small, $a/g = \eta = 0$. In this case, equations (18) and (19) reduce to

$$\tan \Phi = \tan \alpha \quad \text{and} \quad \beta + \gamma = 90^\circ. \quad (20)$$

Hence like in the case of orthogonal side-slipping down to the fall line, the CM inclination line is vertical and the skis are perpendicular to the fall line. This explains why skier's postures in orthogonal and traverse side slipping look very much the same (compare figures 5 and 8.)

Using modern technology, all the parameters appearing in equations (18)-(19) can be measured (e.g. Reid, 2010), and hence the applicability of our model can be addressed quantitatively in field trials on real ski slopes. A relevant study has already been carried by Kaps, Nachbauer, and Mössner (1996), who studied straight traversing with the aim of determining the coefficient of kinetic friction and the aerodynamic drag area in alpine skiing. They set a 25 meters long corridor at the angle of traverse $\beta = 50^\circ$ on a slope with the inclination $\alpha = 18^\circ$. The skier was asked to avoid side-slipping, which raises doubts whether this study is really relevant. However, they comment that the skier could not perform this task exactly and name snow compression, shearing, and ploughing as a possible explanation for the high friction coefficients measured in these trials. This suggests that some degree of side-slipping did occur. In a separate experiment involving gliding down the fall line ($\gamma = 0$) they measured a very low coefficient of friction $\mu = 0.0085$.

In order to calculate the coefficients of Coulomb's friction and aerodynamic drag they needed a dynamical model of skier's motion. As we have commented earlier, in a dynamical model that includes only Coulomb's friction and the gravitational force, straight line traversing is impossible. The incorporation of the aerodynamic drag does not change the outcome because like Coulomb's friction this force is aligned with the velocity vector. In order to resolve this predicament, Kaps et al. (1996) introduced an algebraic constraint on the skier position vector and used the d'Alembert principle to implicitly introduce an additional force of unspecified origin that makes such a straight motion possible. Using this model to fit the experimental data, they estimated the coefficient of kinetic friction for several runs. The results are given in Table 1. They

Table 1

Parameters of straight traverse runs studied in Kaps et al. (1996). v_s , v_f and μ_{eff} are the initial and final speeds of the runs, and the effective coefficient of kinetic friction as measured in that study. The values of the remaining parameters are results of our calculations. These are the angle of attack δ , the angle between the fall line and the longitudinal axis of skis $\gamma + \beta$, the CM inclination angle Φ , and the angle δ between the CM inclination line and the vertical direction.

run	v_s	v_f	μ_{eff}	γ	$\gamma + \beta$	Φ	δ
1	0.6	10.6	0.064	14°	64°	14°	8°
2	11.0	13.4	0.128	27°	77°	16°	4°
3	14.7	16.6	0.153	32°	82°	16°	3°

could not determine the aerodynamic drag force, presumably because it was rather low.

Keeping in mind that these experiments may not have been perfectly tuned to our purposes, we simply check whether their results could be explained in our model with realistic parameters. This exercise should be useful to future trials too. First we notice that equation (19) can be written as

$$a/g = \sin \alpha \cos \beta - \mu_{\text{eff}} \cos \alpha - \eta, \quad (21)$$

where

$$\mu_{\text{eff}} = \tan \Phi \sin \gamma \quad (22)$$

plays the role of the coefficient of Coulomb's friction in the model of Kaps et al. (1996). Solving this equation together with the straight-traverse condition (18) for γ and Ψ we obtain

$$\tan \gamma = \frac{\mu_{\text{eff}}}{\tan \alpha \sin \beta} \quad (23)$$

and

$$\tan \Phi = \frac{\mu_{\text{eff}}}{\sin \gamma}. \quad (24)$$

The angle δ between the CM inclination line and the vertical direction can also be useful. The unit vector of the vertical direction is $\hat{\mathbf{g}} = \cos \alpha \hat{\mathbf{k}} - \sin \alpha \hat{\mathbf{i}}$, whereas the unit vector of the CM inclination line $\hat{\mathbf{c}} = \sin \Phi \hat{\mathbf{n}}_s + \cos \Phi \hat{\mathbf{k}}$. Via scalar multiplication of these vectors we find

$$\sin \delta = \sin \alpha \sin(\gamma + \beta) \sin \Phi + \cos \alpha \cos \Phi. \quad (25)$$

Using these equations we calculate the values of γ , Φ , and δ given in Table 1.

The derived angles of attack are relatively small, particularly in run 1, where the increase in velocity is close to the maximum provided by the drop of the gravitational potential, thus indicating weak braking force. In runs 2 and 3 only about 50% of the gravitational potential is converted into the skier kinetic energy, indicating increased braking. This is in agreement with the increased angle of attack. The CM inclination line is almost vertical in runs 2 and 3. Unfortunately, Kaps et al. (1996) provide no quantitative information that could be used to check these parameters.

Sahashi and Ichino (1998) carried a similar to Kaps et al. (1996) measurements of the effective coefficient of kinetic friction μ_{eff} , but for various types of ski turns, such as snow plow, stem, parallel and wedeln. They obtained the values of μ_{eff} varying between 0.01 and 0.3 throughout the turn. If the lower value is reasonable for normal snow friction, the top one is way too high, even for anisotropic friction. Since in all these turns the turning ski is put on edge, the snow machining is a more realistic explanation of the high values. Using equation (22) we can estimate the maximum inclination angle required by their data. According to their plots, the instantaneous angle of attack in these turns rarely exceeded 20° . For the angle of attack $\gamma = 10^\circ$, the typical value for the phase in the turn where the measured μ_{eff} peaks, we obtain $\Phi_{\text{max}} = 60^\circ$. This is a large inclination for a normal recreational skier, but the skier in the trials was a ski instructor certified by the Ski Association of Japan. Unfortunately, Sahashi and Ichino (1998) do not provide any quantitative information on the CM inclination and the ski edge angle.

Snow spaying in skiing and ice skating

Side-slipping or skidding is a part of many ski turns. Some turns can be fully skidded, others are hybrid and have a skidded section followed by a carved one. At high speed, skidding is accompanied by spectacular sprays of snow. To a lesser degree, it is also noticeable in orthogonal and diagonal side-slipping analysed in the previous sections. A similar phenomenon is observed in ice skating and in water skiing. Whereas water is a fluid and the physical mechanism of spraying can be described entirely by fluid dynamics, the ice and snow are solid and hence the mechanism has to be somewhat different. Following the study of ice-crushing by Jordaan and Timco (1988), we propose the following model based on the very high brittleness of these materials. As the ski (skate blade) machines the snow (ice) it crashed a portion of snow in its vicinity. In figure 10 this volume of crashed (pulverised) snow is represented by the parallelogram aligned with the ski base. Its horizontal width is denoted as W and its length along the ski base as L . After an event of snow crashing, the ski keeps its side-slipping with the mean speed $V_\perp = V \sin \gamma$ and squeezes the snow powder produced during this event out of the crashed layer. Then it hits the wall of undisturbed snow and drives another snow crashing event, and so on. The process continues in a quasi-periodic fashion.

Treating the powder as incompressible fluid, we can estimate the normal to the

Figure 10. *Left panel:* Snow spraying in a high-performance ski turn. *Right panel:* Squeezing pulverised snow out of the layer with the depth D and width W produced in the latest snow-crushing event.

slope component of the spray velocity $V_{\text{sp},n} = V_{\text{sp}} \sin \Psi$ as

$$V_{\text{sp},n} = \frac{D}{W} V_{\perp}, \quad (26)$$

where $D = L \sin \Psi$ is the cutting depth. It is also related to θ_{sp} , the opening angle of the spray fan via

$$V_{\text{sp},n} = V_{\perp} \frac{\tan \theta_{\text{sp}}}{\tan \gamma}. \quad (27)$$

Combining the two expression, we obtain

$$\frac{D}{W} = \frac{\tan \theta_{\text{sp}}}{\tan \gamma}. \quad (28)$$

Naked eye observation of numerous skiing videos, allows us to conclude that $\theta_{\text{sp}} \approx \gamma$ and hence $W \approx D$.

Alternatively, we may use the height of the spray trail, h , to estimate W/D . Once ejected, the ice particles move as projectiles and hence

$$h = \frac{V_{\text{sp},n}^2}{2g}. \quad (29)$$

Combining this result with (26), we obtain

$$\frac{D}{W} = \frac{\sqrt{2gh}}{V \sin \gamma}. \quad (30)$$

Using the typical for competitive slalom values $V = 15 \text{ m/s}$ and $\gamma = 10^\circ$ (Reid, 2010), as well as the realistic value $h = 0.5 \text{ m}$ (see the image in figure 10), we obtain

$W/D = 1.2$. Thus, the transverse length scale of crashed snow is about the same as the cutting depth. For icy snowpack D cannot be more than several millimetres (Mössner et al., 2013). Under these conditions, the layers of pulverised snow are very narrow and side slipping over a distance of one ski width is accompanied by numerous micro-events of snow crashing. A similar phenomenon should occur in ice skating.

Discussion and Implications

In this work, we aimed to understand if it was possible to build a simple approximate model of snow and ice machining that can be used for comprehensible analysis of various manoeuvres in alpine skiing that involve side-slipping. The main complications in the theory of industrial machining originate from the friction between the chip and the rake face of the cutting tool, and the interaction between the very edge of the tool and the workpiece. If these two are ignored then all what is left is the normal reaction force at the rake face (ski base), which is a great simplification allowing simple analytic expressions for the breaking and turning forces in skiing (and ice skating). The comparison between this simplified model and the results of the very limited number of experimental studies on the ice and snow machining shows a good agreement for intermediate edge angles, around $\Psi = 45^\circ$. For very high angle angles, $\Psi \gtrsim 75^\circ$, the ice data show significant deviation from the theoretical expectations, on the level of 15% in terms of the inclination of the ice reaction force (Lieu & Mote, 1984). The origin of this effect is not clear. For very small inclination angles, $\Psi = 15^\circ$, large deviations are exhibited by the snow data (Tada & Kobayashi, 2005). These are most likely due to the contribution of the edge force. A small modification of the model allows a much improved fit of the data.

Due to the highly brittle nature of ice, a characteristic feature of the interaction between moving ice slabs and rigid stationary structures is crashing of ice in the vicinity of the structures (e.g. Jordaan & Timco, 1988). This crashing occurs in a quasi-periodic fashion, with each crashing event followed by an episode of squeezing pulverised ice out of the crashed layer. We conjecture that the same occurs in machining of ice and compacted hard snow in skiing and interpret the phenomenon of ice and snow spraying in skiing and ice skating as a manifestation of these squeezing episodes. Our estimates show that the width of crashed layers is comparable to the cutting depth and hence can be very small. Thus we can be dealing with frequent micro-fracture events, leading to the impression of a continuous process. More experimental studies of the crashing process is required. These could involve high time-resolution video recording and force measurements. In fact, Lieu and Mote (1984) already reported tool chatter and the variability of cutting forces on the level of 10%. However, they give no power spectrum of these variations required to determine the length scales of the crashing process. Tada and Kobayashi (2005) also detected some variability of cutting forces.

In our simplified model we have completely ignored the friction force between the crashed snow (fragmented chip) and the ski base. The comparison of our model

with the data on snow machining by Tada and Kobayashi (2005), hints a disagreement at small edge angles that may be attributed to the friction – the model predicts equal magnitudes for the turning and cutting forces at the angle of attack $\gamma = 45^\circ$, whereas in the experiment this occurs at a somewhat smaller angle of attack. However, the difference is rather small. Given and the complexity of physics behind friction, its incorporation into the model of snow cutting does not seem to be warranted at this stage.

Expert skiers and particularly racers pay great attention to maintaining sharpness of ski edges. Detailed prescriptions exist on how exactly to sharpen them. This is known to be important for providing better grip on icy snow but no proper research into this matter has been done so far. In metal machining, the edge sharpness is known to influence the edge force (e.g. Wyen & K. Wegener, 2010), but no data exist for machining of snow and ice. Racers also consider important the base and side bevels of the ski edges, with different recommendations for different race disciplines. These vary by as little as 0.5° and yet even such a small variation is claimed to have a significant effect on performance.

In industrial setting, the main control parameters of machining is the cutting depth (the thickness of uncut chip), speed of machining, the rake face angle (ski edge angle), the edge inclination angle (the angle of attack) and the sharpness of the edge. These are fixed by the design of the cutting machine and its cutting tool. The forces applied by the workpiece to the tool are the reaction forces dictated by these parameters. Skiing and skating are different in this regard because most of these parameters are no longer fixed. Instead they are dynamical variables that normally change with the phase of a manoeuvre. This makes the problem more complicated.

Here we focused on one particular skiing manoeuvre, the straight-line traverse, where most of the control parameters do not change. Our simplified model of snow machining allows such solutions. This is in contrast to the more simplistic model of the ski-snow interaction, the model of Coulomb friction, where this manoeuvre is impossible.

One of the key constraints of the traverse is its fixed angle to the fall line. In addition, the normal to the slope component of the snow reaction force $\mathbf{N}$ does not depend on the machining process but is completely determined by the gravitational force acting on the skier and the inclination of the ski slope. Finally, the total torque applied to the skier vanishes. These conditions lead to the equation of motion controlled by the angle of attack and the skier (their CM) inclination to the ski slope. Given the skier acceleration along the line of traverse the model yields the corresponding values for the angle of attack and the skier inclination. This simplicity makes the traverse a perfect manoeuvre for confronting the theory with reality in trials on actual ski slopes. The only experimental study of traverse has been carried out by Kaps et al. (1996), with the aim of determining the coefficient of Coulomb friction. Unfortunately, they provided no data on the angle of attack and skier inclination.

In the context of this work, we are interested in the skidded part of generic skiing

turns. Attempts to model fully skidded turns were made by Hirano and Tada (1996) and Tada and Hirano (1999, 2002). To this aim they applied the point-mass dynamics to the skier CM, and assumed that the forces acting on the CM were the same as the ice and snow reaction forces acting on the skis. These forces were described by an empirical model derived from the laboratory experiments with a typical for industrial machining setup. The cutting depth and the edge angle of the ski were prescribed and the angle of attack was governed by the torques which were calculated using some model for the cutting force distribution along the ski. A much more sophisticated model of the ski-skier system was developed in Lieu and Mote (1985) and Renshaw and Mote (1989), using a solid body description of the skier and the Euler-Bernoulli beam model for the ski. However, they did not simulate a ski turn as such but focused on the ski deformation as it moved along a prescribed circular trajectory in a horizontal plane under the action of an artificial “propulsion” force aligned with the trajectory.

It would be very useful, to have a relatively simple dynamical model for a skidded ski turn. A model which can be easy to analyse and to draw instructive conclusions from. In this regard, our simplified model of snow machining could be a step in the right direction. It is interesting that in the application to traverse, we did not need to have a prescription for the edge force (which does not exist). The condition of skier balance (vanishing of the total torque) has completely determined this force. One may use the balance condition to simulate skidded turns as well. Since the skier inclination varies throughout the turn, this will be an approximation whose accuracy depends on how fast the variation actually is. The recent theoretical study of purely carved turns shows that the accuracy can be rather poor (Komissarov, 2020). Moreover, it is completely incompatible with the phase of transition between turns. A proper dynamical model is required to describe consecutive turns.

With regard to the hybrid turn it is interesting that as the angle of attack $\gamma \rightarrow 0$, the turning force in snow machining $F_t \rightarrow N \tan \Phi$, the same as the magnitude of the turning force in balanced carving (Howe, 1983; Komissarov, 2018). At the same time the braking force $F_b \rightarrow 0$. This could be the reason for the seamless transition between the skidding and carving phases of a hybrid turn, when executed by a proficient skier. If the total torque acting on the ski in the plane of a ski slope vanishes, then its orientation vector $\hat{s}$ remains constant throughout the skidding phase, whereas the braking component of the cutting force F_b reduces the velocity component normal to $\hat{s}$. Hence the reduction of the angle of attack is a natural tendency of skidding. However, the torque does not always vanish. First, the skier can introduce it via twisting action of their body and apply to the ski via the boot-binding joint. Second, the torque can arise through the ski-snow interaction itself. For example, if the same cutting force was applied to the mid-points of the front and tail sections of the ski, the resultant torque would not vanish because the front section is normally longer than the tail section. The effect of this torque would be to increase the angle of attack. Skiers can regulate the load distribution between the front and back sections of the

ski via fore/aft displacement of the CM relative to the ski centre. This implies that in basic models of the skidding phase of the hybrid turn, the angle of attack should be treated as a control parameter (a prescribed function), needed to account for the conscious participation of skiers in the skiing process.

Conclusion

We have developed an approximate model of oblique snow machining by a side-slipping (skidding) ski, where we ignored the friction between the ski base and the chipped snow. The theory agrees quite well with the laboratory experiments in ice and snow machining, suggesting that it can be useful in theoretical studies of skiing turns involving a skidding phase. The reduced role of friction compared to machining of metals may be down to two factors: 1) the slippery nature of ice and hard snow and 2) the production of a snow spray instead of a solid chip to due to the high brittleness of snow and ice. We apply this theory to some of the most basic manoeuvres of alpine skiing – side-slipping down the fall-line, traversing, and hockey stop. The results are in qualitative agreement with our observations of skiers executing these manoeuvres and personal skiing experience, as well as with the experimental data on the values of effective coefficient of friction. Rather straightforward field experiments on real ski slopes can be used for the purpose of proper quantitative verification of the model.

References

- Brown, C. (2009). Modeling edge-snow interactions using machining theory. In E. Müller, L. S., & T. Stöggli (Eds.), *Science and skiing iv* (p. 175-182). Maidenhead, UK: Meyer & Meyer Sport.
- Colbeck, S. (1992). *A review of the processes that control snow friction* (Tech. Rep.). Hanover, NH: Cold Regions Research and Engineering Laboratory (CRREL). Retrieved from <https://apps.dtic.mil/sti/citations/ADA252362>
- Guo, Y., & Chou, Y. (2004). The determination of ploughing force and its influence on material properties in metal cutting. *Journal of Materials Processing Technology*, 148(3), 368-375.
- Hirano, Y., & Tada, N. (1996, September). Numerical simulation of a turning alpine ski during recreational skiing. *Medicine and science in sports and exercise*, 28(9), 1209-1213.
- Howe, J. (1983). *Skiing mechanics*. Laporte, Colorado: Poudre Press.
- Jordaan, I., & Timco, G. (1988). Dynamics of the ice-crushing process. *Journal of Glaciology*, 34, 318-326.
- Kaps, P., Nachbauer, W., & Mössner, M. (1996). Determination of kinetic friction and drag area in alpine skiing. In C. Mote, R. Johnson, W. Hauser, & P. Schaff (Eds.), *Skiing trauma and safety: Tenth volume* (p. 165-177). West Conshohocken, PA: ASTM International.
- Kietzig, A.-M., Hatzikiriakos, S., & Englezos, P. (2010). Physics of ice friction. *Journal of Applied Physics*, 107, 081101.

- Komissarov, S. (2018). *Modelling of carving turns in alpine skiing*. (SportRxiv. <https://doi.org/10.31236/osf.io/u4ryc>)
- Komissarov, S. (2020). Dynamics of carving turns in alpine skiing: I: The basic centrifugal pendulum. *Sports Biomechanics*, 25(2), 193-200.
- Landau, L., & Lifshitz, E. (1969). *Mechanics*. Oxford: Pergamon Press.
- Lieu, D., & Mote, C. (1984). Experiments in the machining of ice at negative rake angles. *Journal of Glaciology*, 30, 77-81.
- Lieu, D., & Mote, C. (1985). Mechanics of the turning snow ski. In R. Johnson & C. Mote (Eds.), *Skiing trauma and safety: Tenth volume* (p. 117-140). Philadelphia: ASTM International.
- Lin, G., & Oxley, P. (1972). Mechanics of oblique machining: predicting chip geometry and cutting forces from work material properties and cutting conditions. *Proceedings of the Institution of Mechanical Engineers*, 186, 813-820.
- Lind, D., & Sanders, S. (1996). *The physics of skiing: Skiing at the triple point*. New York: Springer-Verlag.
- Merchant, M. (1945a). Mechanics of the metal cutting process. ii. plasticity conditions in orthogonal cutting. *J. Appl. Phys.*, 16, 318-324.
- Merchant, M. (1945b). Mechanics of the metal cutting process. i. orthogonal cutting and a type 2 chip. *J. Appl. Phys.*, 16, 267-275.
- Michalowski, R., & Mróz, Z. (1978). Associated and non-associated sliding rules in contact friction problems. *Arch. Mech.*, 30, 259-276.
- Mössner, M., Innerhofer, G., Schindelwig, K., Kaps, P., Schretter, H., & Nachbauer, W. (2013). Measurement of mechanical properties of snow for simulation of skiing. *J. Glaciology*, 59, 1170-1178.
- Mróz, Z., & Stupkiewicz, S. (1994). An anisotropic friction and wear model. *International Journal of Solids and Structures*, 31(8), 1113-1131.
- Nachbauer, W., Kaps, P., Hasler, M., & Mössner, M. (2016). Friction between ski and snow. In F. Braghin, C. F., S. Maldifassi, & S. Melzi (Eds.), *The engineering approach to winter sports* (p. 33). New York: Springer.
- Reid, R. (2010). *A kinematic and kinetic study of alpine skiing technique in slalom* (PhD dissertation). Norwegian School of Sport Sciences.
- Renshaw, A., & Mote, C. (1989). A model for the turning snow ski. *International Journal of Mechanical Sciences*, 31, 721-736.
- Sahashi, T., & Ichino, S. (1998). Coefficient of kinetic friction of snow skis during turning descents. *Japanese Journal of Applied Physics*, 37, 720-727.
- Schweizer, J., Michot, G., & Kirchner, H. (2004). On the fracture toughness of snow. *Annals of Glaciology*, 38(1), 1-8.
- Stevenson, R. (1998). The measurement of parasitic forces in orthogonal cutting. *International Journal of Machine Tools and Manufacture*, 38(1), 113-130.
- Tada, N., & Hirano, Y. (1999). Simulation of a turning ski using ice cutting data. *Sports Engineering*, 2(1), 55-64.
- Tada, N., & Hirano, Y. (2002). In search of the mechanics of a turning alpine ski using snow cutting force measurements. *Sports Engineering*, 5, 15-22.
- Tada, N., & Kobayashi, T. (2005). Measurement of snow cutting forces for analysis and design of a snow ski. *Journal of Ski Science*, 3, 11-21.

- Wyen, C.-F., & K. Wegener, K. (2010). Influence of cutting edge radius on cutting forces in machining titanium. *CIRP Annals*, 59(1), 93-96.
- Zmitrowicz, A. (1989). Mathematical descriptions of anisotropic friction. *Int. J. Solids Structures*, 25, 837-862.