Inclusion Problem with Polynomial Data in R^3

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Abstract

In this paper, we studied inclusion problem with polynomials data in R^3 . We provided a Theorem that guaranteed the solution of this problem. Moreover, by using standard potential theory, we approached the solution by using Green's Theorem.

1 Introduction

In this paper, we studied inclusion problem with polynomials data in R^3 . We provided a Theorem that guaranteed the solution of this problem. Moreover, by using standard potential theory, we approached the solution by using Green's Theorem.

Let $x = (x_1, x_2, x_3) \in R^3$, thus for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of non-negative integers we say x^α as monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. The degree of x^α is $\alpha_1 + \alpha_2 + \alpha_3$.

A polynomial is said to be homogeneous of degree m if it is a finite linear combination of monomial x^α of degree m ; $m = 0, 1, 2, \dots$.

Let $\mathcal{P}_m$ denotes the vector space of polynomials in R^3 , homogeneous of degree m , and $\mathcal{H}_m$ is the subspace of harmonic polynomials of degree m , then we have property by Axler and Ramey following

$$\mathcal{P}_m = \mathcal{H}_m + |x|^2 \mathcal{P}_{m-2}.$$

2 Problem Formulation of Inclusion Problem

Let Ω is a unit sphere in R^3 , with the boundary $\partial\Omega = \{x \in R^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.

Let also

$$D = \{x \in R^3 \quad ; 0 \leq |x| \leq R < 1\},$$

then we have inclusion problem as follows

$$\nabla [(1 + (k - 1)\mathcal{X}_D) \nabla u] = 0 \quad (1)$$

$$u|_{\Omega} = p, \quad (2)$$

where $\mathcal{X}_D$ is a characteristic function and p is an n -degree polynomial, $n \in \mathcal{P}_n$.

Let u_0 be the harmonic function on $\Gamma = \omega - D$ with boundary values f , that is

$$\Delta u_o = 0 \quad \text{in} \quad \Gamma. \quad (3)$$

$$u_o|_{\Omega} = p. \quad (4)$$

Let also u_i is a function that satisfies

$$\Delta u_i = 0 \quad \text{in} \quad D, \quad (5)$$

the compatibility condition

$$u_i|_{\partial D} = u_o|_{\partial D}, \quad (6)$$

and

$$\frac{\partial u_o}{\partial n} = k \frac{\partial u_i}{\partial n}. \quad (7)$$

Then the problem (1) is equivalent problem with the problem (3) and (5) with the boundary problem (4) and (6).

The solution of (1) is

$$u = u_o \mathcal{X}_{\Gamma} + u_i \mathcal{X}_D.$$

3 Technical Lemmas

This following Lemmas is needed to solve the Inclusion Problem:

Lemma 3.1 The Poincaré inequality *Let p , so that $1 \leq p \leq \infty$ and Ω a subset with at least one bound. Then there exists a constant C , depending only on Ω and p , so that, for every function u of the $W_0^{1,p}(\Omega)$ Sobolev space,*

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)}$$

Proof: See Lawrence Evans, 1998.

Lemma 2.2 The Lax-Milgram Theorem *Let ϕ be a bounded coercive bilinear form on a Hilbert Space H , then for every bounded linear function f on H , there exists a unique $x_f \in H$ such that*

$$f(x) = \phi(x, x_f),$$

for all $x \in H$.

Proof : See Debnath L and Mikusinsky, 1990.

Lemma 2.3 The Green's Theorem *Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C . If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then*

$$\oint_C (Ldx + Mdy) = \int \int_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy,$$

where the path of integration along C is counterclockwise.

(Steward J)

4 The Main Theorem

Here we provide the Theorem about the solution of the Inclusion problem. The Existence of this solution can be presented in the following Theorem:

Theorem 4.1 *There is exist a unique solution, u , of problem (1) and (2).*

Proof: For $p \in \mathcal{P}_n$, let p denote an extension of the boundary value p such that $p = 0$ in a subdomain $\tilde{D}$ with $D \subset \tilde{D} \subset \Omega$.

Let $u^0 = u - p$ so that $u^0 \in (\mathcal{P})_{\Xi}$ solves

$$\int_{\Omega} a \nabla u^0 \cdot \nabla v dx = - \int_{\Omega} a \nabla p \cdot \nabla v dx,$$

for all $v \in \mathcal{P}_n$, with $a = 1 + (k - 1)\chi_D$.

Since $a > 0$, Poincares's inequality and the Lax-Milgram Theorem imply the existence of a unique solution $u^0 \in \mathcal{P}_n$ (Gilbarg D and Trudinger). Therefore, we have $u = u^0 + p$ is a unique solution. Standard potential theory can be used to obtain a representation of the solution (1) and (2), one that can actually be used as the basis of a numerical approximation scheme.

Let w be the harmonic function on Ω with boundary values $p \in \mathcal{P}_n$, that is

$$\Delta w = 0 \quad \text{in} \quad \Omega. \quad (8)$$

$$w = p \quad \text{on} \quad \partial\Omega. \quad (9)$$

Define $u^0 = u - w$ in Ω , so that u^0 is harmonic in D and $\Omega - \bar{D}$.

$$\Delta u^0 = 0 \quad \text{in} \quad D. \quad (10)$$

$$\Delta u^0 = 0 \quad \text{in} \quad \Omega - \bar{D}. \quad (11)$$

It is easy to check that u^0 vanishes on $\partial\Omega$, because

$$\begin{aligned} u^0|_{\partial\Omega} &= (u - w)|_{\partial\Omega} \\ &= p - p = 0. \end{aligned}$$

Let $G(x, y)$ denote the Green's function for the Laplacian in Ω subject to Dirichlet boundary condition. Then Green's Theorem yields the representations as follows:

for $x \in D$

$$u^0(x) = \int_{\partial D} \left[\frac{u_i^0(y)}{\partial n} G(x, y) - u_i^0(y) \frac{\partial G(x, y)}{\partial n_y} \right] dS_y,$$

for $x \in \Omega - \bar{D}$

$$u^0(x) = - \int_{\partial D} \left[\frac{u_o^0(y)}{\partial n} G(x, y) - u_o^0(y) \frac{\partial G(x, y)}{\partial n_y} \right] dS_y,$$

where $u_i^0 = u_i - w$ and $u_o^0 = u_o - w$.

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