On the Rationality and Transcendentality of Solutions to the Equation $x^n + y^n = z^n$

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Abstract

In this paper, the author develops 2 theorems that serve as extensions to the well-known Fermat's Last Theorem in the field of number theory. The first proposed theorem in this paper states that if there exist numbers x, y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$, then both/either x and/or y must be irrational. The second proposed theorem in this paper states that if there exist a number x , a transcendental number y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$, then x must be irrational. The proposed theorems in this paper expands on the notion that if there exist numbers x, y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$, then at least x or y is not an integer, which is stated in Fermat's Last Theorem.

1 Introduction

For hundreds of years, Fermat's Last Theorem, also called Fermat's Big Theorem, has puzzled countless mathematicians around the world. Believed to be first proposed by Pierre de Fermat, although he had never wrote the so-called "marvelous" proof down himself, the theorem has prompted thousands of attempts to confirm or refute its validity, which has spawned new realms of mathematics as by-products. It was not until the year 1995 did the theorem was proven by Sir Andrew John Wiles, a British mathematician. The proof provided by Wiles is a proof of a special case of the modularity theorem for elliptic curves combined with Ribet's Theorem, and it is widely accepted as an official proof of Fermat's Last Theorem [2][1]. In this paper, the author seeks to expand upon the proven Fermat's Last Theorem, namely identifying the rationality of x and y while assuming z is an integer and all other

conditions of Fermat's Last Theorem is satisfied, in the first half of this paper. In the second half of this paper, the author makes a connection between rationality and transcendentalty of solutions to the equation $x^n + y^n = z^n$, given that all other required conditions are satisfied.

2 On Rationality

2.1 A Connection with Trigonometry Functions:

Lemma 1. *If there exist numbers x, y , and integer z such that $x^n + y^n = z^n$ for any integer n , then $\sin(\pi \sqrt[n]{x^n + y^n})$ must be equal to 0.*

First Proof of Lemma 1. Suppose there are $x, y, z \in R$ such that $x^n + y^n = z^n$:

$$\begin{aligned} z^n &= x^n + y^n \\ z &= \sqrt[n]{x^n + y^n} \end{aligned}$$

Let $\Psi(x)$ be defined as a function such that $\Psi(x) = \sin(z\pi)$ and assume that z is an integer. Because $\sin(z\pi)$ is always equal to 0 with z being an integer such that $z\pi$ is an integer multiple of π , we have:

$$\begin{aligned} \Psi(x) &= 0 \\ \sin(z\pi) &= 0 \\ \sin(\pi \sqrt[n]{x^n + y^n}) &= 0 \end{aligned}$$

□

Second Proof of Lemma 1. Similar to the first proof, suppose there are $x, y, z \in R$ such that $x^n + y^n = z^n$:

$$\begin{aligned} z^n &= x^n + y^n \\ z &= \sqrt[n]{x^n + y^n} \\ (-1)^z &= (-1)^{\sqrt[n]{x^n + y^n}} \\ (-1)^z &= e^{i(\pi \sqrt[n]{x^n + y^n})} \\ (-1)^z &= \cos(\pi \sqrt[n]{x^n + y^n}) + i\sin(\pi \sqrt[n]{x^n + y^n}) \end{aligned}$$

As a matter of fact, if z is not an integer, $(-1)^z$ would have an imaginary part when it is evaluated. Conversely, if z is an integer, $(-1)^z$ would only have the real part without an imaginary part. Therefore, when we assume that z is an integer, $\sin(\pi \sqrt[n]{x^n + y^n})$ must be equal to 0. □

2.2 Graphical Visualization of Lemma 1 and Theorem 1:

Let us examine the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ for $n = 3$ as included below:

In this paper, we are only interested in the first quadrant of the graph, as Fermat's Last Theorem constraints x and y to be positive numbers. Moreover, this graph can be used to represent the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ for any integer $n > 2$ as the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ behaves similarly to the graph of $\sin(\pi \sqrt[3]{x^3 + y^3}) = 0$ for any integer $n > 2$ in the first quadrant. Interestingly, the graph $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ does not behave according to this pattern at $n = 1$ and $n = 2$, hence the fact that Fermat's Last Theorem requires n to be greater than 2.

First of all, it is noticeable that every r^{th} "level" (a coined term) of the graph in the first quadrant of the graph behaves as if it follows two asymptotes $x = r$ and $y = r$ for $0 < x < r$ and $0 < y < r$, in which the graph approaches infinitely close to $y = r$ and $x = r$ as x approaches 0 and r , respectively. This is because the equation for each of the level in the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ is $\sqrt[n]{x^n + y^n} = r$, and thus the derivative of each level is $-\frac{x^{n-1}}{y^{n-1}}$, which makes it impossible to reach $y = r$ and $x = r$ as $-\frac{x^{n-1}}{y^{n-1}}$ cannot be either 0 or $-\infty$ if the argument x and y has to be greater than 0. Therefore, in the first level, we can firmly say that x and y are certainly not integers. However, this raises the question of whether or not other levels of the same graph do pass through a point (x, y) with x, y being integers.

To come up with an answer to this question, we need to consider Lemma 2:

Lemma 2. *If the first level passes through the point (x, y) , the r^{th} level would pass through the point (rx, ry) .*

Proof of Lemma 2: Visually, it is noticeable that if we "stretch" out

the first level by a factor of 2 away from the point $(0,0)$, it will eventually overlap the second level. Similarly, if we stretch out the first level by a factor of 3 away from the point $(0,0)$, it will eventually overlap the third level, and this pattern holds true for every integer factor. This observation leads to a hypothesis that if the first level passes through the point (x,y) and if we stretch the first level by a factor of r away from the point $(0,0)$, the transformed version of it, which is the r^{th} level, would pass through the point (rx,ry) as the point (x,y) is stretched away from the point $(0,0)$ by a factor of r . To confirm this hypothesis, we need to examine at the function in mathematical terms, namely $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$.

Claim 1. *Because $\sin(x)$ is a periodic function, $\sin(x) = 0$ holds true for any x if x is an integer multiple of π . Therefore, $\sqrt[n]{x^n + y^n}$ must be an integer, and this condition can be related to our original assumption that z is an integer and that $z^n = x^n + y^n$.*

Claim 2. *Because the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ is a set of levels, we can express each of the level of the function individually. As a matter of fact, the first level of the function $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ can be expressed as $\sqrt[n]{x^n + y^n} = 1$, the second as $\sqrt[n]{x^n + y^n} = 2$, the third as $\sqrt[n]{x^n + y^n} = 3$, and so on. This fact holds true because each of the levels contains a pair of x and y coordinates that makes $\sqrt[n]{x^n + y^n}$ an integer, which satisfies Claim 1, and thus $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$.*

Based on Claim 2, we can express the ratio of the x and y coordinates of the first level to the r^{th} level as $\frac{1}{r}$. Therefore, if the first level, which is $\sqrt[n]{x^n + y^n} = 1$, is stretch by a factor of r , the equation representing the new level is $r \sqrt[n]{x^n + y^n} = r$, or $\sqrt[n]{(rx)^n + (ry)^n} = r$, hence the statement in Lemma 2. $\square$

Corollary 1. *If there exist numbers x, y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$, then x and y cannot be both rational numbers.*

Proof of Corollary 1 by Contradiction. To start this proof, we assume that there exist rational numbers $x = \frac{a}{b}$ and $y = \frac{c}{d}$, with a, b, c , and d being positive integers, such that x and y :

- are either whole numbers or in reduced form
- satisfy the equation $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$
- in the range from 0 to 1, exclusive

Then, let φ be the product of b and d . Because x and y satisfy the equation $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ and $0 < x < 1, 0 < y < 1$, the point (x, y) must lie on the first level of the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$.

Based on Lemma 2, we can conclude that the point $(\varphi x, \varphi y)$ lies on the φ^{th} level of the graph, as φ is certainly an integer as a product of 2 other integers b and d . Furthermore, because $x = \frac{a}{b}$, $y = \frac{c}{d}$, and $\varphi = bd$, we can express the point $(\varphi x, \varphi y)$ as $(bd\frac{a}{b}, bd\frac{c}{d})$, which simplifies to (ad, cb) . Because a, b, c , and d are defined as integers, both ad and cb must also both be integers. Therefore, if the assumption that x and y are both rational numbers are true, there must be a pair(s) of integer solutions to $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$, and that there are integers x and y such that $x^n + y^n = z^n$ for any integer $n > 2$ and z being an integer. However, this is certainly not possible because according to Fermat's Last Theorem, there are no integer solutions to $x^n + y^n = z^n$ for any integer $n > 2$. As a result, the assumption that there exist rational numbers x and y such that $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ and subsequently $x^n + y^n = z^n$ for any integer $n > 2$ and z being an integer is false. $\square$

Theorem 1. *If there exist numbers x, y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$, then both/either x and/or y must be irrational .*

Proof of Theorem 1: Suppose that there exists a point $(\varphi x, \varphi y)$ that lies on the φ^{th} level of the graph $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$. According to Corollary 1, φx and φy cannot be both integers, that is, at least x or y , when multiplies with an integer φ , has some number of decimal places after the decimal point. Because φ can gets infinitely large, if a number is rational, there will always be a φ such that the product of that number and φ is an integer. The only condition that must exist to counter this dilemma is that x and/or y must be irrational, hence the theorem, so that when it multiplies by φ , even if φ is infinitely large, the result yielded still has some decimal places after the decimal point, or in other words, not an integer. $\square$

3 On Transcendentality

Lemma 3. *For any constant γ , there is a point with coordinates $(\gamma, r \sqrt[n]{1 - (\frac{\gamma}{r})^n})$, with an integer $r > \lfloor \gamma \rfloor$, that satisfy the equation $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$.*

Proof of Lemma 3. As proven in Lemma 1, the equation $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ is satisfied if there exist numbers x, y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$. Taking advantage of this definition, let $x = \gamma$ and $z = r$, with $r > \lfloor \gamma \rfloor$, a condition that ensures the r^{th} level of the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ passes through a

point with x coordinate as γ . If we perform this substitution of x and z , then y , via algebraic manipulations, must be $\sqrt[n]{r^n - \gamma^n}$. Finally, if we factor an r out of the n^{th} root of $r^n - \gamma^n$, all we are left with is $y = r \sqrt[n]{1 - (\frac{\gamma}{r})^n}$, which is the y coordinate of such a point described in Lemma 3. $\square$

Lemma 4. *If $(\frac{\gamma}{r})^n$ is rational, then γ must be able to be written as $\sqrt[n]{\frac{a}{b}}$, with a and b being integers.*

Proof of Lemma 4. Suppose that $(\frac{\gamma}{r})^n$ is rational, then γ^n must also be rational as r^n is rational. Because of this, γ^n must be able to be expressed as $\frac{a}{b}$, with a and b being integers. Finally, we have:

$$\begin{aligned} (\gamma)^n &= \frac{a}{b} \\ \gamma &= \sqrt[n]{\frac{a}{b}} \end{aligned}$$

$\square$

Lemma 5. *If γ is transcendental, then $(\frac{\gamma}{r})^n$, with r and n being integers and $n > 2$, cannot be rational.*

Proof of Lemma 5. Let γ be a transcendental number such that it satisfies the equation $\gamma^n - k = 0$, with n being an integer greater than 2. By definition, transcendental numbers are not solutions to a non-zero polynomial with integer coefficients. Therefore, in order for γ to be transcendental, k must be either a fraction that does not simplify to an integer or an irrational number. However, we can rule out the first possibility because if $k = \frac{c}{d}$ with c and d being integers, the equation $d\gamma^n - c = 0$ must also be true, which is certainly a contradiction to our assumption that γ is a transcendental number because the coefficients of the equation, namely c and d , are defined as integers. Therefore, the only remaining possibility is that k is an irrational number. Moreover, because $\gamma = \sqrt[n]{k}$ and k is an irrational number, γ cannot be expressed as the n^{th} root of some fraction $\frac{a}{b}$, with a and b being integers. Therefore, the assumption that γ is a transcendental number leads to the fact that $(\frac{\gamma}{r})^n$ cannot be rational as proven in Lemma 4. $\square$

Theorem 2. *If there exist a number x , a transcendental number y , and an integer z such that $x^n + y^n = z^n$ for any integer $n > 2$, then x must be irrational.*

Proof of Theorem 2. As proven in Lemma 5, if a constant γ is transcendental, given that it satisfies all other conditions stated in Lemma

5, then $(\frac{\gamma}{r})^n$ must be irrational. Because 1 is a rational number, we can conclude that $1 - (\frac{\gamma}{r})^n$ must be irrational. Lastly, because $1 - (\frac{\gamma}{r})^n$ is irrational, its n^{th} root must also be irrational. Therefore, if there is a pair of solutions (x, y) to the equation $x^n + y^n = z^n$, given that z is an integer, and one of them is a transcendental number such as e or π , then the other solution must be irrational. $\square$

4 Final Remarks

Conjecture 1. *Using the two proposed theorems in this paper, if we are given a graph in the same family as the graph of $\sin(\pi \sqrt[n]{x^n + y^n}) = 0$ and a point on that graph, if we choose to reveal one of the given point's coordinate, we can make an educated guess about the rationality and transcendentality of the other coordinate. If the coordinate that we choose to reveal turns out to be rational, then the other coordinate must be irrational. If the coordinate that we choose to reveal turns out to be irrational but not transcendental, then the other coordinate can either be rational, irrational but not transcendental, and transcendental, which is the worst-case scenario. Lastly, if the coordinate that we choose to reveal turns out to be transcendental, then the other coordinate must be irrational.*

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