Magnitudes Count in Mathematics Education – an Interdisciplinary View.

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Abstract

How do children learn basic skills such as discriminating between different quantities and counting? According to the dominant Approximate Number System (ANS) theory, humans are born with the ability to discriminate between discrete quantities (i.e., numbers). It follows from this that an early mathematics curriculum can or even should focus on discrete quantities. This theory guides many early-mathematics curricula worldwide. We provide a review of empirical evidence challenging the ANS theory and introduced a more recent theoretical framework, the Approximate Magnitude System (AMS). This theory suggests that continuous magnitudes (such as area, density, volume, etc.) are more intuitive and acquired earlier than the ability to understand numbers. Using examples from early mathematics education practices, we discuss the potential benefits of taking the AMS approach and using continuous magnitudes as a scaffolding for understanding numbers and more complex mathematical concepts (as suggested, for instance, by Davydov). Insights gained from studies that combine both cognitive psychology and educational research, with the active participation and contribution of early-mathematics teachers, may be of assistance to both cognitive psychologists, interested in how mathematics abilities develop, and to educators, looking to improve their mathematics curriculum. We hope that this article will inspire others to consider research in this direction and start a productive discussion on the role of continuous magnitudes in teaching mathematics.

Introduction

Many studies, spanning decades, are dedicated to the question of how young children learn mathematics. The idea is that knowing how basic arithmetic skills are acquired, will allow us to teach mathematics more efficiently, more like the way we learn mathematics 'naturally'.

In this paper we will first review the cognitive literature, presenting the consensus, the Approximate *Number* System (ANS) theory and its implications for school curricula. Next, we will present the caveats to this theory and discuss an alternative, the Approximate *Magnitude* System (AMS) theory. The AMS theory is in line with V.V. Davydov's theories, but from a cognitive perspective. Finally, we will discuss the benefits of a magnitude-focused curriculum, as suggested by Davydov and the AMS theory, and propose some activities for implementing the basic principles of Davydov's and the AMS theory in practice.

The Approximate Number System theory

Cognitive studies that investigated the origin of basic numerical skills across individual development and across cultures built a consensus that mathematical abilities rests upon innate basic abilities (such as estimating and comparing quantities). The argument is that such basic skills are made possible by a dedicated neural mechanism (an innate 'sense of number') allowing us to compare, perceive and manipulate non-symbolic quantities (numerosities) from birth. This theory is known as the Approximate Number System (*ANS*) theory. The ANS is assumed to be the precursor for more advanced mathematical skills, such as learning symbolic numbers (e.g., Cantlon, Platt, & Brannon, 2009).

One of the most common methods used to study ANS skills is the numerosity comparison task. In such a task, participants see two groups of items (usually dots) and choose the group containing more items. For example, Halberda and Feigenson (Halberda & Feigenson, 2008) asked 3-6-year-olds to compare which group contains more items, to test the development of the acuity of the ANS with age. Such a task does not require any verbal language skills or

formal knowledge, making it highly suitable not only for studies with western human adults but also for developmental and cross-cultural studies.

The ANS theory influences the way we investigate mathematics abilities and build school curricula. Insofar as studies of mathematical cognition have had an impact on curriculum, the consensus has supported pedagogy that focuses on numeracy development at the expense of other aspects of mathematics.

Number-focused curriculum

Counting, number, and calculating are commonly seen as the royal road in early mathematics education. Some believe it needs to be more so (Ontario Ministry of Education, 2018). An example of such influence of the ANS theory on school curriculum is Dehaene's triple code model. Coles and Sinclair (Coles & Sinclair, 2017) discussed the triple code model (Dehaene, Piazza, Pinel, & Cohen, 2003) where the assumption is that the meaning of number can come only from non-symbolic quantities. This model was highly influential in designing curricula in which cardinality is taught before ordinality. In other words, numbers are taught as absolute and not as relative. In a similar way Dehaene's and others' theories regarding the innateness of numbers (e.g., Dehaene, 1997; Izard, Sann, Spelke, & Streri, 2009; Nieder, 2005), backed up by neurocognitive research, may have tilted the scale for the primacy of number.

While the ANS model promotes an emphasis on number, there are also historical reasons why this is so. Mathematical instruction texts and instruction itself evolved from, among other ancestors, Fibonacci's *Liber Abaci* (see Devlin, 2011), a text that itself sprang out of the need of merchants to calculate easily. There are also frequent political recastings of mathematics education which aim to promote counting, number and calculation over other aspects of mathematics, presumably for reasons that link to political values such as conservatism and authority. For instance, the Department for Education in England is proposing reduced Early Learning Goals for three to five-year olds. The existing version has a Numbers goal, and also a goal for Shape, Space and Measures: "Children use everyday language to talk about size, weight, capacity, position, distance, time and money to compare quantities and objects and

to solve problems. They recognise, create and describe patterns. They explore characteristics of everyday objects and shapes and use mathematical language to describe them." The version now being proposed on the other hand has goals only for number and numerical patterns (Department For Education, 2018).

Similarly, in Canada, the Ontario Ministry of Education's 2018 teacher guidance, *Focusing on the Fundamentals of Maths* (Ontario Ministry of Education, 2018) outlines a new exclusive focus on number, whereas the previous curriculum had extensive sections on measurement and on geometry and spatial sense (Ontario Ministry of Education, 2005).

The Approximate Magnitude System (AMS) theory

Numerosity comparison tasks, which are commonly used in the context of ANS studies, have a serious caveat: Any group of items that has different numerosity also has different *continuous magnitudes*. Continuous magnitudes are any magnitude that need to be measured rather than counted, such as density, volume, total area, etc. Items have different sizes, while groups of items take up different areas on the screen, have different item density, etc. (Figure 1).

Figure 1: Continuous magnitudes change with numerosity. A) The relationship between numerosities and continuous magnitudes. The upper carts demonstrate the common positive correlation: more items take up more space in the shopping cart. The lower carts show that sometimes this correlation is violated. **B)** Numerosity and continuous magnitudes in experimental designs (numerosity comparison task). Each array contains six dots. The continuous magnitudes of these arrays, however, differ from one another. **C)** Different dot arrays will always have different continuous magnitudes. The left-hand side of each rectangle contains six dots and the right-hand side contains ten dots. As can be seen in the different versions, manipulating one magnitude results in differences in other magnitudes as well, making it impossible to control for all the continuous magnitudes at the same time.

In the environment, numerosity and continuous magnitudes are often correlated: A group with more items will usually take up more space, cover more area and be denser than a group with fewer items (Figure 1a-top provides an everyday example). Hence, when participants view two groups of items, as in the study of Halberda and Feigenson mentioned above, not only does the number of items change but there is a change in continuous magnitude such as the total surface area of all the items. Consequently, it is impossible to be sure that participants rely only on numerosity when making their decisions, as suggested by some ANS advocates.

To reduce the possibility of participants using continuous magnitudes rather than numerosity, some studies have attempted to disrupt the correlation between numerosity and continuous magnitudes. The logic is that if continuous magnitudes do not consistently predict numerosity, they will not influence performance. One popular method of reducing the potential influence of continuous magnitudes on performance is to generate a positive correlation between some continuous magnitudes and numerosity in half of the presented pairs of stimuli and a negative correlation in the other half, thus eliminating the overall correlation between numerosity and continuous magnitudes (Figure 1c). So, in half the trials, the group containing more dots occupies a greater total surface area, and the other group containing fewer dots occupies less surface area. This is the natural positive correlation we encounter in everyday life. In other words, this condition is congruent with our expectations and for that reason is known as a ***congruent*** condition. In the other half of the trials, the group with fewer dots takes up more total surface area. This goes against our expectations and is incongruent with them. This condition is therefore known as ***incongruent***. When asked to choose the group containing more dots, both children and adults were found to be faster and more accurate in congruent conditions than in incongruent conditions. This is known as the ***size congruity effect*** (e.g., Gilmore et al., 2016).

The congruity effect suggests that even when continuous magnitudes are not a predictive cue of numerosity throughout the experiment, they nevertheless influence performance. As a result, it is incredibly difficult (if not impossible) to determine on what basis participants select

the larger numerosity (Leibovich, Katzin, Harel, & Henik, 2017 for more details). This hints at a possibly important role for continuous magnitude in these tasks and in everyday life.

Evidence supporting the role of continuous magnitudes in numerosity processing

Empirical evidence suggests that continuous magnitudes are processed and influence performance even when they are irrelevant to the task and do not predict numerosity (Clearfield & Mix, 1999; Mix, Huttenlocher, & Levine, 2002; Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013). For example, in one study (Leibovich & Henik, 2014), adult participants compared the number of dots in two groups. The ratio between the number of dots in these groups, was manipulated, with the ratio calculated as the smaller divided by the larger number of dots. For example, the numerical ratio between a group of five dots and a group of 10 dots is $5/10 = 0.5$. The closer a ratio is to one, the more similar are the to-be-compared magnitudes. Performance is worse when the ratio is close to 1, an effect known as the ratio effect (Moyer & Landauer, 1967). In addition, the ratio between five continuous magnitudes of the compared groups of dots was calculated. The results demonstrated that both numerosity ratio and continuous magnitudes ratio influence response times (RT) and accuracy rates, even though the continuous magnitudes were task-irrelevant, and none of the continuous magnitudes could have been used as a cue to predict numerosity. Consistent with this finding, other studies, with children and adults, have demonstrated that when number and continuous magnitudes are negatively correlated or incongruent (Figure 1c), RT is slower and accuracy is lower than when the correlation between number and continuous magnitudes is positive or congruent (Gilmore et al., 2016; Leibovich & Ansari, 2017).

Figure 2: Theoretical model for development of the concept of number. The model suggests that people are born with the ability to discriminate continuous magnitudes, not numbers, and that the ability to understand the concept of number depends on understanding the positive correlation between number and continuous magnitudes. For more details about the model, see (Leibovich et al., 2017).

New theoretical model for the development of the concept of number

Based on empirical studies and a review of the literature, Leibovich et al (2017) proposed a new theoretical model to describe the development of the concept of numbers (**Figure 2**).

This model suggests that humans are born with the ability to discriminate between continuous magnitudes, when the difference is large enough. According to this model, it is impossible for a newborn to differentiate numerosities for two main reasons: first, the visual acuity of newborns is 25 times worse than that of an adult, therefore, at a distance of over ~40cm (breastfeeding distance) for an infant younger than five months, the environment appears as a blur. Second, until the age of five months, infants cannot individuate items; they cannot tell that an object is separated from the background, or where one object ends, and another object begins. Under these conditions, it is difficult to imagine how, even if there were an innate 'number sense', it could be useful before key physiological and cognitive systems mature.

After the age of about five months, two important things happen: first, the infant is able to individuate items. Second, the development of language skills, and specifically acquisition of the first number words (for numbers <5). Both of these together emphasize numerosity, and the attention of the infant is now directed not only to the whole group of items, but also to the individual items. At this stage, the model suggests, children learn, by experimentation with the environment, the natural correlation between numerosities and continuous magnitudes. Namely, more items will occupy more space, will be heavier to lift, etc., compared with fewer items. At first, children over-apply this correlation to every situation, even to exceptional situations like that described on **Figure 1a**. It is only later, with the development of cognitive control, and specifically, the ability to ignore irrelevant information, that children can start to "break" the correlation and understand that while continuous magnitudes and numerosity correlate, there are exceptions, and that sometimes (as in the case with the grocery store cart), number is more important than continuous magnitudes to the task at hand (for instance in choosing a line to the cashier). It is only then that the child is ready to understand the concept of number, as something that is correlated with continuous magnitudes, but independent of it. For example, two coins and two buildings have the same numerosity although their continuous magnitudes are vastly different. We argue that before that understanding occurs, a child cannot understand basic features required to learn counting and symbolic numbers. For example, if a child's idea of "where there is more" is tied to continuous magnitude, then s/he cannot understand that 3 dots is more than 2 dots, if the 2 dots are physically larger.

Educational outcomes of the suggested model

If, as we suggest, continuous magnitude rather than number is the fundamental, in the sense of being the foundation on which both the spatial and the numerical knowledge is built, there will be implications for instruction.

Children have built up a wealth of practical knowledge about magnitude by the time they come to school. We build our sense of continuous magnitudes as we reach or step, judge distances for a jump, or judge a gap for whether we can get through it. While playing with

playdough, water or sand, children construct a sense of how quantities behave. They experience turning as they run or ride a scooter, length as they whirl a rope around, time as they wait, or have to stop. It might be asked; how does all this serve them in their journey through the school mathematics curriculum? Don't teachers have to steer the subject, gently or abruptly towards counting, and then to arithmetic?

Magnitudes are everywhere in young children's play. A common list of play schemas is (Grimmer, 2017): transporting, enveloping, enclosing, trajectory, rotation, connecting, positioning, and transforming. Every one of these involves continuous magnitudes of some kind (as well as arrangements and geometry and topology and pattern - but counting not so much). If we have a number-skewed conception of mathematics, we do not credit these play types for all the mathematics they contain.

As Tony Cotton (2019) writes:

"I have described previously the enjoyment in observing my grandchildren creating patterns, experimenting with filling and emptying containers or loading toy trucks with rods. Interestingly, what they don't do is count. They only count when asked by teachers or other adults. Counting is their lowest mathematical priority."

The merits of including continuous magnitudes in school curriculum

As children enter school with an extensive prior sense of magnitudes, gained through experiment with the environment from a very early age, we suggest that teachers value and build upon that experience. For example, a four-year-old is in the sandpit constructing a tower from wooden blocks. Its top is now above his head, but he is trying to make the tower higher. He shows no sign of counting the blocks. What he seems to be interested in is its height, perhaps its height compared to him. Maybe he wants to see where the limit of his reach can take him. What does the teacher interested in talking with him, and also interested in touching on the mathematics, say? As number and counting is so much to the forefront of our thinking, perhaps it's tempting to say, 'I wonder how many blocks you've used?' Following the interest

of the child, it might be apt to comment on magnitude: 'Wow, that tower goes up really high! Are you taller or shorter than the tower? Stand next to it; Let me see.'

We would also like to argue that activities with continuous magnitudes in general, and spatial materials in particular, can help children more easily acquire complex mathematical concepts. Such concepts may include counting, early perception of ratio and proportions, and estimation of inexact quantities.

This approach is not altogether new to those familiar with the work and the theoretical framework put forward by the Russian psychologist V.V. Davydov (1930-1998). Davydov argued that numbers cannot be the elementary building blocks of mathematics because they are defined based on other, more fundamental mathematical concepts. Davydov suggested in the 1970's that early mathematics education should start with comparing physical objects' magnitude properties, such as length, mass, area and volume. First the questions are of equality or inequality (e.g., 'who has more milk?'). Guidance and instruction through these naturally occurring experiences teaches children relationships between different magnitudes, and important properties such as $a+b = b+a$, that will help them understand more complex algebra in the future. Davydov claimed that numbers should be introduced when the question moves from 'who has more' to 'how much more', and there is a need to accurately define the relationship between quantities. In other words, numbers are to be taught as relative, not absolute, and as measuring units of magnitudes (Venenciano & Dougherty, 2014). Accordingly, Davydov (1990) developed a mathematics curriculum in Russia that rested on continuous magnitudes. Davydov is not the only one to have designed structured materials to allow systematic explorations of magnitudes: Friedrich Froebel's Gifts in the nineteenth century (e.g., Lange, 2018) and Maria Montessori's sensorial materials in the early twentieth century (Lillard & Stoll, 2006) are influential examples of magnitude-based curricula.

Another example demonstrating the principles of Davydov's and the AMS theory are the Cuisenaire rods developed by George Cuisenaire, in the mid-twentieth century. These are wooden rods where length (or area or volume) represents discrete magnitudes. Cuisenaire

rods are not demarcated to show units and are not compared through counting. Instead, magnitudes (length, volume) are being compared to one another and the relative relationships between different magnitudes are emphasized before symbolic numbers are introduced (Gattegno, 1963). When five year olds, asked to make 'trains' of Cuisenaire rods that are the same length, make an arrangement like the one in **Figure 3**, as well as experiencing yellow + pink = blue, they might become aware of commutativity (yellow + pink = pink + yellow) and the distributive property ($3(\text{pink} + \text{yellow}) = 3 \text{ pink} + 3 \text{ yellow}$). The children will also begin to know that the blue rod is nine times the length of the smallest rod, while the yellow is five and the pink is four. When they have become aware of this, they experience $9 = 4 + 5$ and $9 = 5 + 4$ too. They do this without counting, and indeed, if they had to count, say with connecting cubes, the direct experience of these gestalts could easily get lost in counting units.

Figure 3 – Cuisenaire rods

Continuous magnitudes serve later mathematical learning

Counting and addition are usually seen as first steps in children's learning of number. But if we focus on these to the exclusion of continuous magnitudes, might we make later developments harder? Consider these scenarios:

We are looking at how much two plants have grown. Both the smaller one and the bigger one have grown two centimetres. Which has grown the most? Going through the usual path of counting then addition, children may have left aside whatever awareness they have of how comparisons are often multiplicative. Our natural logarithmic sense is that two dogs are the same height if they are 1 cm difference in their heights, but two ants with the same difference are very different in size. Are there ways to keep our unschooled sense of ratio alive during our foray into addition?

There is evidence that young children can think about proportional or multiplicative situations when they are considering continuous magnitudes in a way they cannot with discrete magnitudes. Jeong, Levine and Huttenlocher (2007) found that six-year olds were able to make a choice to optimise the proportion of one colour in a probability situation when the colour was continuous, without units that could be counted and not when there were discrete parts (see **Figure 4**). They suggest that, in the discrete condition, the children's counting may interfere with the gestalt perception of the proportions.

Figure 4: After Jeong, Levine, Huttenlocher, *The Development of Proportional Reasoning: Effect of Continuous Versus Discrete Quantities*, 2007. Six-year old children were more successful in selecting the choice with the higher proportion of red in the continuous condition.

The continuous condition here could be adopted in instructional practice. Spinners are a regular feature in primary classrooms and could be adapted for this. But the implications go

beyond the particular materials used.

Another example: Creating a growing pattern from, say, blocks, children plot points on a graph. Then, there is the option of adding a line through all the points. Does this idea make sense to the children? We look at how the area of a leaf can be estimated by tracing it onto squared paper. We could count all the squares inside as an approximation. We see how, if we make the squares small, our approximation becomes more accurate. We imagine continuing to make the squares smaller until, at the limit, we have the correct area of the leaf. We have come the long way around to being able to talk about the size of an irregular area, something we already have a 'sense of' a young age.

Conclusions

In the current manuscript, we have provided evidence-based arguments for the importance of continuous magnitudes to numerosity comparison and estimation, and for the difficulty of claiming that numbers are innate and processed automatically regardless of continuous magnitudes. We have introduced a recent theoretical framework, based on empirical evidence, for the importance and possible contribution of continuous magnitudes to the acquisition of the concept of number. This line of research, alongside much classroom practice, points to a major role for 'how much?' rather than 'how many?' in scaffolding and supporting all classroom mathematics.

Davydov's framework and curricula based on these ideas were tested in the education literature and were found to be successful in aiding children with different levels of prior attainment in mathematics to learn basic concepts (Dougherty & Slovin, 2004; Gattegno, 1963; Venenciano & Dougherty, 2014). However, as far as we know, there were no attempts to study Davydov's theory from a cognitive perspective. This would be interesting because some of Davydov's suggestions speak indirectly to the role of cognitive control – such as ignoring the physical size of the whole and understanding what the whole is regardless of its physical size. One can look at the activities sketched above, in the spirit of Davydov's theoretical framework, such as playing with sand, water, etc., alongside instructional activities with

Cuisenaire rods and similar materials in the light of the suggested AMS model: the relationship between different magnitudes is learned first, and later, if necessary, the understanding of this relationship is translated to discrete exact numerosities. Empirical study is necessary to investigate whether a causal connection exists between the activities mentioned above and a later success in acquisition of basic and advanced mathematics skills, and whether a curriculum based on such activities would improve and promote deeper and more intuitive understanding of mathematical concepts.

We would like to suggest that teaching that restricts children's classroom learning to the discrete, the numeric and the additive, narrows the learners' experience, making the transition to the continuous, the uncertain, the algebraic and the multiplicative harder. Moreover, it fails to use a powerful resource that students already possess and on which their numeric learning is based. Further studies, that combines both cognitive psychology and educational research, with the active participation and contribution of early-mathematics teachers, would allow us to develop our claims. The insights gained from such an endeavor could be of assistance to both cognitive psychologists, interested in how mathematical abilities develop, and to educators, looking to improve mathematics curricula. We hope that this review will inspire others to consider research in this direction and contribute to a productive discussion on the role of continuous magnitudes in teaching mathematics.

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